# The Probabilistic Method

### Techniques

- Union bound
- **Argument from expectation**
- Alterations
- The second moment method
- The (Lovasz) Local Lemma

### And much more

- Alon and Spencer, "The Probabilistic Method"
- Bolobas, "Random Graphs"

# The Argument from Expectation: Main Idea

- $X$ a random variable with $\mathsf{E}[X] = \mu$, then
  - There must exist a sample point $\omega$ with $X(\omega) \geq \mu$
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Have we seen this?

# Example 1: Large Cuts in Graphs

## Intuition & Question

Intuition: every graph must have a "sufficiently large" cut $(A, B)$.
Question: How large?

## Line of thought

On average, a *random* cut has size $\mu$, hence there must exist a cut of size $\geq \mu$.

- Put a vertex in either $A$ or $B$ with probability $1/2$
- Expected number of edges $X$ with one end point in each is

$$\mathsf{E}[X] = \mathsf{E}\left[\sum_e X_e\right] = \sum_e \mathsf{Prob}[X_e] = |E|/2$$

## Theorem

*For every graph $G = (V, E)$, there must be a cut with $\geq |E|/2$ edges*

# Example 2: $\pm 1$ Linear Combinations of Unit Vectors

**Theorem**

*Let $\mathbf{v}_1, \cdots, \mathbf{v}_n$ be $n$ unit vectors in $\mathbb{R}^n$.*
*There exist $\alpha_1, \cdots, \alpha_n \in \{-1, 1\}$ such that*

$$|\alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n| \leq \sqrt{n}$$

*and, there exist $\alpha_1, \cdots, \alpha_n \in \{-1, 1\}$ such that*

$$|\alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n| \geq \sqrt{n}$$

Simply because on average these combinations have length $\sqrt{n}$.
Specifically, choose $\alpha_i \in \{-1, 1\}$ independently with prob. $1/2$

$$\mathsf{E}\left[|\alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n|^2\right] = \sum_{i,j} \mathbf{v}_i \cdot \mathbf{v}_j \mathsf{E}[\alpha_i \alpha_j] = \sum_i \mathbf{v}_i^2 = n.$$

# Example 3: Unbalancing Lights

**Theorem**

*For $1 \le i, j \le n$, we are given $a_{ij} \in \{-1, 1\}$. Then, there exist $\alpha_i, \beta_j \in \{-1, 1\}$ such that*

$$\sum_i \sum_j a_{ij}\alpha_i\beta_j \ge \left(\sqrt{\frac{2}{\pi}} + o(1)\right) n^{3/2}$$

- Choose $\beta_j \in \{-1, 1\}$ independently with prob. $1/2$.
- $R_i = \sum_j a_{ij}\beta_j$, then

$$\mathsf{E}[|R_i|] = 2\frac{n\binom{n-1}{\lfloor (n-1)/2 \rfloor}}{2^n} \approx \left(\sqrt{\frac{2}{\pi}} + o(1)\right) n^{1/2}$$

- Choose $\alpha_i$ with the same sign as $R_i$, for all $i$