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Teaching Mathematical Induction: An Alternative Approach

Partly as a result of an increase in computer technology, iterative and recursive methods are becoming more common. If taught well, mathematical induction can help students' understanding of these methods grow. *Principles and Standards for School Mathematics* recommends that "students should learn that certain types of results are proved using the technique of mathematical induction" (NCTM 2000, p. 345).

When students are learning mathematical induction, they usually begin by proving such summation formulas as

$$\sum_{i=1}^{n} i = \frac{(n)(n+1)}{2}$$

and similar formulas. This standard way of teaching induction may be common in both high school and college classrooms because these types of problems appear in textbooks. Unfortunately, Baker (1996) found that several difficulties arise with this type of learning. Students often have little experience with summation notation, so before they even begin, the traditional instruction presents a stumbling block. Another problem frequently arises during the inductive step of the proof because students think that they are assuming what they are trying to prove. Even students who have experience in logic and proof, which most high school students lack, may have this belief. Another danger is that students may become proficient at induction algorithms while gaining little or no understanding of why induction works. In particular, students can learn to verify summation formulas by induction without understanding what they are doing or why.

Students think that they are assuming what they are trying to prove

This article presents my experience in a classroom setting using a new approach to teaching induction that is being developed by the Mathematical Methods in High School project. I recorded my detailed observations and an assessment in journal form. The basic idea behind the new approach is to use induction to prove that two formulas, one in recursive form and the other in a closed or explicit form, will always agree for whole numbers.

I taught the unit on induction for five days in a precalculus class at Delavan High School, in Delavan, Illinois. It is a rural high school with a total enrollment of 171 students. Precalculus is the highest-level mathematics class offered at the school. No class at Delavan has an "honors" designation, but the students are predominantly college bound. The class consisted of ten girls and eight boys; six were juniors, and twelve were seniors. Little technology was used in the class; however, students who had graphing calculators were allowed to use them. The lack of technology required adjustments in teaching the recursion section because the textbook encourages the use of computers to show examples with large numbers.

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Day 1—functions, patterns, and rules

We began class with a brief review of functions. Then we made an input-output table for $f(n) = n^2$, as shown in **figure 1.** Next we looked at the three tables shown in **figure 2.** I instructed the class to find functions that matched each table. To make sure that everyone understood, we did table A first and discussed it as a class. As I had anticipated, the students looked at the problems differently. For example, most students quickly observed that table A could describe the function $f(n) = 2n$, but at least

n	$f(n)$
0	0
1	1
2	4
3	9

Fig. 1
A standard input-output table for $f(n) = n^2$

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one student claimed that the answer was $f(n) = n + n$ before quickly realizing that the answers were equivalent. The convention that we used was $f(n)$ for the explicit form, or what we called the *closed form,* and $h(n)$ for the recursive form, where n is a whole number.

After the students broke into groups, most of them quickly obtained functions that fit tables B and C. The majority gave the functions for table B as $f(n) = n^2 + n$ and for table C as $f(n) = n^2 + 2n$. One group, however, gave the solution as the product of n and "the next one"; that group wrote the answers as $f(n) = n(n + 1)$ and $f(n) = n(n + 2)$. I asked one student who made this realization to present it at the board. Most of the students still proclaimed that they preferred the first method, but more than one student applied the latter method on the homework.

While students worked the problems, they began to develop their own strategies for completing them. For example, they quickly realized that if $f(0) = C$, then $f(n)$ should end in $+C$. Also, many students commented that if a table was decreasing, "it had to have a minus in it." The classroom teacher noted that students who usually struggled in his class were holding their own during this lesson and sometimes outperformed students who regularly outpaced them in class. Many students took the initiative to complete the unassigned problems.

Day 2—recursive definitions

I was apprehensive going into the second day's lesson because recursive thinking is very different from what most high school students have experienced. As I had expected, they found the second day's material more difficult than that covered on the previous day.

As suggested in the Mathematical Methods in High School materials, we began class by looking at table D, shown in **figure 3.** I explained that a new way to match a function to a table would be to use the previous term in the sequence instead of simply manipulating the input, n.

A few students pointed out that this function could be generalized to the form $h(n) = h(n - 1) + 5$, but not all the students saw it immediately. The textbook suggested proceeding to problems where the students are asked to generate input-output tables from a recursive function. I thought that working on problems where the students looked at a table and found the related function would be more beneficial. However, that decision was probably a mistake.

The students found these problems to be particularly difficult. The students who attempted to work the problem by the method shown in the materials—that is, writing it out in the form $h(4) = h(3) + 1$, $h(3) = h(2) + 1, \ldots$ —were far more successful than the students who tried to skip straight to the answer. I asked one student who had mastered this method to demonstrate it on the board, and that presentation seemed to help some of the students who had been confused.

We went back to the previous page of the materials to discuss problems similar to the one in **figure 4.** The students were given a recursive function and asked to generate the input-output table in that figure. To my surprise, students picked up on this process much more quickly. A common comment that I heard during group work was that "you just have to keep going down until you hit zero and then go back up." I then began to recognize a shortcoming. Several students proclaimed that they could do these problems, but finding the function from the table still confused them. They did not realize that the two problems were tightly connected. I had hoped that they would realize that problems of the second type are the reverse of the first type. Generating tables apparently comes more naturally than finding the function.

Table A

n	$f(n)$
0	0
1	2
2	4
3	6

Table B

n	$f(n)$
0	0
1	2
2	6
3	12

Table C

n	$f(n)$
0	0
1	3
2	8
3	15

Fig. 2
Examples of tables used to generate functions of the closed, or explicit, form

Table D

n	$f(n)$
0	1
1	6
2	11
3	16
4	21

Students noticed this trend:
$f(4) = f(3) + 5$
$f(3) = f(2) + 5$
$f(2) = f(1) + 5$
$f(1) = f(0) + 5$

Fig. 3
Students found a recursive pattern for this table.

Recursive thinking is very different from what most high school students have experienced

Given:

$$f(n) = \begin{cases} 2 \text{ if } n = 0 \\ f(n-1) + 4 \text{ if } n > 0 \end{cases}$$

Find $f(3)$

Answer:

n	$f(n)$	
3	$\downarrow f(2) + 4$	$10 + 4 = 14$
2	$\downarrow f(1) + 4$	$6 + 4 = 10\uparrow$
1	$\downarrow f(0) + 4$	$2 + 4 = 6\uparrow$
0	2	$2\uparrow$

Fig. 4
Most students organized their work similarly to this.

The textbook includes problems and examples of difference tables similar to the one shown in **figure 5.** I had planned to discuss this method with the students and even assign a problem or two of this type in the homework. I was surprised to find that doing so seemed unnecessary. Because almost every student began generating these difference charts with no prompting on my part, I believe that searching for these differences is an idea that comes naturally to the students.

Some students thought that if f(n) and h(n) are equal for every whole-number value of n, then the functions are equal

n	$f(n)$	
0	0	
1	2	2
2	6	4
3	12	6

Fig. 5
Students should recognize that an increase of $2n$ occurs from one term to the next.

Overall, most students seemed to grasp the basics of the ideas presented in the recursion section, but they were still more comfortable with the closed form. I believe that having access to computer technology would have been especially useful on this day. With only a simple recursive program or a spreadsheet, the students could have obtained results for much larger values of n.

Day 3—review and introduction to mathematical induction

The students had difficulty doing the homework problem shown in **figure 6.** We discussed it as a class, and they found two distinct ways of looking at the problem. Some preferred to use a function similar to the rest of the homework: $h(n) = h(n - 1) + (2n + 3)$. A surprisingly large number of students, though, had instead realized that another possibility was $h(n) = (\sqrt{h(n-1)} + 1)^2$. Since each solution was recursive and fit the chart, I accepted both solutions. I used this opportunity to point out that many equivalent solutions may exist for a given function.

One student made the interesting observation that in the problems that he had worked, the additive term in the recursive definition related to the derivative of the closed definition. For example, if $f(x) = x^2$, then

$$\begin{aligned} h(x) &= h(x-1) + f'(x) + C \\ &= h(x-1) + 2x + C, \end{aligned}$$

where $C = -1$. The classroom teacher asked the student to explain why he thought that the derivative was related, and the student explained that the recursive definition depends on the rate of change of the function.

n	$f(n)$
0	4
1	9
2	16
3	25

Fig. 6
A homework problem

We next moved on to an introduction to induction. We looked at **figure 7.** With relative ease, students were able to deduce both the closed and recursive definitions of the function, which we named $f(n) = 3n + 1$ and $h(n) = h(n - 1) + 3$, respectively. I asked the students whether the two functions were equal. "No," came the initial response, "because they don't look the same." I then asked what it means for two functions to be equal. Some students thought that if $f(n)$ and $h(n)$ are equal for every whole-number value of n, then the functions are equal. They were reasonably certain that $f(n)$ and $h(n)$ are equal for these values, even though the table went to only $n = 3$. I said that we should try to verify that $f(n)$ and $h(n)$ are the same for a value of n that was not on the chart.

n	$f(n)$
0	1
1	4
2	7
3	10

Fig. 7
An introduction to induction

I next asked the following question: "Since we know that $f(3) = h(3)$, can we verify whether $f(4) = h(4)$?" The class agreed that we could easily do so by calculating $f(4)$ and $h(4)$. However, the students also agreed that this approach would be more difficult for really large values of n. I said that we would develop a way to show that if $f(3) = h(3)$, then $f(4) = h(4)$. I wrote $h(4)$ on the board, skipped a few lines, and wrote $= f(4)$. With a little prodding, the students obtained the following proof:

$$\begin{aligned} h(4) &= h(3) + 3 \\ &= f(3) + 3 && \text{(because } f(3) = h(3)\text{)} \\ &= (3 \cdot 3 + 1) + 3 && \text{(because } f(3) = 3 \cdot 3 + 1\text{)} \\ &= (3 \cdot 3 + 3) + 1 \\ &= (3 \cdot 3 + 3 \cdot 1) + 1 \\ &= 3(3 + 1) + 1 \\ &= 3(4) + 1 \\ &= f(4) \end{aligned}$$

Day 4—introduction to mathematical induction

To begin the day, we finished the argument involving the closed form $f(n) = 3n + 1$ and the recursive form

$$h(n) = \begin{cases} 1 \text{ if } n = 0 \\ h(n-1) + 3 \text{ if } n > 0. \end{cases}$$

I asked students to find both the closed and recursive forms and show that knowing that $f(4) = g(4)$ implied that $f(5) = g(5)$. Most students were able to come up with the argument, and those who did not seemed to understand it when it was written on the board.

I next tried to help the class generalize the method. I pointed out that we knew that $f(5) = h(5)$, and I asked whether we could do anything with that information. Several students pointed out that the same argument would work for $f(6) = h(6)$. I asked what we could do with that information. The students began to realize that the method could be done enough times to show that $f(n) = h(n)$ for any n that they wanted to show. One student said that this argument meant that the functions were equal everywhere. I asked him how many cases we would have to show. The class agreed that we would have to show every number, and that exercise was not practical.

Someone stated that we should put in n's. I asked him to clarify what he meant, and I then led the class to the following proof:

$$\begin{aligned} h(n+1) &= h(n) + 3 \\ &= f(n) + 3 \\ &= 3n + 1 + 3 \\ &= (3n + 3) + 1 \\ &= 3(n + 1) + 1 \\ &= f(n + 1) \end{aligned}$$

I explained that because we had a starting point of $f(0) = h(0)$ and because we had a rule that said that if the functions were equal at one value of n, they were equal at the next one, we could truly say that the functions were equal. I did not have time to discuss the "starting point" as thoroughly as I had wanted to, so I saved that discussion for the final day of the lesson.

The students continued to gain insight into the similarities between closed and recursive forms and the idea of induction. One girl pulled me aside to point out that she had noticed that on a linear function in closed form, the constant in front of x would be the constant added in the recursive form. She said that she was sure that this result had occurred on every problem that we had attempted. The classroom teacher and I briefly discussed with her that the constant slope in a linear function determines how much of an increase in $f(x)$ exists for an increase of 1 in the x-value.

Another student asked me after class why she could not start with $f(n + 1)$ and end up with $h(n + 1)$. I explained that either order was acceptable; I had started with $h(n + 1)$ only because the materials used it and because I thought that it might be the easier direction to understand.

Day 5—introduction to mathematical induction concluded

On the last day, all that remained was to formalize the induction argument. Almost all students were confident—perhaps too much so—of their ability to show the inductive step of the proofs from the homework problems. Common remarks included, "This was easy," or "I taught this to a student who missed class yesterday."

We still had to formalize the induction proof, so I used the old-fashioned domino argument. We looked at a problem from the homework. The inductive step was on the board, but the basis step was not. Before I had a chance to ask the class, one student asked me whether the inductive step was enough to show that the functions were equal. I opened discussion of this question up to the students, some of whom disagreed. I claimed that even though we had a rule that would show that if our claim held for n, then it would hold for $n + 1$, we needed something more. We needed an initial point that would get the rule started.

I offered the domino example. I set up a row of dominoes on the desk and asked the class what usually happens when someone sets up a row of dominoes. They gave me the obvious answer that each domino knocks over the next one. Then I asked why the dominoes were still standing if we had a rule that says that each domino knocks over the next. "Ohhh . . . ," came the reply, "we have to knock over the first." The entire class seemed satisfied with this explanation of why we had to show $f(0) = h(0)$ for the proof to be complete.

In retrospect, I realize that I should have offered a counterexample. Finding one that results in a false proof is not difficult. For example, let $f(n) = 3n + 1$ and

$$h(n) = \begin{cases} 2 \text{ if } n = 0 \\ h(n-1) + 3 \text{ if } n > 0. \end{cases}$$

We can still prove the inductive step, but the two functions are not equal. Clearly, the basis step is necessary.

Because the students had primarily been using linear functions, all of which have similar inductive proofs, I thought that I would finish by showing them a situation that had a different proof. We looked at the exponential function defined by the table in **figure 8.** This problem is analogous to one that was assigned in the homework. I used induction to show that the closed form $f(n) = 4^n$ is equiv-

Students were confident of their ability to show the inductive step of the proof

alent to the recursive form $h(n) = 4 \cdot h(n - 1)$. The students were uncomfortable with the slightly different proof, but they seemed to understand it. They had more trouble finding the two functions than doing the inductive proof.

n	$f(n)$
0	1
1	4
2	16
3	64

Fig. 8
An exponential function

ASSESSMENT

My teaching experiment concluded with an examination that included four questions. The first question asked the students to find the closed form of a function for two input-output tables; the second question asked for the recursive form; the third question asked the students to do two induction proofs, one on a linear function and one on an exponential function; and finally, question 4 asked them to explain the two steps of induction.

Not surprisingly, the students had little trouble with the first and second questions. During the week, they had become comfortable with both the closed and recursive forms. In the third question, the students successfully did the inductive proof with the linear function. However, only half the students could do the exponential problem, which had been discussed only briefly in class. All but a few students gave a coherent explanation of the inductive step in the final question, and many gave excellent answers. Interestingly, students had a harder time explaining the reason for the basis step. Still, most of them again gave a reasonable answer.

The approach circumvented many difficulties in the traditional method for teaching induction

REACTION

The outcome of this project suggests that this approach may interest teachers who teach induction, especially teachers whose students have learned to define functions iteratively or recursively. The approach circumvented many difficulties encountered in using the traditional method for teaching induction. The students seemed to become comfortable with closed and recursive forms in a short period of time. No student objected that he or she started by assuming what had to be proved. After doing the proof with several specific values, the students could clearly see why doing so with variables is safe.

Of course, the most important question is whether the students really learned induction. The results from the examination were encouraging but not overwhelmingly favorable. Few students made incorrect statements in their explanations of the steps of induction, but many of them did not include enough information. They perhaps have not had enough practice putting mathematical phenomena into words.

A major concern about the induction section of the textbook that uses the new approach was the lack of variety in the problems. Most problems involved linear functions, and a few were exponential. Perhaps high school students can identify only so many functions from an input-output table. Regardless, more types of functions are necessary so that students can see the variety of ways in which induction can be applied.

With a good introduction to mathematical induction, teachers should find that their students are comfortable with it in a variety of contexts. Another use that should not present notational problems is using induction to prove that certain quantities have certain factors. For example, one can use induction to show that 3 is a factor of $4^n - 1$ (UCSMP 1992). The author hopes that this introduction might make other, more traditional problems, such as proving summations, easier for students to understand. Of course, a creative teacher will use induction in appropriate places throughout the curriculum.

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For more information on the approach being developed by Education Development Center (EDC), contact Helen Lebowitz, EDC, 55 Chapel Street, Newton, MA 02458 (hlebowitz@edc.org), and ask about the Mathematical Methods in High School project.

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