CSE 531 Homework Assignment 5

Due in class on Tuesday, Nov 20.

November 5, 2007

There are totally 6 problems, 10 points each. You should do them all. We will grade only 4 problems chosen at my discretion. If it so happens that you don't do one of the problems we don't grade, then no points will be deducted.

Note: this homework is on solving problems with network flow formulation. You should describe the formulation (along the line of the maximum matching in bipartite graph problem), show why solving the flow problem gives a solution to the problem at hand. Make sure the the formulation does not induces an exponential time algorithm in terms of the problem you're trying to solve. You do not need to be precise on the running time, as long as you're certain that it is a polynomial-time algorithm. This requires, at the very least, that the "transformation" from an instance of the problem you're solving to the corresponding network flow instance can be done in polynomial time.

Problem 1. We define the *Escape Problem* as follows. We are given a directed graph $G = (V, E)$ (picture a network of roads). A certain collection of nodes $X \subset V$ are designated as *populated nodes*, and a certain other collection $S \subset V$ are designated as *safe nodes*. (Assume that X and S are disjoint.) In case of an emergency, we want evacuation routes from the populated nodes to the safe nodes. A set of evacuation routes is defined as a set of paths in G so that (i) each node in X is the starting point of one path, (ii) the last node on each path lies in S, and (iii) the paths do not share any edges. Such a set of paths gives a way for the occupants of the populated nodes to "escape" to S, without overly congesting any edge in G.

(a) Given G, X, and S, show how to decide in polynomial time whether such a set of evacuation routes exists.

(b) Suppose we have exactly the same problem as in (a), but we want to enforce an even stronger version of the "no congestion" condition (iii): we change (iii) to say "the paths do not share any *nodes*."

With this new condition, show how to decide in polynomial time whether such a set of evacuation routes exists.

(c) Provide an example with the same G, X, and S, in which the answer is yes to the question in (a) but no to the question in (b).

Problem 2. Consider the problem faced by a hospital that is trying to evaluate whether its blood supply is sufficient.

The basic rule for blood transfusion is the following. A person's own blood supply has certain *antigens* present (we can think of antigens as a kind of molecular signature); and a person cannot receive blood with a particular antigen if their own blood does not have this antigen present.

Concretely, this principle underpins the division of blood into four *types*: A, B, AB, and O. Blood of type A has the A antigen, blood of type B has the B antigen, blood of type AB has both, and blood of type O has neither. Thus, patients with type A can receive only blood types A or O in a transfusion, patients with type B can receive only B or O, patients with type O can receive only O, and patients with type AB can receive any of the four types.

(a) Let s_O, s_A, s_B, and s_{AB} denote the supply in whole units of the different blood types on hand. Assume that the hospital knows the projected demand for each blood type d_O, d_A, d_B and d_{AB} for the coming week. Give a polynomial-time algorithm to evaluate if the blood on hand would suffice for the projected need.

(b) Consider the following example. The typical distribution of blood types in U.S. patients is roughly 45% type O, 42% type A, 10% type B, and 3% type AB. The hospital wants to know if the blood supply it has on hand would be enough if 100 patients arrive with the expected type distribution. There is a total of 105 units of blood on hand. The table below gives these demands, and the supply on hand.

Blood type	supply	demand
O	50	45
A	36	42
B	11	10
AB	8	3

Is the 105 units of blood on hand enough to satisfy the 100 units of demand? Find an allocation that satisfies the maximum possible number of patients. Use an argument based on a minimum-capacity cut to show why not all patients can receive blood. Also, provide an explanation for this fact that would be understandable to the clinic administrators, who have not taken a course on algorithms. (So, for example, this explanation should not involve the words *flow, cut* or *graph* in the sense we use them in our course.)

Problem 3. Let m and k be positive integers. Let $G = (L, R; E)$ be a bipartite graph satisfying the following conditions: (a) all vertices in L have degree m, (b) all vertices in R have degree mk.

Show that we can color the edges of G using only m colors (each edge gets one color) such that vertices in L are incident to edges with different colors, and vertices in R are incident to exactly k edges of each color.

Problem 4. There are n students in a certain university in the Midwest. The ith student can only take at most c_i courses in a semester. (For simplicity, let's assume that every student can take as many courses as she/he wants subject to the upper bounds c_i.) There are m courses that are going to be offered next Spring. Every course needs at least k registered students, otherwise it will be canceled. Moreover, the ith student has a set S_i of courses that she/he prefers. No student registers for a course she/he does not prefer.

(a) Describe a polynomial-time algorithm which evaluates whether the students can register for Spring courses so that no course will be canceled.

(b) Now, suppose each student belongs to one of l ethnic groups. For cultural diversity, the university imposes the constraint that a course will be canceled if all registered students are from the same ethnic group. Describe a polynomial-time algorithm to evaluate whether there is a way for students to register for courses so that no course is canceled.

Problem 5. Let $G = (V, E)$ be a directed graph, with source $s \in V$, sink $t \in V$, and nonnegative edge capacities $\{c_e\}$. Give a polynomial-time algorithm to decide whether G has a *unique* minimum s, t-cut (i.e., an s, t-cut of capacity strictly less that that of all other s, t-cut).

Problem 6. Two optical switches R and S are connected by f optical fibers. There are totally w different wavelengths $\lambda_1, \ldots, \lambda_w$. However, due to various physical limitations, the jth fiber can only accommodate up to n_j different wavelengths (any subset of at most n_j wavelengths is OK), for $1 \le j \le f$.

A set C of connections are to be routed from switch R to switch S. Each connection in C is to be carried on a pre-assigned wavelength. In the set C, there are m_i connections which were pre-assigned with wavelength λ_i, $1 \le i \le w$.

We are to route the connections in C through (R, S), namely each connection in C is assigned to one of the f fibers such that no two connections with the same wavelength are assigned on the same fiber, and that the jth fiber does not get assigned to more than n_j connections.

Suppose $m_1 \ge \cdots \ge m_w$, and $n_1 \le \cdots \le n_f$. Show that the routing can be done if and only if, for all k, and l, where $0 \le k \le w$, $0 \le l \le f$, it holds that $k(f-l) + \sum_{j=1}^{l} n_j \ge \sum_{i=1}^{k} m_i$.