PAPER FOLDING AND CONVERGENT SEQUENCES

Paper folding can help in understanding some infinite sequences and in finding their limits. A simple physical model useful at all levels of ability is presented and infinite sequences of interest to senior high school students are explored.

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THERE are many well-known physical representations of convergent sequences. A standard example is to traverse the length of a room by walking half the remaining distance each time a "step" is taken. One sequence corresponding to this model $(0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots)$ has 1 as its obvious limit.

There is a sequence (see table 1) that also has a simple and interesting physical model but whose limit is not so obvious. And precisely because of these facts, the sequence and its physical representation can be used as the basis and motivation for several interesting lessons.

The physical model requires the student (or the teacher) to fold a strip of paper according to the directions given in the next section. One lesson that can be based on this exercise involves discovery of a mathematical description of the resulting sequence and finding the limit of this sequence. This is done in the section called "Mathematical Description of the Sequence." In the last section, "Other Related Lessons," some other lessons that can be built around the physical model are suggested.

Obtaining a Sequence by Folding

Take a strip of adding-machine tape at least twelve inches long. (Any size piece

of paper that is suitable for folding will do, but at least one of its edges must be straight). Label the left edge A and the right edge B (see fig. 1a).

The sequence of folds is as follows:

1. Fold B to the *left* to coincide with A (see fig. 1b); call the crease that is created by this fold, C (see fig. 1c).

2. Without unfolding the paper, fold B to the *right* to coincide with C, forming a second crease, D (see fig. 1d). Just as a check, the paper (when viewed from the side) should now look like fig. 1e).

3. Without unfolding, fold B to the *left* to coincide with D, creating crease E (top view in fig. 1f, side view in fig. 1g).

4. Continue as before, successively folding B to the right, left, and so on, so that at the end of each folding operation, B coincides with the crease made last.

After three more folds, for example, the side view should look like figure 1h. (Making the folds is actually quite simple, certainly much simpler than the verbal description.)

An interesting question comes to mind immediately: Where will edge B appear if one could continue folding indefinitely?

Some students might not be convinced that this question is meaningful (unlike the room-traversing example where most students are certain that the other side of the room would eventually be reached). One way to clarify the question is to unfold the paper so that it resembles figure 1a again, except that now it is creased. Make a cut with scissors from B to fold
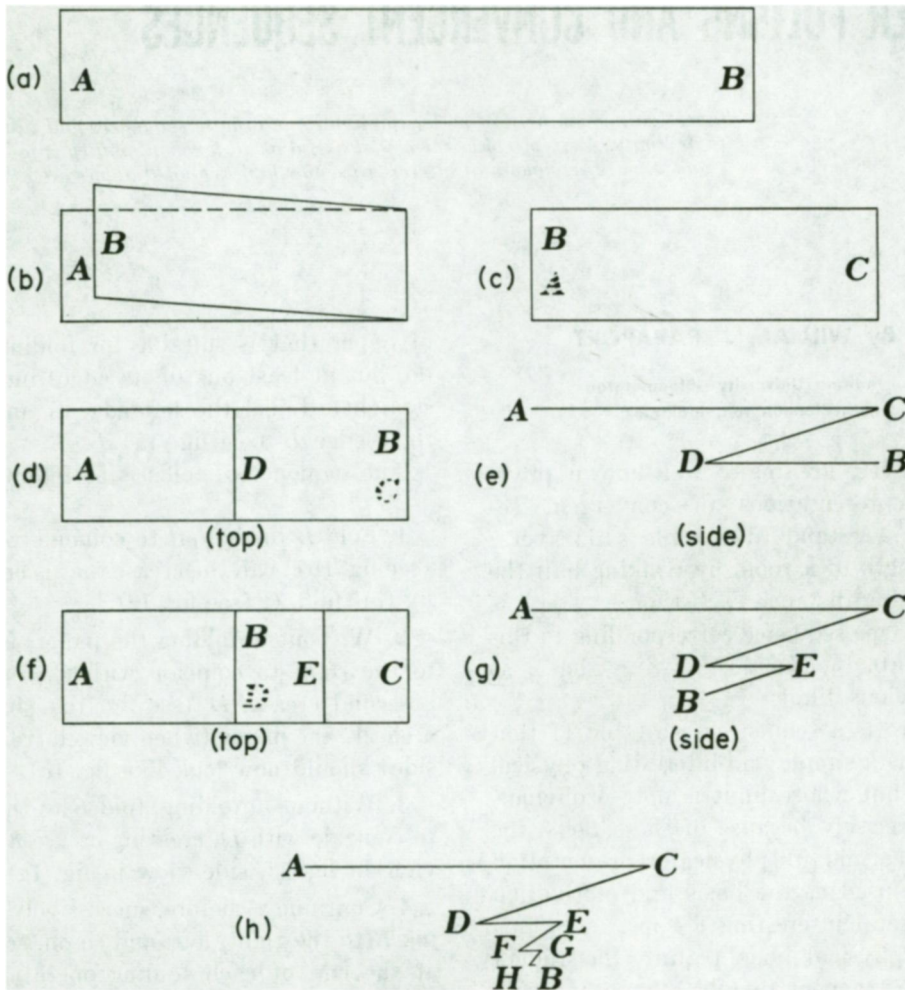
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Fig. 1

C and then up to the top edge (see fig. 2), thus cutting off one-fourth of the paper. Now refold the paper as before, this time marking a dot after each fold is made in order to show where B lands (see fig. 3).

If the dots are thought of as points on a number line, you should be able to see how they begin to cluster around one point. (This can be made even clearer by using a longer piece of paper, thus making it possible to make more folds.) It is the cluster, or limit, point that we desire to find. You might guess, by inspection, that the limit point is a certain, very simple, fraction of the way from A to B. Let us now find out exactly.
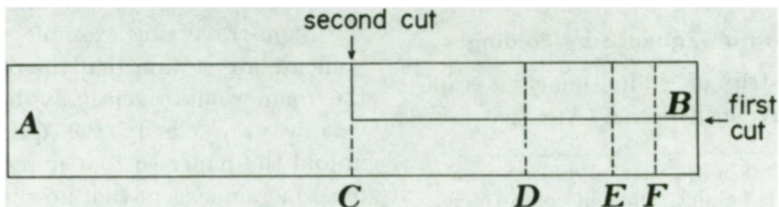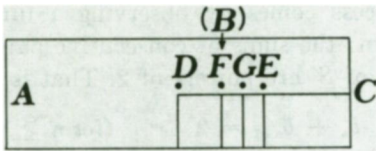

Fig. 2

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Fig. 3

Mathematical Description of the Sequence

Think of the top edge of the paper as a number line, with 0 at edge A and 1 at edge B. We have already mapped the sequence of folds onto a sequence of dots or points. We now want to map this point sequence onto a sequence of numbers so that we can employ numerical techniques to answer our question. The trick is to do this in an efficient way. Allowing each student in a laboratory situation to do this in his or her own way is advisable. Alternatively, some class time could be spent on deciding precisely what mapping to use. I suggest the following.

After fold 1, B is at 0 on the number line. After fold 2, B is at $\frac{1}{2}$, since it then coincides with crease C (which was the result of folding the paper in half). After fold 3, B coincides with crease D, which was obtained by folding the halved paper in half; B is now at $\frac{1}{4}$. Although the fourth fold halves again, the result is not $\frac{1}{8}$, but $\frac{3}{8}$. The reason for this may be seen by realizing that the exercise is a physical model for *averaging*: after the fourth fold, B is in the *middle* of its last two positions ($\frac{1}{4}$ and $\frac{1}{2}$). Folding, that is, averaging, a fifth time yields $\frac{5}{16}$. Table 1 may be derived.

TABLE 1

Fold	Position of B
1	0
2	1/2
3	1/4
4	3/8
5	5/16
6	11/32
7	21/64
8	43/128
9	85/256
10	171/512
11	341/1024
.	.
.	.
.	.

We thus obtain the following sequence (of positions of B):

$$S: 0, \frac{1}{2}, \frac{1}{4}, \frac{3}{8}, \frac{5}{16}, \cdots .$$

Our question may now be phrased more precisely: What is the limit of sequence S?

In order to answer this question, it is necessary to find a formula for arbitrary terms of S. Clearly, the denominators of the terms are powers of 2. In fact, for fold n, the denominator (of the fraction representing the position of B after the nth fold) is 2^{n-1}. Let the first term (i.e., 0) in sequence S be called a_1, the second term (i.e., $\frac{1}{2}$) a_2, and so on, so that we may say that the denominator of a_n is 2^{n-1} (for $n \geq 1$; it is assumed throughout that n is a natural number). Now the numerator of a_n for each n is needed.

It is at this point that the students' problem-solving abilities are put to the test, for although there are many patterns to be found in this sequence, a pattern that will be useful for our purposes can prove to be quite elusive. One method (out of many) begins by expressing S recursively (where a_1 and a_2 are given) and then finding an equivalent formula in which a_n depends only on n (i.e., a formula that does not require knowing in advance any terms of S).

We have already seen that each term is the average of the two preceding terms. (Note that before folding, i.e., at "fold" 0, B is located at 1. Question: Where is B at "fold" -1?) Thus,

(1)
$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= \tfrac{1}{2} \\
a_{n+2} &= \frac{a_n + a_{n+1}}{2} \quad \text{(if } n \geq 1\text{).}
\end{aligned}
$$

Formula (1) is a recursive formula (with two initial conditions) that generates S. It allows us to imagine folding as many times as we wish, eliminating the practical limitations of the thickness and length of the paper. The problem, though, is to find

$\lim\limits_{n\to\infty} a_n$, and, as interesting as (1) may be, it does not help much.

Recall that a formula is needed for the numerator of a_n for each n. That is, a formula is wanted for an arbitrary term of the sequence S' of numerators:

$$S': \quad 0, 1, 1, 3, 5, 11, \ldots .$$

Call each term of this new sequence b_n (for $n \geq 1$). Thus, $a_n = b_n/2^{n-1}$ (for $n \geq 1$). Next is the derivation of a simple recursive formula for any term of S', wherein each term depends on the two previous terms. The determination of this formula is a nice short exercise by itself. The result follows:

$$\begin{aligned} b_1 &= 0 \\ (2) \qquad b_2 &= 1 \\ b_{n+2} &= 2b_n + b_{n+1} \qquad \text{(if } n \geq 1\text{)}. \end{aligned}$$

However, this is not just what was wanted.

There is a more interesting line of attack. Is there any property of either sequence, S or S', that reflects the physical fact that the folds alternate from right to left? There are at least two such properties.

First, observe that each term of S' (after the first) is either one more than or one less than twice the preceding term; that is, $b_{n+1} = 2b_n \pm 1$ (for $n \geq 1$). One can then verify (or discover) formula (3):

$$\begin{aligned} (3) \qquad b_1 &= 0 \\ b_{n+1} &= 2b_n + (-1)^{n-1} \qquad \text{(if } n \geq 1\text{)}. \end{aligned}$$

Here, the alternation in folding direction is reflected in the alternating parity of the terms of the sequence $(1, -1, 1, -1, \ldots)$ generated by $(-1)^{n-1}$. Formula (3) is halfway to the general formula; it represents an improvement over (2) in that it has only *one* initial condition. In order to eliminate the need for *any* such initial condition, it is necessary to express the nth term of S' (or S) in terms of n only.

Success comes by observing a further pattern: the sums of consecutive pairs of terms of S' are powers of 2. That is,

$$(4) \qquad b_n + b_{n+1} = 2^{n-1} \qquad \text{(for } n \geq 1\text{)}.$$

Since (4) is equivalent to

$$(5) \qquad b_{n+1} = 2^{n-1} - b_n \qquad \text{(for } n \geq 1\text{)},$$

the right-hand sides of (3) and (5) may be set equal to each other in order to find the desired formula ($n \geq 1$, throughout).

$$\begin{aligned} 2b_n + (-1)^{n-1} &= 2^{n-1} - b_n \\ \therefore \quad 3b_n &= 2^{n-1} - (-1)^{n-1} \\ (6) \qquad \therefore \quad b_n &= \frac{2^{n-1} - (-1)^{n-1}}{3} \\ (7) \qquad \therefore \quad a_n &= \frac{2^{n-1} - (-1)^{n-1}}{3 \cdot 2^{n-1}}. \end{aligned}$$

With (7) we have reached our goal. It now remains to find $\lim\limits_{n\to\infty} a_n$. This may be done as follows:

$$\begin{aligned} \lim_{n\to\infty} a_n \\ = \lim_{n\to\infty} \frac{2^{n-1} - (-1)^{n-1}}{3 \cdot 2^{n-1}} \\ = \frac{1}{3} \cdot \lim_{n\to\infty} \frac{2^{n-1} - (-1)^{n-1}}{2^{n-1}} \\ = \frac{1}{3} \left(\lim_{n\to\infty} \frac{2^{n-1}}{2^{n-1}} - \lim_{n\to\infty} \frac{(-1)^{n-1}}{2^{n-1}} \right) \\ = \frac{1}{3} \left(1 - \lim_{n\to\infty} (-\tfrac{1}{2})^{n-1} \right). \end{aligned}$$

But $|-1/2| < 1$; therefore, $\lim\limits_{n\to\infty} (-1/2)^{n-1} = \lim\limits_{n\to\infty} (-1/2)^n = 0$ (cf. Walter Rudin, *Principles of Mathematical Analysis*, theorem 3.20). Hence, $\lim\limits_{n\to\infty} a_n = 1/3$, as the reader may have conjectured. (An interesting point to consider is this: Where did the 3 come from, when it was powers of 2 that seemed to play such an important role? *Hint:* Look at (6).)

A different method of solution arises from the aspect of S' that reflects the alternating property of the folding rule. Two subsequences can be derived from S'—one for even values of n and one for odd values of n (see table 2).

Clearly, the odd values correspond to

TABLE 2

Odd		Even	
n	b_n	n	b_n
1	0	2	1
3	1	4	3
5	5	6	11
7	21	8	43
9	85	10	171
.	.	.	.
.	.	.	.
.	.	.	.

folds to the *left* and the even values to folds to the *right*. The student should discover (or be led to see) that the pair-wise differences of the terms of the odd subsequence are 1, 4, 16, 64, . . . , and those for the even subsequence are 2, 8, 32, 128, ... That is, for the *odd* subsequence, the pair-wise differences are $2^0, 2^2, 2^4, \ldots ,$ $2^{2k}, \ldots$ $(k \geq 0)$, whereas for the *even* subsequence, the pair-wise differences are $2^1, 2^3, 2^5, \ldots , 2^{2k+1}, \ldots$ $(k \geq 0)$.

The students should then find (following the methods suggested above) formulas for these subsequences, calculate the limits for each subsequence, and see that each limit is ⅓. This alternative approach affords a good example of the fact that if a sequence converges to a limit L, then all of its subsequences also converge to L.
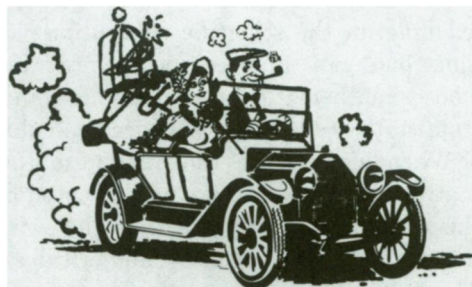
Other Related Lessons

One of the nicer aspects of this exercise is that it can be used at many levels of ability and for many different purposes.

At an elementary level, all the numerical manipulations can be ignored and the emphasis placed on making the folds, in order to give the students an intuitive idea of limits that differs from the more familiar example mentioned at the beginning.

A lesson could also be developed for a unit on measuring. For such a lesson, a twelve-inch length of adding-machine tape and a twelve-inch ruler for each student (or a thirty-six-inch length and a yard-stick for a larger group) would be ideal. The student could then measure, after each fold, the distance of each crease from edge A, getting a sequence whose limit is 4″ (or 12″ for the 36″ length). This would not only afford practice in measurement, but also give the students a good feel for the limiting process. (One warning for metric enthusiasts: ⅓ of 12″ is a very "clean" 4″, but ⅓ of 10 cm. might prove impractical. Try a 15 cm. length or some other multiple of 3; after all, a teacher's materials must be prepared in advance just as much as a magician's!)

For a class studying fractions, this exercise could be used to give the students practice in working with fractions, using formula (1) to derive terms in the sequence. Moreover, by changing each fraction into its decimal equivalent, the students not only will have practice in division, but also will be able to "see" the terms of the sequence approach .333 . . . , both from "above" and from "below." And, of course, it may serve as a motivating exercise for a unit on averaging.