LOGIC
Techniques of Formal Reasoning

Second Edition

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The Greek Alphabet

Greek Letter	Name
α	Alpha
β	Beta
γ	Gamma
δ	Delta
ϵ	Epsilon
ζ	Zeta
η	Eta
θ	Theta
ι	Iota
κ	Kappa
λ	Lambda
μ	Mu
ν	Nu
ξ	Xi
ο	Omicron
π	Pi
ρ	Rho
σ	Sigma
τ	Tau
υ	Upsilon
ϕ	Phi
χ	Chi
ψ	Psi
ω	Omega

Logical Symbols

Symbol	Name
~	Negation sign
→	Conditional sign
∧	Conjunction sign
∨	Disjunction sign
↔	Biconditional sign
⋀	Universal quantifier
⋁	Existential quantifier
=	Identity sign
℩	Descriptive operator

Because the characterization of the class of symbolic sentences on page 4 is stipulated to be exhaustive, No. 2, if a symbolic sentence, must result from one of the three clauses of that characterization. Since neither of the clauses (1) or (2) can generate an expression beginning with a parenthesis, clause (3) must yield No. 2. For this to happen, either both 'P' and 'Q → R', or both 'P → Q' and 'R', must be symbolic sentences. But in view of considerations like the foregoing, neither 'Q → R' nor 'P → Q' is, strictly speaking, a symbolic sentence; each fails to be a conditional by virtue of lacking a pair of peripheral parentheses. Hence No. 2 is not a symbolic sentence.

3. ((~(~P → Q) → Q) → R)
4. (~(~P) → Q)
5. ((P → Q) → ~G)
6. (~((P → Q) → R) → (P → ~Q))

2. From symbols to English and back. Frequently it will be desirable to *translate* a symbolic sentence into English and perform the reverse process of *symbolizing* an English sentence. We consider first the passage from symbols to English.

As was mentioned earlier, the correlation between sentence letters and the English sentences which they abbreviate is quite arbitrary. Thus the passage from a symbolic sentence to a sentence of English must proceed on the basis of a *scheme of abbreviation*, which will establish such a correlation.

More explicitly, let us understand by an *abbreviation* an ordered pair of sentences, the first of which is a sentence letter and the second an English sentence. A *scheme of abbreviation* is a collection of abbreviations such that no two abbreviations in the collection have the same first member.

For example, the collections

(1)
Q : the lectures are dull
T : the text is readable
P : Alfred will pass

and

S : the lectures are dull
Q : the lectures are dull
T : the text is readable
P : Alfred will pass

are schemes of abbreviation; but the collection

Q : the lectures are dull
Q : the text is readable
P : Alfred will pass

is not. (The two abbreviations whose first member is 'Q' are at fault.)

The process of *literal translation into English on the basis of a given scheme of abbreviation* begins with a symbolic sentence and if successful ends with a sentence of English. The process consists of the following steps:

(*i*) *Restore any parentheses that may have disappeared as a result of applying the informal conventions of the last section.*

(*ii*) *Replace sentence letters by English sentences in accordance with the given scheme of abbreviation; that is, each sentence letter is to be replaced by the English sentence with which it is paired in the scheme.*

(*iii*) *Replace all occurrences of*

$$\sim\phi \quad ,$$

where ϕ *is a sentence, by*

it is not the case that ϕ .

(*iv*) *Replace all occurrences of*

$$(\phi \rightarrow \psi) \quad ,$$

where ϕ *and* ψ *are sentences, by*

(*if* ϕ, *then* ψ) .

For example, under the scheme of abbreviation (1) the sentence

(2) $Q \rightarrow (\sim T \rightarrow \sim P)$

becomes in step (i)

$$(Q \rightarrow (\sim T \rightarrow \sim P)) \quad ,$$

in step (ii)

(The lectures are dull → (~ the text is readable → ~ Alfred will pass)) ,

in step (iii)

(The lectures are dull → (it is not the case that the text is readable → it is not the case that Alfred will pass)) ,

and in step (iv)

(3) (If the lectures are dull, then (if it is not the case that the text is readable, then it is not the case that Alfred will pass)) .

(The parentheses in sentences such as (3), as in symbolic sentences, serve as marks of punctuation.)

We shall generally wish to be more liberal in translating from symbols to English than the notion of a literal translation will permit. For example, we should like to consider the sentence

(4) Assuming that the lectures are dull, if the text is not readable, then Alfred will not pass

as a translation of (2) on the basis of the scheme (1). Accordingly, we say that an English sentence is a *translation* (or, when a distinction is to be drawn, a *free translation*) of a symbolic sentence ϕ on the basis of a given scheme of abbreviation if it is a stylistic variant of the literal translation of ϕ into English on the basis of that scheme.

Because (4) differs only in style from (3), the former as well as the latter qualifies as an English translation of (2) on the basis of scheme (1).

In the realm of free translations, we countenance looseness. Specifically, we attempt no precise description of *stylistic variance;* in this connection, intuition (here identified with linguistic insight) rather than exact rules must guide the reader. To remove this source of looseness would require systematic exploration of the English language, indeed of what might be called the 'logic of ordinary English', and would be either extremely laborious or perhaps impossible. In any case, we do not consider such an exploration appropriate material for the present book (however, see Montague [4] and Partee [1]).

Although no exact definition of stylistic variance will be offered, we shall not leave the readers entirely to their own devices. Two stylistic variants of

It is not the case that Alfred concentrated

are

Alfred did not concentrate

and

Alfred failed to concentrate .

English idiom provides a number of stylistic variants for

(5) (If Diogenes is a canine, then Diogenes is carnivorous) ,

for example,

(6) Diogenes is carnivorous if Diogenes is a canine ,

(7) Diogenes is carnivorous provided that Diogenes is a canine ,

(8) Diogenes is a canine only if Diogenes is carnivorous ,

(9) Only if Diogenes is carnivorous is Diogenes a canine .

The only difference (apart from the use of parentheses) between (5) and (6) is the order in which the clauses occur. Alternative (7) comes from (6) upon

replacement of 'if' by its intuitive equivalent 'provided that'. To see the intuitive equivalence between (5) and (8), the following consideration should be of assistance: to assert that Diogenes is canine only if carnivorous is to deny that Diogenes is a canine and not carnivorous, which is to assert that if Diogenes is a canine, then he is also carnivorous; but this amounts to (5). Alternative (9) comes from (8) by inversion of the order of clauses. (Note that here English idiom also requires an inversion of the word order in the antecedent clause of (5).)

If ϕ and ψ are any sentences, a partial list of stylistic variants of

(if ϕ, then ψ)

is the following:

if ϕ, ψ	ψ if ϕ
provided that ϕ, ψ	ψ provided that ϕ
given that ϕ, ψ	ψ given that ϕ
in case ϕ, ψ	ψ in case ϕ
assuming that ϕ, ψ	ψ assuming that ϕ
on the condition that ϕ, ψ	ψ on the condition that ϕ
ϕ only if ψ	

Further instances of stylistic variance may be obtained by introducing pronouns in place of nouns and by altering word order. Examples of these and other sorts of stylistic variance will be found among the exercises of this and later chapters.

We shall also be interested in the passage from English to symbols. Accordingly, we say that ϕ is a *symbolization* of an English sentence ψ on the basis of a given scheme of abbreviation if and only if ϕ is a symbolic sentence that has ψ as a translation on the basis of that scheme, in other words, if and only if ψ is a stylistic variant of the literal translation of ϕ on the basis of the scheme.

To find a symbolization of a given English sentence on the basis of a given scheme of abbreviation, the reader will find it useful to proceed roughly as follows:

(1) Introduce 'it is not the case that' and '(if . . . , then)' in place of their respective stylistic variants.

(2) Reverse the steps leading from a symbolic sentence to a literal English translation; that is,

(2a) replace all parts of the form

(if ϕ, then ψ) ,

where ϕ and ψ are sentences, by

$(\phi \rightarrow \psi)$;

(2b) *replace all parts of the form*

it is not the case that ϕ ,

where ϕ *is a sentence, by*

$\sim\phi$;

(2c) *replace English components by sentence letters in accordance with the scheme of abbreviation; that is, replace each English component by a sentence letter with which it is paired in the scheme of abbreviation;*

(2d) *omit peripheral parentheses and replace parentheses by brackets in accordance with the informal conventions of the preceding section.*

EXERCISES

On the basis of the scheme of abbreviation

P : logic is enjoyable
Q : Alfred will pass
R : Alfred concentrates
S : the text is readable
T : Alfred will secure employment
U : Alfred will marry
V : the lectures are exciting

translate the following symbolic sentences into idiomatic English:

7. $P \rightarrow (Q \rightarrow R)$
8. $(R \rightarrow Q) \rightarrow P$
9. $S \rightarrow [P \rightarrow (\sim Q \rightarrow \sim R)]$

On the basis of the scheme of abbreviation above, symbolize the following English sentences. In solving exercises 10–17 (as well as later exercises), the reader may find the following succinct summary a helpful (but not infallible) guide to stylistic variants of conditionals:

(i) replacement in a sentence of 'provided that', 'given that', 'in case', 'assuming that', 'on the condition that' by 'if' yields a stylistic variant of that sentence;

(ii) 'if' (not preceded by 'only') introduces an antecedent, whereas 'only if' introduces a consequent.

Exercise 10 is solved for illustration.

10. Only if Alfred concentrates will he pass, provided that the lectures are exciting.

Sentence No. 10 becomes, in step (1) of the procedure given on pages 11 and 12,

(If the lectures are exciting, then (if Alfred will pass, then Alfred concentrates))

In taking this step we first made minor stylistic changes in accordance with the above summary and replaced 'will he pass' by 'Alfred will pass'; then, in conformity with the discussion on pages 10 and 11, we inverted the order of the component clauses. In step (2a) the sentence becomes

(the lectures are exciting $\rightarrow$ (Alfred will pass $\rightarrow$ Alfred concentrates)) ;

step (2b) is inapplicable; step (2c) leads to

$$(V \rightarrow (Q \rightarrow R)) \quad ;$$

and step (2d) to

$$V \rightarrow (Q \rightarrow R) \quad .$$

11. If Alfred will pass if he concentrates, then logic is enjoyable.

12. Alfred will pass on the condition that if he will pass only if he concentrates then he will pass.

13. If Alfred will pass only on the condition that he concentrates, then logic is enjoyable provided that the lectures are exciting.

14. It is not the case that if Alfred will secure employment provided that logic is enjoyable, then he will marry only if he concentrates.

The following sentences are ambiguous, in the sense that the placement of parentheses in their symbolizations is not uniquely determined. Give all plausible symbolizations of these sentences on the basis of the scheme of abbreviation that appears above.

15. Alfred will pass only if he concentrates provided that the text is not readable.

16. It is not the case that Alfred concentrates if the lectures are not exciting.

17. It is not the case that Alfred will secure employment if he fails to concentrate on the condition that the lectures are not exciting.

3. Derivability and validity of symbolic arguments. An *argument*, as we shall understand it, consists of two parts: first, a sequence of sentences called its *premises*, and second, an additional sentence called its *conclusion*. An *English argument* is an argument whose premises and conclusion are sentences of English; similarly, a *symbolic argument* is an argument whose premises and conclusion are symbolic sentences. Any group of sentences may constitute an argument if that group is broken down into premises and conclusion—no matter how unrelated the premises and conclusion may seem. Furthermore, 'premise' and 'conclusion' are relative terms; that is, a sentence which occurs as a premise in one argument may be a conclusion in another, and conversely. Ordinarily, we shall present an omitted in certain contexts, as in the following examples. The symbolic sentence

$$(P \wedge Q) \rightarrow (R \vee S)$$

becomes

$$P \wedge Q \rightarrow R \vee S \quad ,$$

and

$$(P \vee Q) \leftrightarrow (R \wedge S)$$

becomes

$$P \vee Q \leftrightarrow R \wedge S \quad .$$

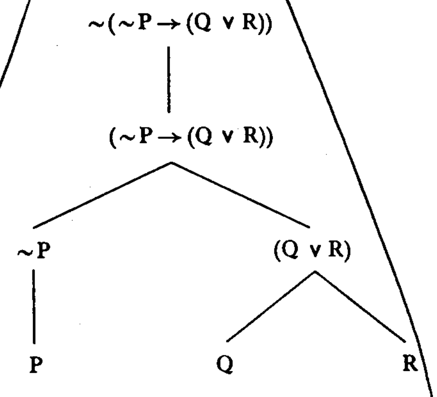
Thus we arrive informally at a larger class of symbolic sentences than that provided by clauses (1)–(3) above. In those few cases when it is necessary to draw a distinction, symbolic sentences of the smaller class may be called *symbolic sentences in the official sense.*

EXERCISES

Which of the following are symbolic sentences in the official sense? Exercises 1 and 2 are solved for illustration.

1. $\sim(\sim P \rightarrow (Q \vee R))$

According to clauses (1) and (2) of the characterization of symbolic sentences, '$\sim P$' is a symbolic sentence; and according to clauses (1) and (3), '$(Q \vee R)$' is a symbolic sentence. By clause (3), then, '$(\sim P \rightarrow (Q \vee R))$' is a symbolic sentence; thus, by clause (2), No. 1 is a symbolic sentence in the official sense. The following tree displays graphically the generation of No. 1:



2. $(P \vee (Q \rightarrow R) \wedge \sim P)$

Because the characterization of symbolic sentences given by clauses (1)–(3) is stipulated to be exhaustive, and clauses (1) and (2), together with the fourth part of clause (3), are inapplicable, No. 2, if a symbolic sentence, must be either a disjunction, a conditional, or a conjunction of symbolic sentences. Thus either

$$P$$

and

(5) $(Q \rightarrow R) \wedge \sim P$

must be symbolic sentences, or

(6) $P \vee (Q$

and

(7) $R) \wedge \sim P$

must be symbolic sentences, or

(8) $P \vee (Q \rightarrow R)$

and

$$\sim P$$

must be symbolic sentences. But (5), (6), and (8) (and, incidentally, (7)) are not symbolic sentences in the official sense, as the reader can easily verify. Thus No. 2 is not a symbolic sentence in the official sense.

3. $((P \leftrightarrow Q) \leftrightarrow ((P \wedge Q) \vee (\sim P \wedge \sim Q)))$
4. $\sim((P \vee Q \rightarrow R) \rightarrow ((P \rightarrow R) \wedge (Q \rightarrow R)))$
5. $\sim(\sim(\sim\sim P \vee Q) \vee R) \rightarrow (P \leftrightarrow (Q \leftrightarrow R))$

From each of the following symbolic sentences obtain a symbolic sentence in the official sense by restoring parentheses omitted by informal conventions. Exercise 6 is solved for illustration.

6. $(P \wedge Q \rightarrow P \vee Q) \leftrightarrow P \vee (Q \wedge R)$

This sentence becomes

$$(((P \wedge Q) \rightarrow (P \vee Q)) \leftrightarrow (P \vee (Q \wedge R))) \quad .$$

7. $Q \vee R \rightarrow (P \wedge R \rightarrow (Q \leftrightarrow R \vee P))$
8. $(P \wedge Q) \vee (\sim P \wedge \sim Q) \rightarrow ((P \leftrightarrow Q) \leftrightarrow (Q \leftrightarrow P))$
9. $(P \rightarrow Q \vee R) \vee (Q \vee R \rightarrow P) \leftrightarrow (Q \vee R \leftrightarrow P)$
10. $\sim(P \vee (Q \wedge R) \rightarrow ((P \vee Q) \wedge (P \vee R) \leftrightarrow P \wedge Q))$

2. Translation and symbolization.

2. Translation and symbolization. The process of *literal translation into English on the basis of a given scheme of abbreviation* begins as before with a symbolic sentence and if successful ends with a sentence of English.

The process consists of the following steps:

(*i*) *Restore any parentheses that may have disappeared as a result of the informal conventions of the last section.*
(*ii*) *Replace sentence letters by English sentences in accordance with the scheme of abbreviation.*
(*iii*) *Eliminate sentential connectives in favor of the corresponding phrases of connection, preserving all parentheses.*

As before, we call an English sentence a *free translation* (or simply a *translation*) of a symbolic sentence ϕ on the basis of a given scheme of abbreviation if it is a stylistic variant of the literal English translation of ϕ based on that scheme.

Let us consider, for example, the scheme of abbreviation

P : Alfred dances with Alice
Q : Alfred dances with Mary
R : Alfred will enjoy the party

and the symbolic sentence

(1) $$P \wedge Q \rightarrow R \quad .$$

In step (i) of the process of literal translation into English, this sentence becomes

$$((P \wedge Q) \rightarrow R) \quad ,$$

in step (ii)

((Alfred dances with Alice ∧ Alfred dances with Mary) → Alfred will enjoy the party) ,

and in step (iii)

(2) (If (Alfred dances with Alice and Alfred dances with Mary), then Alfred will enjoy the party) .

Thus (2) is the literal translation of (1) into English on the basis of the scheme above, and the more idiomatic sentence

(3) Alfred will enjoy the party if he dances with both Alice and Mary ,

being a stylistic variant of (2), qualifies as a free translation of (1) on the basis of the same scheme.

As regards stylistic variance, we persist in our loose practices, giving no exact definition but only a few examples. 'But', 'although', 'even though', 'both . . . and', and sometimes the relative pronouns 'who', 'which', 'that', serve as stylistic variants of 'and'; 'unless', 'either . . . or', as stylistic variants of 'or'; and 'exactly on condition that', and sometimes 'just in case', as stylistic variants of 'if and only if'. Thus each of

Alfred slept through class, but he passed ,
Even though Alfred slept through class, he passed ,
Alfred, who slept through class, passed

is regarded as a stylistic variant of

Alfred slept through class and Alfred passed ;

The needle on the ammeter will deflect unless the battery is dead

as a stylistic variant of

The needle on the ammeter will deflect or the battery is dead ;

and

Alfred will be elected exactly on condition that he stand for office ,

Just in case Alfred stands for office will he be elected

as stylistic variants of

Alfred will be elected if and only if Alfred stands for office .

It should be noted that 'or', and thus also its stylistic variant 'unless', has two senses in English. When a parent who is attempting to limit a child's consumption of sweets says 'ice cream or cake', the intent is to exclude the option of eating both ice cream and cake; but the host who says 'cream or sugar' is offering to a guest either cream, sugar, or both. For example, in passing from

The picnic will occur unless it rains

to

The picnic will occur or it rains ,

'or' is to be understood in the former, *exclusive*, sense if the intent is not to have the picnic if it rains; whereas, in passing from

Alfred does not pass unless he studies

to

Alfred does not pass or he studies .

'or' is to be understood in the latter, *inclusive*, sense if studying is considered necessary but not sufficient for Alfred to pass.

As a further illustration of stylistic variance, observe that the phrase 'neither . . . nor' is expressible by means of phrases of connection.

Neither ϕ nor ψ ,

where ϕ and ψ are sentences, has the stylistic variants

(It is not the case that ϕ and it is not the case that ψ)

and

It is not the case that (ϕ or ψ) .

Thus, for example,

(4) Neither Alice nor Mary danced with Alfred

could pass into

(5) It is not the case that Alice danced with Alfred *and* it is not the case that Alfred danced with Mary .

A more idiomatic form of (5), and thus also a stylistic variant of (4), is the sentence

(6) Both Alice and Mary did not dance with Alfred .

The sentence

(7) Not both Alice and Mary danced with Alfred

is sometimes confused with (6) but must be distinguished from it. For (7) has the sentence

(8) It is not the case that Alice danced with Alfred *or* it is not the case that Alfred danced with Mary

as a stylistic variant, and (8) clearly differs in meaning from (5). Thus (7), in contrast to (6), is not a stylistic variant of (4).

It should be emphasized that the *binary* sentential connectives, that is, '$\rightarrow$', '$\wedge$', '$\vee$', '$\leftrightarrow$', may stand only between sentences (or, in later chapters, between formulas). English usage, however, provides other contexts for the corresponding phrases of connection, as in the sentence (3) above,

(9) Socrates is either snub-nosed or bald ,

and

(10) Arcadia lies between Laconia and Achaea .

(3) and (9) have *canonical* stylistic variants, that is, stylistic variants in which phrases of connection operate only on sentences. We have seen this already in the case of (3), and (9) can be expanded into

(Socrates is snub-nosed or Socrates is bald) .

But (10) cannot be similarly treated; it is clearly not synonymous with

Arcadia lies between Laconia and Arcadia lies between Achaea .

As in chapter I, we say that ϕ is a *symbolization* of an English sentence ψ on the basis of a given scheme of abbreviation just in case ϕ is a symbolic sentence which has ψ as a translation on the basis of that scheme.

To find a symbolization of a given English sentence on the basis of a given scheme of abbreviation, the reader will find it useful to proceed roughly as follows:

(*1*) *Introduce phrases of connection, accompanied by parentheses and occurring canonically* (*that is, standing only before or between sentences*), *in place of their stylistic variants.*

(*2*) *Reverse the steps leading from a symbolic sentence to a literal English translation; that is,*

(*2a*) *replace all parts having one of the forms*

it is not the case that ϕ ,
(*if* ϕ, *then* ψ) ,
(ϕ *and* ψ) ,
(ϕ *or* ψ) ,
(ϕ *if and only if* ψ) ,

where ϕ *and* ψ *are sentences, by*

$\sim\phi$,
$(\phi \rightarrow \psi)$,
$(\phi \wedge \psi)$,
$(\phi \vee \psi)$,
$(\phi \leftrightarrow \psi)$

respectively;

(*2b*) *replace English components by sentence letters in accordance with the scheme of abbreviation;*

(*2c*) *omit parentheses and insert brackets in accordance with the informal conventions of the preceding section.*

EXERCISES

11. On the basis of the scheme of abbreviation

P : Alice will dance with Alfred
Q : Mary will dance with Alfred
R : Alfred will improve his deportment

translate the following symbolic sentence into idiomatic English:

$$([\sim P \wedge \sim Q] \vee R) \wedge (R \leftrightarrow P \wedge Q) \quad .$$

Symbolize each of the following sentences on the basis of the scheme of abbreviation that accompanies it. Exercise 12 is solved for illustration.

12. Errors will decrease in the subject's performance just in case neither motivation is absent nor learning has not occurred. (P : errors will decrease in the subject's performance; Q : motivation is absent; R : learning has occurred)

In step (1) of the informal procedure for symbolizing English sentences (p. 57), we may transform No. 12 into

> (Errors will decrease in the subject's performance if and only if it is not the case that (motivation is absent or it is not the case that learning has occurred)) .

In step (2a) the sentence becomes

> (Errors will decrease in the subject's performance $\leftrightarrow \sim$(motivation is absent $\vee \sim$learning has occurred)) ;

in step (2b),

$$(P \leftrightarrow \sim(Q \vee \sim R)) \quad ;$$

and in step (2c),

$$P \leftrightarrow \sim(Q \vee \sim R) \quad .$$

13. Assuming that either logic is difficult or the text is not readable, Alfred will pass only if he concentrates. (P : logic is difficult; Q : the text is readable; R : Alfred will pass; S : Alfred concentrates)

14. Unless logic is difficult, Alfred will pass if he concentrates. (P : logic is difficult; R : Alfred will pass; S : Alfred concentrates)

15. Mary will arrive at 10:30 A.M. unless the plane is late. (P : Mary will arrive at 10:30 A.M.; Q : the plane is late)

16. Assuming that the professor is a Communist, he will sign the loyalty oath; but if he is an idealist, he will neither sign the loyalty oath nor speak to those who do. (P : the professor is a Communist; Q : the professor will sign the loyalty oath; R : the professor is an idealist; S : the professor will speak to those who sign the loyalty oath)

17. If Alfred and Mary are playing dice together, it is the first throw of the game, and Mary is throwing the dice, then she wins the game on the first throw if and only if she throws 7 or 11. (P : Alfred is playing dice; Q : Mary is playing dice; R : Alfred and Mary are playing dice together; S : it is the first throw of the game; T : Mary is throwing the dice; U : Mary wins on the first throw; V : Mary throws 7 or 11; W : Mary throws 7; X : Mary throws 11)

18. If the world is a progressively realized community of interpretation, then either quadruplicity will drink procrastination or, provided that the Nothing negates, boredom will ensue seldom more often than frequently. (P : the world is a progressively realized community of interpretation; Q: quadruplicity will drink procrastination; R : the Nothing negates; S : boredom will ensue seldom more often than frequently)

The following sentences are ambiguous in the sense that the placement of parentheses in their symbolizations is not uniquely determined. Give all plausible symbolizations of each on the basis of the given scheme of abbreviation. Exercise 19 is solved for illustration.

19. Errors will occur in the subject's performance if and only if motivation is absent or learning has not taken place.

Given the scheme of abbreviation

> P : errors will occur in the subject's performance
> Q : motivation is absent
> R : learning has taken place ,

No. 19 becomes in step (1) either

> (Errors will occur in the subject's performance if and only if (motivation is absent or it is not the case that learning has taken place))

or

> ((Errors will occur in the subject's performance if and only if motivation is absent) or it is not the case that learning has taken place) ;

and in step (2) it correspondingly becomes either

$$P \leftrightarrow Q \vee \sim R$$

or

$$(P \leftrightarrow Q) \vee \sim R \quad .$$

20. If either a war or a depression occurs then neither science nor music and literature will flourish unless the government supports research and provides patronage for artists. (P : a war occurs; Q : a depression occurs; R : science will flourish; S : music will flourish; T : literature will flourish; U : the government will support research; V : the government will provide patronage for artists)

21. Neither Alfred will listen nor Rudolf will listen if Alonzo is talking to either. (P : Alfred will listen; Q : Rudolf will listen; R : Alonzo is talking to Alfred; S : Alonzo is talking to Rudolf)

3. Inference rules and derivability. The *sentential calculus* is that branch of logic which essentially involves the sentential connectives. For a complete formulation of the sentential calculus, we must add to the

3. Informal notational conventions. We shall continue to use the conventions of chapter II (pp. 51, 64, and 79) for omitting parentheses and replacing them by brackets. For practical purposes, then, we shall deal with a larger class of symbolic formulas than that given in section 1. It should be emphasized, however, that in theoretical discussions—for instance, in the criteria of bondage and freedom given in the last section and in the rules of derivation that will appear in section 5—the word 'formula' is always to be understood in the official sense characterized in section 1.

4. Translation and symbolization. The English sentence

(1) For each x, x is bald

may be paraphrased in a variety of ways, for instance, as

Everything is bald ,
Each thing is bald ,
All things are bald ,
For all x, x is bald ;

and the English sentence

(2) There is an object x such that x is bald

may be paraphrased as

Something is bald ,
At least one thing is bald ,
There is a bald thing ,
For some x, x is bald .

Using phrases of quantity together with the phrases of connection of chapters I and II, we can develop stylistic variants of a number of English expressions in addition to those just mentioned. For instance,

Nothing is bald

may be paraphrased as either

(3) For each x, it is not the case that x is bald

or

(4) It is not the case that there is an object x such that x is bald

Other examples are afforded by sentences of the familiar Aristotelian forms.

(5) All dogs are canine ,
(6) Some dogs are cats ,
No dogs are canine ,
Some dogs are not cats

are stylistic variants, respectively, of

(7) For each x (if x is a dog, then x is a canine) ,
(8) There is an object x such that (x is a dog and x is a cat) ,
(9) For each x (if x is a dog, then it is not the case that x is a canine) ,
(10) There is an object x such that (x is a dog and it is not the case that x is a cat) .

The supposition that the true sentence (5) has

(11) For each x (x is a dog and x is canine)

as a stylistic variant can be rejected by observing that (11) is false, for it logically implies the falsehood 'for each x, x is a canine'. (See exercise 102, section 11.) Similarly, the supposition that the false sentence (6) has

(12) There is an object x such that (if x is a dog, then x is a cat)

as a stylistic variant can be rejected by observing that (12) is true, for it is implied by the truth 'there is an object x such that x is a cat'. (See exercise 103, section 11.)

Somewhat less obvious examples of stylistic variance are provided by the sentences

Only citizens are voters

and

None but citizens are voters ,

both of which may be paraphrased as

For each x (if x is a voter, then x is a citizen) .

It is impractical to list all the combinations that are expressible in terms of phrases of quantity; however, further examples of stylistic variance will be given when we illustrate the processes of translation and symbolization and also in connection with the exercises for this section.

The processes of translation and symbolization are more interesting here than in the sentential calculus. Some English sentences that previously entered into these processes as unanalyzed units can now have their internal structure represented in the expanded symbolic language. First, we must extend the notion of an abbreviation, for now names of English and formulas of English with one variable as well as sentences of English will require symbolic representation. Accordingly, we presently understand by an *abbreviation* either an ordered pair of which the first member is a sentence letter and the second member is a sentence of English, an ordered pair of which the first member is a name letter and the second member a name of English, or an ordered pair of which the first member is a predicate letter and the second member a formula of English whose only variable is '*a*'. (The adequacy of employing only the variable '*a*' in an abbreviation will become apparent when we characterize the process of translation.) Again, a *scheme of abbreviation* is to be a collection of abbreviations such that no two abbreviations in the collection have the same first member. For example, the following is a scheme of abbreviation:

P : appearances are deceptive
A : Socrates
F : *a* is bald .

On the basis of this scheme of abbreviation, we shall regard the English sentences

Socrates is bald
Either Socrates is bald or appearances are deceptive

as literal translations, respectively, of the symbolic sentences

FA
FA ∨ P ;

and the latter as symbolizations, respectively, of the former. Also, on the basis of this scheme, we shall regard the English sentences (1)–(4) above as literal translations, respectively, of the symbolic sentences

$\bigwedge x Fx$
$\bigvee x Fx$
$\bigwedge x \sim Fx$
$\sim \bigvee x Fx$.

Further, we shall count the latter symbolic sentences as symbolizations not only of their literal translations but also of all the stylistic variants of their literal translations. Thus, for example, '$\bigwedge x Fx$' is a symbolization of 'Everything is bald' as well as of (1) above.

The pairs

F : *a* is a dog
G : *a* is a canine
H : *a* is a cat

are another example of a scheme of abbreviation. On the basis of this scheme we shall consider the English sentences (7)–(10) above literal translations, respectively, of the symbolic sentences

(13) $\bigwedge x(Fx \rightarrow Gx)$
(14) $\bigvee x(Fx \wedge Hx)$
$\bigwedge x(Fx \rightarrow \sim Gx)$
$\bigvee x(Fx \wedge \sim Hx)$.

And we shall count these symbolic sentences as symbolizations not only of their literal translations but also as symbolizations of all stylistic variants of their literal translations. Thus, for example, (13) is a symbolization of both (7) and (5), and (14) is a symbolization of both (8) and (6). With these examples as an introduction, we now turn to a general characterization of the processes of translation and symbolization.

The process of *literal translation into English on the basis of a given scheme of abbreviation* now begins with a symbolic formula and if successful ends with a formula of English. The process consists of the following steps:

(*i*) *Restore any parentheses that may have disappeared as a result of informal conventions.*

(*ii*) *Replace name letters and sentence letters by English names and sentences in accordance with the scheme of abbreviation.*

(*iii*) *Replace each predicate letter by the formula of English with which it is paired in the scheme of abbreviation, flanking the latter with a pair of braces. (The result of this step will contain parts of the form*

$\{\phi\}\zeta$,

where ϕ is a formula of English and ζ is either the variable or the name of English that accompanies the predicate letter being replaced.)

(*iv*) *Replace all parts of the form*

$\{\phi\}\zeta$,

where ϕ is a formula of English and ζ is either a variable or a name of English, by the result of replacing in ϕ all occurrences of the variable 'a' by ζ.

(*v*) *Eliminate sentential connectives and quantifier phrases in favor of the corresponding phrases of connection and quantity, preserving all parentheses.*

As before, we say that an English formula is a *free translation* (or simply a *translation*) of a symbolic formula ϕ on the basis of a given scheme of abbreviation if it is a stylistic variant of the literal English translation of ϕ based on that scheme.

Consider, for example, the symbolic sentence

(15) $\bigwedge x(Fx \rightarrow Gx) \rightarrow P \vee GA$.

Let us translate it into English on the basis of the following scheme of abbreviation:

(16) P : there are exceptions
A : Socrates
F : *a* is a person
G : *a* is mortal .

In step (i) of the process of literal translation into English, (15) becomes

$$(\Lambda x(Fx \rightarrow Gx) \rightarrow (P \vee GA)) \quad ;$$

in step (ii)

$$(\Lambda x(Fx \rightarrow Gx) \rightarrow (\text{there are exceptions} \vee \text{G Socrates})) \quad ;$$

in step (iii)

$$(\Lambda x(\{a \text{ is a person}\}\, x \rightarrow \{a \text{ is mortal}\}\, x) \rightarrow (\text{there are exceptions} \vee \{a \text{ is mortal}\} \text{ Socrates})) \quad ;$$

in step (iv)

$$(\Lambda x(x \text{ is a person} \rightarrow x \text{ is mortal}) \rightarrow (\text{there are exceptions} \vee \text{Socrates is mortal})) \quad ;$$

and in step (v)

(17) (If for each *x* (if *x* is a person, then *x* is mortal), then (there are exceptions or Socrates is mortal)) .

Thus (17) is a literal translation of (15) into English on the basis of the scheme (16); and the more idiomatic sentence

> If all persons are mortal, then either there are exceptions or Socrates is mortal ,

being a stylistic variant of (17), qualifies as a free translation of (15) on the basis of the same scheme.

The reader should note that the single variable '*a*' of an abbreviation passes into the various variables and name letters of the symbolic formula being translated into English in the passage from step (iii) to step (iv) of the process of translation. Thus the variable '*a*' in an abbreviation serves as a *place holder*. The utility of this service will be emphasized in the exercises that follow.

We say that ϕ is a *symbolization* of a formula ψ of English on the basis of a given scheme of abbreviation just in case ϕ is a symbolic formula which has ψ as a translation on the basis of that scheme.

To find a symbolization of a given formula of English on the basis of a given scheme of abbreviation, it is useful to proceed roughly as follows:

(1) *Introduce phrases of quantity and connection, the latter accompanied by parentheses and occurring canonically (that is, standing only before or between formulas), in place of their stylistic variants.*

(2) *Reverse the steps leading from a symbolic formula to a literal English translation.*

For example, consider the sentence

(18) If a woman is married and all married women are wives, then she is a wife ,

together with the scheme of abbreviation

F : *a* is a woman
G : *a* is married
H : *a* is a wife .

Some points should be noted before a symbolization is attempted. First, the indefinite article is often used as a stylistic variant of a phrase of quantity, and this is true of the first occurrence of 'a' in (18). Secondly, despite the fact that (18) begins with 'if', its symbolization should clearly be not a conditional but a universal generalization of a conditional. Thirdly, English pronouns often play a role like that of variables in our symbolic language; this is the case with 'she' in (18). These points suggest that (18) should become in step (1)

> For each *x* (if ((*x* is a woman and *x* is married) and for each *y* (if (*y* is a woman and *y* is married), then *y* is a wife)), then *x* is a wife) .

Let us now perform step (2) of the process of symbolization. Reversing step (v) of the process of literal translation into English, we obtain

$$\Lambda x(((x \text{ is a woman} \wedge x \text{ is married}) \wedge \Lambda y((y \text{ is a woman} \wedge y \text{ is married}) \rightarrow y \text{ is a wife})) \rightarrow x \text{ is a wife}) \quad .$$

When step (iv) is reversed, this becomes

$$\Lambda x(((\{a \text{ is a woman}\}\, x \wedge \{a \text{ is married}\}\, x) \wedge \Lambda y((\{a \text{ is a woman}\}\, y \wedge \{a \text{ is married}\}\, y) \rightarrow \{a \text{ is a wife}\}\, y)) \rightarrow \{a \text{ is a wife}\}\, x).$$

Reversing step (iii), we obtain

$$\Lambda x((Fx \wedge Gx) \wedge \Lambda y((Fy \wedge Gy) \rightarrow Hy) \rightarrow Hx) \quad .$$

Finally, reversing step (i) (for step (ii) is irrelevant), we obtain

(19) $\Lambda x(Fx \wedge Gx \wedge \Lambda y(Fy \wedge Gy \rightarrow Hy) \rightarrow Hx)$.

(The use of two variables, '*x*' and '*y*', is not necessary for a symbolization of (18); it could equally well become '$\Lambda x(Fx \wedge Gx \wedge \Lambda x(Fx \wedge Gx \rightarrow Hx) \rightarrow Hx)$', which is, however, somewhat less perspicuous than (19).)

EXERCISES, GROUP I

On the basis of the scheme of abbreviation

F : *a* is an even number
G : *a* is a prime number
H : *a* is honest
J : *a* is a person
A : 2
B : the son of Lysimachus ,

translate the following symbolic formulas into idiomatic English. Exercises 18–20 are solved for illustration.

18. $\bigwedge x(Jx \wedge Hx \rightarrow HB)$

In steps (i) and (ii) of the process of literal translation into English, No. 18 becomes

$\bigwedge x((Jx \wedge Hx) \rightarrow$ H the son of Lysimachus) ,

in steps (iii) and (iv)

$\bigwedge x$((*x* is a person $\wedge$ *x* is honest) $\rightarrow$ the son of Lysimachus is honest) ,

and in step (v)

(20) For each *x* (if (*x* is a person and *x* is honest), then the son of Lysimachus is honest) .

Thus (20) is the literal English translation of No. 18, and the following stylistic variant of (20) is a translation of that symbolic formula into idiomatic English:

If anyone is honest, then the son of Lysimachus is honest .

19. $GA \rightarrow \bigvee x(Fx \wedge Gx)$
20. $GA \rightarrow \bigvee xFx \wedge Gx$

On the basis of the given scheme, the literal translations into English of Nos. 19 and 20 are, respectively,

(If 2 is a prime number, then there is an object *x* such that (*x* is an even number and *x* is a prime number))

and

(If 2 is a prime number, then (there is an object *x* such that *x* is an even number, and *x* is a prime number)) ;

and the following stylistic variants are respective translations into idiomatic English of the symbolic formulas:

If 2 is a prime number, then there is an even prime number ,
If 2 is a prime number, then there is an even number and *x* is a prime number .

To account for the fact that No. 19 has a different translation from No. 20, it is sufficient to note that the latter, in contrast with the former, is not a symbolic *sentence*. In the translation of the symbolic formula (20), the variable '*x*' is comparable to a third person singular pronoun that lacks an antecedent referent and is a counterpart to the free occurrence of '*x*' in (20).

21. $GA \leftrightarrow \bigwedge x(Fx \rightarrow Gx)$
22. $\bigvee x(Fx \wedge Gx) \rightarrow \bigvee xFx \wedge \bigvee xGx$
23. $GA \wedge \bigvee xFx \rightarrow \bigvee x(Fx \wedge Gx)$
24. $\bigwedge x(Fx \wedge Gx \rightarrow GA)$
25. $\bigvee x(Fx \wedge Gx) \rightarrow GA$

EXERCISES, GROUP II

Symbolize each of the following sentences on the basis of the scheme of abbreviation that accompanies it. Exercises 26–29 are solved for illustration.

26. Only citizens are voters. (F : *a* is a citizen; G : *a* is a voter)

To symbolize No. 26, we recall the remarks concerning that sentence given on page 127 and obtain in step (1) of the process of symbolization

For each *x* (if *x* is a voter, then *x* is a citizen) .

In step (2), we first reverse step (v) of the process of literal translation into English to obtain

$\bigwedge x$(*x* is a voter $\rightarrow$ *x* is a citizen) ;

and then reverse steps (iv) and (iii) to obtain as a symbolization of No. 26 (for steps (ii) and (i) of the process of literal translation are irrelevant here) the sentence

$$\bigwedge x(Gx \rightarrow Fx) \quad .$$

27. All and only philosophers love wisdom. (F : *a* is a philosopher; G : *a* loves wisdom)

Here we recall first our comments in chapter II concerning canonical stylistic variants (see p. 56) and paraphrase No. 27 by

All philosophers love wisdom and only philosophers love wisdom ,

and then recall our earlier remarks in this section in order to obtain in step (1) of the process of symbolization

(For each *x* (if *x* is a philosopher, then *x* loves wisdom) and for each *x* (if *x* loves wisdom, then *x* is a philosopher)) .

In step (2), we first reverse step (v) of the process of literal translation into English to obtain

$(\bigwedge x(x$ is a philosopher $\rightarrow x$ loves wisdom) $\wedge \bigwedge x(x$ loves wisdom $\rightarrow x$ is a philosopher)) ;

next we reverse steps (iv) and (iii) to obtain

$$(\bigwedge x(Fx \rightarrow Gx) \wedge \bigwedge x(Gx \rightarrow Fx)) \quad ;$$

and finally, reversing step (i) (for step (ii) is irrelevant), we obtain as a symbolization of No. 27 the sentence

$$\bigwedge x(Fx \rightarrow Gx) \wedge \bigwedge x(Gx \rightarrow Fx) \quad .$$

28. Among snakes, only copperheads and rattlers are poisonous. (F : a is a snake; G : a is a copperhead; H : a is a rattler; J : a is poisonous)

To symbolize No. 28 we must supplement our preceding comment on 'only' (p. 127) with an intuitive observation about English; this intuition suggests that the sentence pass in step (1) of the process of symbolization into either

For each x (if x is a snake, then (if x is poisonous, then (x is a copperhead or x is a rattler)))

or

For each x (if (x is a snake and x is poisonous), then (x is a copperhead or x is a rattler)) .

Let us now perform step (2) of the process of symbolization, using the first alternative above. Reversing step (v) of the process of literal translation into English, we obtain

$\bigwedge x(x$ is a snake $\rightarrow$ (x is poisonous $\rightarrow$ (x is a copperhead $\vee$ x is a rattler))) ;

reversing steps (iv) and (iii), we obtain

$$\bigwedge x(Fx \rightarrow (Jx \rightarrow (Gx \vee Hx))) \quad ;$$

and finally, reversing step (i) (for step (ii) is irrelevant), we obtain

$$\bigwedge x(Fx \rightarrow [Jx \rightarrow Gx \vee Hx]) \quad .$$

(Idiomatic English often employs the definite article 'the' as a phrase of quantity; for example,

The only poisonous snakes are copperheads and rattlers

is a stylistic variant of No. 28 and thus has the same symbolization as that sentence.)

29. No one is civilized unless he or she loves classical music. (F : a is a person; G : a is civilized; H : a loves classical music)

Intuition, perhaps aided by some of our earlier and immediately preceding comments on stylistic variance, suggests that No. 29 pass in step (1) of the process of symbolization into

For each x (if x is a person, then (either it is not the case that x is civilized or x loves classical music)) .

and then in step (2) into

$$\bigwedge x(Fx \rightarrow \sim Gx \vee Hx) \quad .$$

Alternative first steps, based on such paraphrases of No. 29 as

Only those who love classical music are civilized

It is not the case that someone is civilized and does not love classical music ,

would yield as symbolizations of No. 29 the symbolic sentences

$$\bigwedge x(Fx \wedge Gx \rightarrow Hx)$$

$$\sim \bigvee x(Fx \wedge Gx \wedge \sim Hx) \quad .$$

It will sometimes be our (admittedly questionable) practice to treat 'no one', 'someone', 'everyone' as stylistic variants of 'nothing', 'something', 'everything'. Adoption of this practice would be indicated, for example, by omitting the pair 'F : a is a person' from the scheme of abbreviation that accompanied No. 29 and accepting the sentences

$$\bigwedge x(\sim Gx \vee Hx)$$

$$\bigwedge x(Gx \rightarrow Hx)$$

$$\sim \bigvee x(Gx \wedge \sim Hx)$$

as symbolizations of No. 29.

As a summary of the immediately preceding and earlier examples of symbolization, we give below a list of frequently encountered patterns of idiomatic English, each accompanied by one or more symbolizations (based on a natural scheme of abbreviation). This list should provide a useful but not infallible guide to symbolization in these and subsequent exercises.

Everything is bald.	$\bigwedge xFx$
Nothing is bald.	$\bigwedge x \sim Fx$, $\sim \bigvee xFx$
Something is bald	$\bigvee xFx$
Something is not bald.	$\bigvee x \sim Fx$
All dogs are canine.	$\bigwedge x(Fx \rightarrow Gx)$
Some dogs are canine.	$\bigvee x(Fx \wedge Gx)$
No dogs are canine.	$\bigwedge x(Fx \rightarrow \sim Gx)$, $\sim \bigvee x(Fx \wedge Gx)$
Some dogs are not canine.	$\bigvee x(Fx \wedge \sim Gx)$
Only (none but) citizens are voters.	$\bigwedge x(Gx \rightarrow Fx)$

All and only philosophers love wisdom.	$\bigwedge x(Fx \rightarrow Gx) \wedge \bigwedge x(Gx \rightarrow Fx)$
Among snakes, only copperheads and rattlers are poisonous.	$\bigwedge x(Fx \rightarrow [Jx \rightarrow Gx \vee Hx])$
No one is civilized unless he or she loves classical music.	$\bigwedge x(Fx \rightarrow \sim Gx \vee Hx)$, $\bigwedge x(Fx \wedge Gx \rightarrow Hx)$, $\sim \bigvee x(Fx \wedge Gx \wedge \sim Hx)$

30. Something is round and something is square, but it is not the case that something is a round square. (F : *a* is round; G : *a* is square)

31. If everything is mental, then nothing is physical unless something is both mental and physical. (F : *a* is mental; G : *a* is physical)

32. None but the brave deserve the fair. (F : *a* is brave; G : *a* deserves the fair)

33. Kettles boil quickly only if not watched. (F : *a* is a kettle; G : *a* boils quickly; H : *a* is watched)

34. Kettles boil quickly if not watched. (F : *a* is a kettle; G : *a* boils quickly; H : *a* is watched)

35. Some soldiers love war, but not all who love war are soldiers. (F : *a* is a soldier; G : *a* loves war)

36. If all persons are mortal, then Christ is not a person. (F : *a* is a person; G : *a* is mortal; A : Christ)

37. If Socrates is a philosopher and all philosophers are risible, then he is risible. (F : *a* is a philosopher; G : *a* is risible; A : Socrates)

38. No one under seventeen is admitted unless accompanied by a parent. (F : *a* is under seventeen; G : *a* is admitted; H : *a* is accompanied by a parent)

39. Married persons and only married persons have spouses. (F : *a* is a married person; G : *a* has a spouse)

40. Among equine mammals, all and only zebras have dark stripes on a white or buffy ground. (F : *a* is an equine mammal; G : *a* is a zebra; H : *a* has dark stripes on a white or buffy ground)

41. Democrats and Republicans who put loyalty above legality are detrimental to democracy. (F : *a* is a Democrat; G : *a* is a Republican; H : *a* puts loyalty above legality; J : *a* is detrimental to democracy)

42. Professors are good lecturers if and only if they are both witty and intellectually competent. (F : *a* is a professor; G : *a* is a good lecturer; H : *a* is witty; J : *a* is intellectually competent)

43. If a registered voter has not declared a party and only those who have declared a party can vote in a primary election, then he or she cannot vote in a primary election. (F : *a* is a registered voter; G : *a* has declared a party; H : *a* can vote in a primary election)

44. If only Republicans support the incumbent and no Democrat supports the candidate, then if anyone is a Democrat, someone supports neither the incumbent nor the candidate. (F : *a* is a Republican; G : *a* supports the incumbent; H : *a* is a Democrat; J : *a* supports the candidate)

45. If those who believe in God have immortal souls, then, given that God exists, they will have eternal bliss. (F : *a* believes in God; G : *a* has an immortal soul; H : *a* will have eternal bliss; P : God exists)

EXERCISES, GROUP III

Symbolize each of the following pairs of sentences on the basis of the scheme of abbreviation that accompanies the pair. Because these sentences exhibit some of the more subtle aspects of the symbolization of idiomatic English, we solve exercises 46–49 for illustration.

46. (i) Nothing that glitters is gold. (ii) All that glitters is not gold. (F : *a* glitters; G : *a* is gold)

The sentence (i), but not (ii), may be paraphrased

(21) For each x (if x glitters, then it is not the case that x is gold) .

The sentence (ii) is not correctly paraphrased by (21), for (21) entails that gold does not glitter—a consequence clearly not intended by the familiar proverb (which might be more accurately worded as 'Not all that glitters is gold'). Intuition suggests that the proverb may instead be paraphrased

(22) It is not the case that for each x (if x glitters, then x is gold).

On the basis of the scheme provided, (21) and (22), and hence (i) and (ii), are respectively symbolized by

$$\bigwedge x(Fx \rightarrow \sim Gx)$$

and

$$\sim \bigwedge x(Fx \rightarrow Gx) \quad .$$

47. (i) Alfred can solve every puzzle. (ii) Alfred can solve any puzzle. (F : *a* is a puzzle; G : *a* can be solved by Alfred)

The sentences (i) and (ii) are stylistic variants of each other; thus on the given scheme they can both be symbolized by

$$\bigwedge x(Fx \rightarrow Gx) \quad .$$

However, consider these English sentences modified with a phrase of negation:

48. (i) Alfred cannot solve every puzzle. (ii) Alfred cannot solve any puzzle. (F : *a* is a puzzle; G : *a* can be solved by Alfred)

Intuition tells us that it would be a mistake to regard (i) and (ii) of this exercise as stylistic variants of each other. Indeed, if Alfred is like most of us, (i) is true but (ii) is not. (i) can be paraphrased

(23) It is not the case that for each x (if x is a puzzle, then x can be solved by Alfred) ;

and (ii) can be paraphrased

(24) For each x (if x is a puzzle, then it is not the case that x can be solved by Alfred) .

On the basis of the scheme provided, (23) and (24), and hence (i) and (ii) of this exercise, pass respectively into

$$\sim \bigwedge x(Fx \rightarrow Gx)$$

and

$$\bigwedge x(Fx \rightarrow \sim Gx) \quad .$$

49. (i) If any witness lied, then he or she committed perjury. (ii) If any witness lied, then all of them committed perjury. (F : a is a witness; G : a lied; H : a committed perjury)

Although (i) appears to be a conditional, it must be interpreted as a universal generalization of a conditional and may be paraphrased by

(25) For each x (if (x is a witness and x lied), then x committed perjury) .

(To see why (i) cannot be interpreted as an existential generalization of a conditional the reader should review the remarks made on page 127.) However, the sentence (ii) may be paraphrased as a conditional. In this case, intuition must be consulted to determine whether 'any' should pass into a universal or an existential quantifier phrase; the latter seems more appropriate. Thus we obtain as a paraphrase of (ii)

(26) (If there is an object x such that (x is a witness and x lied), then for each x (if x is a witness, then x committed perjury)) .

On the basis of the scheme provided, (25) and (26), and hence (i) and (ii) of this exercise, pass respectively into

$$\bigwedge x(Fx \wedge Gx \rightarrow Hx)$$

and

$$\bigvee x(Fx \wedge Gx) \rightarrow \bigwedge x(Fx \rightarrow Hx) \quad .$$

(The sentence (ii) could also be paraphrased as a universal generalization of a conditional, in which case we would obtain first

For each x (if (x is a witness and x lied), then for each x (if x is a witness, then x committed perjury))

and then

$$\bigwedge x[Fx \wedge Gx \rightarrow \bigwedge x(Fx \rightarrow Hx)]$$

as an alternative symbolization of (ii).)

50. (i) Not everyone who talks a great deal has a great deal to say. (ii) No one who talks a great deal has a great deal to say. (F : a talks a great deal; G : a has a great deal to say)

51. (i) If anyone will spot an error, Alonzo will. (ii) If everyone will spot an error, Alonzo will. (F : a will spot an error; A : Alonzo)

52. (i) If something can go wrong, it will go wrong. (ii) If something can go wrong, then everything will go wrong. (F : a can go wrong; G : a will go wrong)

53. (i) A doctor should uphold the Hippocratic Oath. (ii) A doctor is in the house. (F : a is a doctor; G : a should uphold the Hippocratic Oath; H : a is in the house)

54. (i) Alfred married an intelligent woman, and Rudolf married one too. (ii) Alfred married an intelligent woman, and Rudolf married her too. (F : Alfred married a; G : Rudolf married a; H : a is an intelligent woman)

55. (i) There are diamonds. (ii) There are fake diamonds. (We leave the scheme of abbreviation as well as the symbolization to the reader.)

56. (i) Some students were protesting, but not all of them were arrested. (ii) If some students were protesting, they were arrested. (F : a is a student; G : a was protesting; H : a was arrested)

57. (i) The politician who always promises prosperity is a prevaricator. (ii) Politicians, who always promise prosperity, are prevaricators. (F : a is a politician; G : a always promises prosperity; H : a is a prevaricator)

5. **Inference rules.** For the logic of quantifiers we shall add three inference rules to our original stock. To facilitate the formulation of these new rules we introduce an auxiliary notion; and for perspicuity in the characterization of this notion we write one Greek letter as a subscript to another. We say that a symbolic formula ϕ_ζ *comes from* a symbolic formula ϕ_α *by proper substitution of a name letter* ζ *for a variable* α if ϕ_ζ is like ϕ_α except for having occurrences of ζ wherever ϕ_α has *free* occurrences of α; and we say that a symbolic formula ϕ_β *comes from* a symbolic formula ϕ_α *by proper substitution of a variable* β *for a variable* α if ϕ_β is like ϕ_α except for having *free* occurrences of β wherever ϕ_α has *free occurrences* of α. Consider for example the formula

(1) $$\bigvee z(Fx \wedge Gy) \rightarrow \bigvee xHx \vee Gx \quad .$$

The formula

$$\bigvee z(FA \wedge Gy) \rightarrow \bigvee xHx \vee GA$$

comes from (1) by proper substitution of 'A' for 'x', and

$$\bigvee z(Fy \wedge Gy) \rightarrow \bigvee xHx \vee Gy$$

comes from (1) by proper substitution of 'y' for 'x'.

11. In this formula identify (perhaps with the aid of a modified grammatical tree) each occurrence of a term as bound or free.

12. Which terms are bound in the formula? Which are free in the formula?

3. Informal notational conventions. We shall again employ the conventions of chapter II (pp. 51, 64, and 79) for omitting parentheses and replacing them by brackets.

In addition, it will be our usual practice to omit superscripts from operation letters and predicate letters, inserting parentheses and brackets to avoid ambiguity. In the case of 0-place operation letters, and 0-place and 1-place predicate letters, no parentheses or brackets will be employed. In the case of operation letters of one or more places, and predicate letters of two or more places, we shall enclose the terms accompanying the letter in question in a pair of parentheses or brackets. For instance, the terms (12) and (13) of section 1 (p. 204) may become

$$C[D(xy)] \quad ,$$
$$D[C(x)y]$$

respectively, and the formulas (14)–(17) on page 205 may become

$$G(xy) \quad ,$$
$$F\,A \quad ,$$
$$G(A\,x) \quad ,$$
$$G[x\,B(xy)]$$

respectively. Restoration of official notation is completely automatic; the superscript of a letter is determined by the sequence of terms following it.

Again, we must emphasize that in theoretical discussions—for instance, the definitions of bondage and freedom and the ensuing characterization of derivability—the words 'term' and 'formula' are always to be understood in the official sense.

EXERCISES

13. For each of the following formulas in unofficial notation, delete the parentheses and brackets inserted, and restore the superscripts and parentheses omitted, in accordance with the informal notational conventions introduced in this section.

(i) $F\,A(x)$
(ii) $G(B\,x)$
(iii) $H[C(B\,B)\,A(B)\,B]$
(iv) $\bigwedge xFx \lor P\text{-}\; G(xy) \land H(xyz)$

4. Translation and symbolization. Many symbolic sentences of our present language cannot be translated into English on the basis of the schemes of abbreviation used in chapter III; an example is

(1) $\sim \bigvee xF(xA)$.

For present purposes we must admit, in addition to the abbreviations of the last chapter, abbreviations involving operation letters of more than zero place as well as predicate letters of more than one place.

It is convenient to establish a standard order of variables; the order we will use is represented by the following list:

$$a, \ldots, z, a_0, \ldots, z_0, a_1, \ldots, z_1, \ldots \quad .$$

Thus the *first variable* will be 'a', the *second variable* 'b', the *twenty-sixth variable* 'z', and so on.

We shall admit abbreviations only of those formulas and terms of English whose variables are the first k variables, for some nonnegative integer k. We shall abbreviate such a formula or term by a k-place predicate letter or operation letter.

Thus an *abbreviation* will now be either (1) an ordered pair whose first member is a k-place predicate letter, for some $k \geqslant 0$, and whose second member is a formula of English containing exactly the first k variables, or (2) an ordered pair whose first member is a k-place operation letter, for some $k \geqslant 0$, and whose second member is a term of English containing exactly the first k variables.

As in earlier chapters, a *scheme of abbreviation* is to be a collection of abbreviations such that no two abbreviations in the collection have the same first member. The following is an example:

(2) A^0 : Adam
F^2 : a is father of b .

The process of *literal translation into English on the basis of a given scheme of abbreviation* begins, as before, with a symbolic formula and if successful ends with a formula of English. The process consists of the following steps:

(*i*) *Restore official notation by reversing the conventions of the preceding section.*

(*ii*) *Replace 0-place operation letters and 0-place predicate letters by English names and sentences in accordance with the scheme of abbreviation.*

(*iii*) *Replace each operation letter and predicate letter of one or more places by the term or formula of English with which it is paired in the scheme of abbreviation, flanking the latter with a pair of braces.* (*The result of this step will contain parts of the form*

$$\{\phi\}\zeta_1 \ldots \zeta_k \quad ,$$

where ϕ is a term or formula of English containing exactly the first k variables and $\zeta_1, \ldots, \zeta_k$ are variables or terms of English.)

(*iv*) *Successively replace all parts of the form*

$$\{\phi\}\zeta_1 \ldots \zeta_k \quad ,$$

where ϕ is a term or formula of English containing exactly the first k variables and $\zeta_1, \ldots, \zeta_k$ are variables or terms of English, by the term or formula of English obtained from ϕ by simultaneously replacing all occurrences of 'a' by ζ_1, 'b' by ζ_2, etc., up to the kth variable, which is to be replaced by ζ_k.

(*v*) *Eliminate sentential connectives and quantifier phrases in favor of the corresponding phrases of connection and quantity, preserving all parentheses.*

As before, we say that an English formula is a *free translation* (or simply a *translation*) of a symbolic formula ϕ on the basis of a given scheme of abbreviation if it is a stylistic variant of the literal English translation of ϕ based on that scheme.

Let us, for example, translate the sentence (1) into English on the basis of the scheme (2). In step (i) of the process of literal translation into English, (1) becomes

$$\sim \bigvee x \; F^2 \, x \, A^0 \quad ,$$

in step (ii)

$$\sim \bigvee x \; F^2 \, x \; \text{Adam} \quad ,$$

in step (iii)

$$\sim \bigvee x \; \{a \text{ is father of } b\} \; x \; \text{Adam} \quad ,$$

in step (iv)

$$\sim \bigvee x \; x \text{ is father of Adam} \quad ,$$

and in step (v)

(3) It is not the case that there is an object *x* such that *x* is father of Adam .

Thus (3) is the literal translation of (1) into English on the basis of the scheme (2); and the more idiomatic sentence

Adam has no father ,

being a stylistic variant of (3), qualifies as a free translation of (1) on the basis of the same scheme.

As a slightly more involved illustration, consider the symbolic sentence

(4) $$\bigwedge x \bigwedge y(Fx \wedge Fy \wedge Gy \rightarrow G\,A(xy))$$

and the scheme of abbreviation

(5)

F^1	:	*a* is a number
G^1	:	*a* is even
A^2	:	the product of *a* and *b* .

Let us find the literal English translation of (4) based on the scheme (5). Applying step (i), we obtain

$$\bigwedge x \bigwedge y(((F^1x \wedge F^1y) \wedge G^1y) \rightarrow G^1\, A^2xy) \quad .$$

Step (ii) is irrelevant. In step (iii) we obtain

$$\bigwedge x \bigwedge y(((\{a \text{ is a number}\}\, x \wedge \{a \text{ is a number}\}\, y) \wedge \{a \text{ is even}\}\, y) \rightarrow \{a \text{ is even}\}\{\text{the product of } a \text{ and } b\}\, x\, y) \quad .$$

Applying step (iv), we obtain first

$$\bigwedge x \bigwedge y(((\{a \text{ is a number}\}\, x \wedge \{a \text{ is a number}\}\, y) \wedge \{a \text{ is even}\}\, y) \rightarrow \{a \text{ is even}\} \text{ the product of } x \text{ and } y) \quad ,$$

and then

$$\bigwedge x \bigwedge y(((x \text{ is a number} \wedge y \text{ is a number}) \wedge y \text{ is even}) \rightarrow \text{the product of } x \text{ and } y \text{ is even}) \quad .$$

Finally, by step (v), we obtain as the literal translation of (4) the English sentence

(6) For each *x*, for each *y* (if ((*x* is a number and *y* is a number) and *y* is even), then the product of *x* and *y* is even) .

Moreover,

The product of any number and any even number is even ,

being a stylistic variant of (6), is a free English translation of (4) on the basis of (5).

As before, ϕ is said to be a *symbolization* of an English formula ψ on the basis of a given scheme of abbreviation just in case ϕ is a symbolic formula that has ψ as a translation on the basis of that scheme.

To find a symbolization of a given formula of English on the basis of a given scheme of abbreviation the reader will again find it useful to proceed roughly as follows:

(*1*) *Introduce phrases of quantity and connection, the latter accompanied by parentheses and occurring canonically, in place of their stylistic variants.*

(*2*) *Reverse the steps leading from a symbolic formula to a literal English translation.*

For an example of the process of symbolization, consider the sentence

(7) There is no composer whose contrapuntal ingenuity surpasses that of Bach ,

together with the scheme of abbreviation

F^1 : a is a composer
G^2 : a surpasses b
A^1 : the contrapuntal ingenuity of a
B^0 : Bach .

In step (1), with intuition as guide, we transform (7) into

It is not the case that there is an object x such that (x is a composer and the contrapuntal ingenuity of x surpasses the contrapuntal ingenuity of Bach) .

Let us now perform step (2) of the process of symbolization. Reversing step (v) of the process of literal translation into English, we obtain

$\sim\bigvee x$(x is a composer $\wedge$ the contrapuntal ingenuity of x surpasses the contrapuntal ingenuity of Bach) .

Reversing step (iv), this becomes first

$\sim\bigvee x$({a is a composer} x $\wedge$ {a surpasses b} the contrapuntal ingenuity of x the contrapuntal ingenuity of Bach) ,

and then

$\sim\bigvee x$({a is a composer} x $\wedge$ {a surpasses b} {the contrapuntal ingenuity of a} x {the contrapuntal ingenuity of a} Bach) .

Reversing step (iii), we obtain

$$\sim\bigvee x(F^1x \wedge G^2\, A^1x\, A^1\text{Bach}) \quad ;$$

reversing step (ii),

$$\sim\bigvee x(F^1x \wedge G^2\, A^1x\, A^1B^0) \quad ;$$

and reversing step (i),

$$\sim\bigvee x(Fx \wedge G[A(x)A(B)]) \quad .$$

An example of stylistic variance of rather frequent incidence is the mutual conversion of active and passive voice. Consider, for example, the sentence

If someone is loved by one whom he does not love and by nobody else, then he does not love his lover ,

together with the following scheme of abbreviation:

L^2 : a loves b
T^2 : a differs from b
A^1 : the lover of a .

In step (1), if we construe 'x is loved by y' as a stylistic variant of 'y loves x' and ignore gender, the sentence above becomes

For each x (if there is an object y such that ((y loves x and it is not the case that x loves y) and it is not the case that there is an object z such that (z differs from y and z loves x)), then it is not the case that x loves the lover of x) ;

and carrying through the successive parts of step (2), we obtain the symbolization

$$\bigwedge x[\bigvee y(L(yx) \wedge \sim L(xy) \wedge \sim\bigvee z[T(zy) \wedge L(zx)]) \rightarrow \sim L(x\, A(x))] \quad .$$

The order of quantifiers in a symbolization is now of particular importance. For example, if we adopt the scheme of abbreviation

T^2 : a differs from b ,

then

$$\bigwedge x \bigvee y\, T(xy)$$

is a symbolization of the true sentence

Each thing differs from something ,

whereas

$$\bigvee y \bigwedge x\, T(xy)$$

is a symbolization of the false sentence

Something is such that everything differs from it .

EXERCISES, GROUP I

14. On the basis of the scheme of abbreviation

F^1 : a is a person
G^2 : a loves b ,

find for each sentence of group A a sentence of group B that is a symbolization of it.

GROUP A

(1) Everyone loves someone.
(2) Someone loves someone.
(3) If anything is a person, then someone loves himself.
(4) Someone loves everyone.
(5) Everyone loves everyone.

GROUP B

(a) $\bigvee x(Fx \wedge \bigwedge y[Fy \rightarrow G(xy)])$
(b) $\bigwedge x(Fx \wedge \bigwedge y[Fy \rightarrow G(xy)])$
(c) $\bigvee y(Fy \wedge \bigvee x[Fx \wedge G(yx)])$
(d) $\bigwedge x(Fx \rightarrow \bigvee x[Fx \wedge G(xx)])$
(e) $\bigwedge y(Fy \rightarrow \bigwedge x[Fx \rightarrow G(yx)])$
(f) $\bigvee x(Fx \rightarrow \bigvee y[Fy \wedge G(xy)])$
(g) $\bigwedge x(Fx \rightarrow \bigvee y[Fy \wedge G(xy)])$

15. On the basis of the scheme of abbreviation

F^1 : *a* is a student
G^1 : *a* is a teacher
H^1 : *a* is a subject
S^3 : *a* studies *b* with *c* ,

find for each sentence of group B a sentence of group A that is a translation of it.

GROUP A

(1) Every student studies every subject with every teacher.
(2) Every student studies some subject with some teacher.
(3) No student studies every subject with a teacher.
(4) Only students study every subject with a teacher.
(5) No subject is such that every student studies it with every teacher.
(6) Every teacher has some subject that some student studies with him or her.
(7) Each subject has some student who studies it with some teacher.
(8) None but teachers are such that all students study all subjects with them.
(9) Anyone who studies any subject with any teacher is a student.
(10) There is a teacher such that every student studies some subject with him or her.
(11) There is a teacher such that some student studies every subject with him or her.
(12) Teachers who study some subject with a teacher are students.
(13) No student who does not study every subject with a teacher is a teacher.
(14) Any student who studies any subject with himself or herself is a teacher.
(15) Some teacher who studies every subject with himself or herself is a student.
(16) There is no teacher with whom any student studies all subjects.

GROUP B

(a) $\bigwedge x(\bigwedge y \bigwedge z[Fy \wedge Hz \rightarrow S(yzx)] \rightarrow Gx)$
(b) $\sim \bigvee x(Gx \wedge \bigvee y[Fy \wedge \bigwedge z(Hz \rightarrow S(yzx))])$
(c) $\bigwedge x(Fx \rightarrow \bigwedge y \bigwedge z[Hy \wedge Gz \rightarrow S(xyz)])$
(d) $\bigwedge x(Hx \rightarrow \bigvee y \bigvee z[Fy \wedge Gz \wedge S(yxz)])$

(e) $\bigwedge x(Fx \rightarrow \bigvee y \bigvee z[Hy \wedge Gz \wedge S(xyz)])$
(f) $\bigwedge x(Fx \wedge \bigvee y[Hy \wedge S(xyx)] \rightarrow Gx)$
(g) $\bigvee x(Gx \wedge \bigwedge y[Fy \rightarrow \bigvee z(Hz \wedge S(yzx))])$
(h) $\bigwedge x \bigwedge y \bigwedge z(Fx \wedge Hy \wedge Gz \rightarrow S(xyz))$
(i) $\sim \bigvee x(Hx \wedge \bigwedge y \bigwedge z[Fy \wedge Gz \rightarrow S(yxz)])$
(j) $\bigwedge x(Fx \rightarrow \sim \bigwedge y[Hy \rightarrow \bigvee z(Gz \wedge S(xyz))])$
(k) $\bigwedge x(Fx \wedge \sim \bigwedge y[Hy \rightarrow \bigvee z(Gz \wedge S(xyz))] \rightarrow \sim Gx)$
(l) $\bigwedge x(Fx \rightarrow \bigwedge y[Hy \rightarrow \bigwedge z(Gz \rightarrow S(xyz))])$
(m) $\bigwedge x(\bigvee y \bigvee z[Hy \wedge Gz \wedge S(xyz)] \rightarrow Fx)$
(n) $\bigvee x(Gx \wedge \bigwedge y[Hy \rightarrow S(xyx)] \wedge Fx)$
(o) $\bigwedge x(Fx \rightarrow \bigvee y(Hy \wedge \bigvee z(Gz \wedge S(xyz))])$
(p) $\bigwedge x(\bigwedge y[Hy \rightarrow \bigvee z(Gz \wedge S(xyz))] \rightarrow Fx)$
(q) $\sim \bigvee x(Fx \wedge \bigvee y[Hy \wedge \sim \bigvee z(Gz \wedge S(xyz))] \wedge Gx)$
(r) $\bigvee x \bigvee y(Gx \wedge Fy \wedge \bigwedge z[Hz \rightarrow S(yzx)])$
(s) $\bigwedge x(Gx \rightarrow \bigvee y \bigvee z[Fy \wedge Hz \wedge S(yzx)])$
(t) $\sim \bigvee x(Fx \wedge \bigwedge y[Hy \rightarrow \bigvee z(Gz \wedge S(xyz))])$
(u) $\bigwedge x(Gx \wedge \bigvee y \bigvee z[Hy \wedge Gz \wedge S(xyz)] \rightarrow Fx)$

EXERCISES, GROUP II

Translate each of the following symbolic formulas into idiomatic English on the basis of the scheme of abbreviation that accompanies it. An example is solved for illustration.

$$\bigwedge x \bigwedge y \bigwedge z \bigwedge w[Fx \wedge G(yx) \wedge G(zx) \wedge G(wx) \wedge H(yz) \wedge H(yw) \wedge H(zw) \rightarrow I(A(B[y]\ B[z])\ B[w])]$$

(F^1 : *a* is a triangle; G^2 : *a* is a side of *b*; H^2 : *a* is different from *b*; I^2 : *a* is greater than *b*; A^2 : the sum of *a* and *b*; B^1 : the length of *a*)

In step (i) of the process of literal translation into English, the symbolic formula becomes

$$\bigwedge x \bigwedge y \bigwedge z \bigwedge w(((((((F^1x \wedge G^2yx) \wedge G^2zx) \wedge G^2wx) \wedge H^2yz) \wedge H^2yw) \wedge H^2zw) \rightarrow I^2\ A^2B^1yB^1z\ B^1w) \quad ;$$

in step (iii), for no change occurs in step (ii),

$\bigwedge x \bigwedge y \bigwedge z \bigwedge w((((((($\{*a* is a triangle\} *x* ∧ \{*a* is a side of *b*\} *y x*) ∧ \{*a* is a side of *b*\} *z x*) ∧ \{*a* is a side of *b*\} *w x*) ∧ \{*a* is different from *b*\} *y z*) ∧ \{*a* is different from *b*\} *y w*) ∧ \{*a* is different from *b*\} *z w*) → \{*a* is greater than *b*\} \{the sum of *a* and *b*\} \{the length of *a*\} *y* \{the length of *a*\} *z* \{the length of *a*\} *w*) ;

in step (iv), first

$\bigwedge x \bigwedge y \bigwedge z \bigwedge w((((((($*x* is a triangle ∧ *y* is a side of *x*) ∧ *z* is a side of *x*) ∧ *w* is a side of *x*) ∧ *y* is different from *z*) ∧ *y* is different from *w*) ∧ *z* is different from *w*) → {*a* is greater than *b*} {the sum of *a* and *b*} the length of *y* the length of *z* the length of *w*) ,

next

⋀*x*⋀*y*⋀*z*⋀*w*((((((*x* is a triangle ∧ *y* is a side of *x*) ∧ *z* is a side of *x*) ∧ *w* is a side of *x*) ∧ *y* is different from *z*) ∧ *y* is different from *w*) ∧ *z* is different from *w*) → {*a* is greater than *b*} the sum of the length of *y* and the length of *z* the length of *w*) ,

and finally

⋀*x*⋀*y*⋀*z*⋀*w*((((((*x* is a triangle ∧ *y* is a side of *x*) ∧ *z* is a side of *x*) ∧ *w* is a side of *x*) ∧ *y* is different from *z*) ∧ *y* is different from *w*) ∧ *z* is different from *w*) → the sum of the length of *y* and the length of *z* is greater than the length of *w*) ;

and in step (v)

For each *x*, for each *y*, for each *z*, for each *w* (if (*x* is a triangle and *y* is a side of *x*) and *z* is a side of *x*) and *w* is a side of *x*) and *y* is different from *z*) and *y* is different from *w*) and *z* is different from *w*), then the sum of the length of *y* and the length of *z* is greater than the length of *w*) ,

which is the literal translation of the symbolic formula into English; and the following stylistic variant is a free translation:

The sum of the lengths of any two sides of a triangle is greater than the length of the third side .

16. ⋁*x*(F*x* ∧ ⋀*y*[F*y* → GA(*xy*)]) → ⋁*x*(F*x* ∧ G*x*)
(F^1 : *a* is a number; G^1 : *a* is even; A^2 : the product of *a* and *b*)

17. ⋀*x*(F*x* → ⋁*y*G(*yx*) ∧ ⋁*y*H(*yx*)) ∧ ⋁*x*(F*x* ∧ ~[⋁*y*G(*xy*) ∨ ⋁*y*H(*xy*)])
(F^1 : *a* is a person; G^2 : *a* is father of *b*; H^2 : *a* is mother of *b*)

18. ⋀*x*(F(*x* E) → ⋁*y*[G(*y* E) ∧ H(*xy*) ∧ I(*y* B(C(E)))])
(F^2 : *a* is a student of *b*; E^0 : the course; G^2 : *a* is a workshop of *b*; H^2 : *a* attends *b*; I^2 : *a* is taught by *b*; B^1 : the teaching assistant of *a*; C^1 : the instructor of *a*)

EXERCISES, GROUP III

Symbolize each of the following formulas on the basis of the scheme of abbreviation that accompanies it. An example is solved for illustration.

A person pays a dollar for the banquet of the club only if he or she belongs to at least one of the club's committees and attends every meeting of that committee. (F^1 : *a* is a person; G^2 : *a* pays a dollar for *b*; H^2 : *a* is a committee of *b*; I^2 : *a* belongs to *b*; J^2 : *a* is a meeting of *b*; K^2 : *a* attends *b*; A^1 : the banquet of *a*; B^0 : the club)

Linguistic insight would suggest that this example become in step (1) of the process of symbolization

For each *x* (if (*x* is a person and *x* pays a dollar for the banquet of the club), then there is an object *y* such that ((*y* is a committee of the club and *x* belongs to *y*) and for each *z* (if *z* is a meeting of *y*, then *x* attends *z*))) .

By performing the successive stages of step (2), we obtain first

⋀*x*((*x* is a person ∧ *x* pays a dollar for the banquet of the club) → ⋁*y*((*y* is a committee of the club ∧ *x* belongs to *y*) ∧ ⋀*z*(*z* is a meeting of *y* → *x* attends *z*))) ,

next

⋀*x*(({*a* is a person} *x* ∧ {*a* pays a dollar for *b*} *x* {the banquet of *a*} the club) → ⋁*y*(({*a* is a committee of *b*} *y* the club ∧ {*a* belongs to *b*} *x y*) ∧ ⋀*z*({*a* is a meeting of *b*} *z y* ∧ {*a* attends *b*} *x z*))) ,

next

⋀*x*((F^1x ∧ G^2 *x* A^1 the club) → ⋁*y*((H^2 *y* the club ∧ I^2xy) ∧ ⋀*z*(J^2zy → K^2xz))) ,

next

⋀*x*((F^1x ∧ G^2 *x* A^1B^0) → ⋁*y*((H^2y B^0 ∧ I^2xy) ∧ ⋀*z*(J^2zy → K^2xz))) ,

and finally

⋀*x*[F*x* ∧ G(*x* A(B)) → ⋁*y*(H(*y* B) ∧ I(*xy*) ∧ ⋀*z*[J(*zy*) → K(*xz*)])] .

19. Every husband and wife has a spouse. (F^1 : *a* is a husband; G^1 : *a* is a wife; H^2 : *a* is a spouse of *b*)

20. No one lacks a father, but not everyone is a father. (F^1 : *a* is a person; G^2 : *a* is father of *b*)

21. Every number has some number as its successor. (F^1 : *a* is a number; G^2 : *a* is a successor of *b*)

22. Some number is a successor of every number. (F^1 : *a* is a number; G^2 : *a* is a successor of *b*)

23. A teacher has no scruples if he or she assigns a problem that has no solution. (F^1 : *a* is a teacher; G^1 : *a* has scruples; H^2 : *a* assigns *b*; J^1 : *a* is a problem; K^2 : *a* is a solution of *b*)

24. The net force acting on a particle is equal to the product of its mass and its acceleration. (F^1 : a is a particle; A^1 : the net force acting on a; G^2 : a is equal to b; B^2 : the product of a and b; C^1 : the mass of a; D^1: the acceleration of a)

25. No principle is innate unless everyone who hears it assents to it. (F^1 : a is a principle; G^1 : a is innate; H^1 : a is a person; J^2 : a hears b; K^2 : a assents to b)

26. The cube of the square of 2 is even. (F^1 : a is even; A^1 : the cube of a; B^1 : the square of a; C^0 : 2)

27. No monk likes every course he takes unless the only courses he takes are theological studies. (F^1 : a is a monk; G^1 : a is a course; H^1 : a is a theological study; J^2 : a takes b; K^2 : a likes b)

28. A man who loves himself more than he loves anyone else is not loved by anyone other than himself. (F^1 : a is a man; G^1 : a is a person; H^2 : a loves b; J^2 : a is different from b; L^4 : a loves b more than c loves d)

29. If a father has only male children, then he does not have to provide a dowry for any one of them. (F^1 : a is a father; G^1 : a is male; H^2 : a is a child of b; J^2 : a has to provide a dowry for b)

30. The wife of anyone who marries the daughter of the brother of his father marries the son of the brother of the husband of her mother. (F^2 : a marries b; A^1 : the wife of a; B^1 : the daughter of a; C^1 : the brother of a; D^1 : the father of a; E^1 : the son of a; A^1_1 : the husband of a; B^1_1 : the mother of a)

31. If x is an integer greater than or equal to zero and every integer is divisible by x, then x is equal to 1. (F^1 : a is an integer; G^2 : a is greater than b; H^2 : a is equal to b; I^2 : a is divisible by b; A^0 : zero; B^0 : 1)

32. The square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the lengths of the other two sides. (F^1 : a is a right triangle; G^2 : a is a side of b; H^2 : a is equal to b; A^1 : the square of a; B^1 : the hypotenuse of a; C^2 : the sum of a and b; D^1 : the length of a)

EXERCISES, GROUP IV

On the basis of the scheme of abbreviation provided, symbolize each of the following sentences. A sentence may be ambiguous, in which case all plausible symbolizations should be provided. Exercise 33 is solved in part for illustration.

33. Someone is hit by a car every day. (F^1 : a is a person; G^1 : a is a car; H^1: a is a day; J^3 : a is hit by b on c)

Two of the four readings for this sentence can be expressed as follows:

> Every day there is some person (or other) who is hit by some car (or other) ,
>
> There is some person (in particular) who is hit every day by some car (or other) .

These sentences, on the basis of the given scheme, are respectively symbolized by

$$\Lambda z(Hz \rightarrow \vee x \vee y[Fx \wedge Gy \wedge J(xyz)])$$

and

$$\vee x(Fx \wedge \Lambda z[Hz \rightarrow \vee y(Gy \wedge J(xyz))]) \quad .$$

We leave the other two readings of this sentence and their symbolizations to the reader.

34. (i) You can fool some of the people all of the time. (ii) You cannot fool all of the people all of the time. (F^2 : you can fool a at b; G^1 : a is a person; H^1 : a is a time)

35. The product of 3 and the square of 6 plus 1 is even. (A^0 : 3; B^0 : 6; C^0 : 1; D^2 : the product of a and b; E^1 : the square of a; A^2_1 : a plus b; F^1 : a is even)

5. Revised inference rules.

5. Revised inference rules. We must extend the notion of proper substitution for a variable. In chapter III only variables and name letters (presently 0-place operation letters) could be substituted for variables; now we shall admit the substitution of arbitrary symbolic terms for variables. Thus we say that a symbolic formula $\phi\zeta$ *comes from* a symbolic formula ϕ_α *by proper substitution of a symbolic term* ζ *for a variable* α if $\phi\zeta$ is like ϕ_α except for having free occurrences of ζ wherever ϕ_α has free occurrences of α.

In chapter III the instantial term for UI and EG had to be either a variable or a name letter. We now reformulate these two rules to permit as the instantial term an arbitrary symbolic term.

Universal instantiation (UI): $\dfrac{\Lambda\alpha\phi_\alpha}{\phi\zeta}$

Existential generalization (EG): $\dfrac{\phi\zeta}{\vee\alpha\phi_\alpha}$

In both cases α is to be a variable, ζ is to be *any symbolic term*, and $\phi\zeta$ is to be a symbolic formula that comes from the symbolic formula ϕ_α by proper substitution of ζ for α.

All previous applications of UI and EG are comprehended under the present formulation; there are additional applications of the following sort. From the sentence 'ΛxFx' we can now infer by UI such formulas as

(1) $FB(A)$,
(2) $FB(x)$,
(3) $FC(Ax)$.

will be counted among the symbolic formulas of our language. To be completely explicit, the class of *symbolic formulas* is exhaustively characterized as follows:

(1) *If π is a k-place predicate letter and $\zeta_1, \ldots, \zeta_k$ are symbolic terms, then*

$$\pi\zeta_1 \ldots \zeta_k$$

is a symbolic formula. (In particular, a 0-place predicate letter is itself a symbolic formula.)

(2) *If ϕ and ψ are symbolic formulas, so are*

$$\sim\phi$$
$$(\phi \rightarrow \psi) \ ,$$
$$(\phi \wedge \psi) \ ,$$
$$(\phi \vee \psi) \ ,$$
$$(\phi \leftrightarrow \psi) \ .$$

(3) *If ϕ is a symbolic formula and α a variable, then*

$$\bigwedge\alpha\phi \ ,$$
$$\bigvee\alpha\phi$$

are symbolic formulas.

(4) *If ζ and η are symbolic terms, then*

$$\zeta = \eta$$

is a symbolic formula.

Formulas obtained by clause (1) or by clause (4) are *atomic formulas* (for neither a quantifier nor a sentential connective occurs in them); and we shall also refer to formulas obtained by clause (4) as *identity formulas*. We shall continue to use the informal notational conventions introduced in section 3 of chapter IV. In addition we shall adopt the convention of writing the negation of an identity formula

$$\zeta \neq \eta$$

rather than

$$\sim\zeta = \eta \ .$$

Again we emphasize that official notation must be mentally restored before decisions concerning bondage and freedom and applications of inference rules can be made.

2. Translation and symbolization. An *abbreviation* and a *scheme of abbreviation* are to be understood in the same way as in the preceding chapter (p. 209). And *literal translation into English on the basis of a given*

scheme of abbreviation is to be handled as in that chapter, section 4, with one modification. In step (v) (p. 210), we replace '=' by 'is identical with', in addition to eliminating sentential connectives and quantifier phrases in favor of the corresponding phrases of connection and quantity.

As before, we say that an English formula ψ is a *free translation* (or simply a *translation*) of a symbolic formula ϕ (or that ϕ is a *symbolization* of ψ) on the basis of a given scheme of abbreviation if ψ is a stylistic variant of the literal English translation of ϕ based on that scheme.

For example, consider the scheme of abbreviation

(1) A^0 : Mark Twain
B^0 : Samuel Clemens

and the symbolic sentence

(2) $A = B$.

The literal English translation of (2) on the basis of the scheme (1) is

(3) Mark Twain is identical with Samuel Clemens ,

and a free translation of (2) would be

(4) Mark Twain is Samuel Clemens .

Thus we regard (4) as a stylistic variant of (3). 'Is' does not, however, always have the sense of *identity*. In some contexts, for example,

(5) Socrates is wise ,

'is' has the sense of *predication*. (5) is true just in case Socrates has the characteristic, possibly common to many individuals, of wisdom; it is not asserted that the words 'Socrates' and 'wise' designate the same object (or even that 'wise' designates any object), as the sense of identity would require.

The logic of predication can best be regarded as part of the theory developed in chapters III and IV. The present chapter concerns the logic of identity.

Sentence (3) has other stylistic variants than (4); for example,

Mark Twain is the same as Samuel Clemens ,
Mark Twain and Samuel Clemens are the same ,
Mark Twain and Samuel Clemens are identical ,
Mark Twain equals Samuel Clemens .

The negation of (2) has free translations other than those suggested by the free translations of (2). For example,

Mark Twain differs from Samuel Clemens ,
Mark Twain is distinct from Samuel Clemens ,

and

Mark Twain is other than Samuel Clemens

are all free translations of

$$A \neq B \quad ,$$

or, in official notation, of

$$\sim A = B \quad ,$$

on the basis of the scheme (1).

To find a symbolization of a given formula of English on the basis of a given scheme of abbreviation, the reader will find it useful to proceed roughly as follows:

(1) Introduce 'is identical with', phrases of quantity, and phrases of connection, the latter accompanied by parentheses and occurring canonically, in the place of their stylistic variants.

(2) Reverse the steps leading from a symbolic formula to a literal English translation.

For example, consider the scheme of abbreviation

F^2 : *a* is married to *b*
G^1 : *a* is a monogamous citizen
A^1 : the spouse of *a*

and the sentence

Anyone married to a monogamous citizen is the spouse of such a citizen

In step (1) we obtain

For each *x* (if there is an object *y* such that (*x* is married to *y* and *y* is a monogamous citizen), then there is an object *y* such that (*y* is a monogamous citizen and *x* is identical with the spouse of *y*)) ;

and carrying through the successive parts of step (2), we obtain the symbolization

$$\bigwedge x(\bigvee y[F(xy) \wedge Gy] \rightarrow \bigvee y[Gy \wedge x = A(y)]) \quad .$$

As another example, consider the scheme of abbreviation

F^2 : *a* is a member of *b*
G^2 : *a* has defeated *b*
A^0 : the team
B^1 : the captain of *a*

and the sentence

The captain of the team has defeated every other member of the team .

The symbolization in this case is

$$F(B(A)A) \wedge \bigwedge x[F(xA) \wedge x \neq B(A) \rightarrow G(B(A)x)] \quad .$$

Closely related to the preceding example is the symbolization of superlatives. On the basis of the scheme of abbreviation

F^2 : *a* is a member of *b*
G^2 : *a* is taller than *b*
A^0 : the team
B^1 : the captain of *a* ,

the sentence

The captain of the team is the tallest member of the team

is symbolized by

$$F(B(A)A) \wedge \bigwedge x[F(xA) \wedge x \neq B(A) \rightarrow G(B(A)x)] \quad .$$

Frequently in English sentences, the definite article 'the' indicates the identity sense of 'is', which we symbolize by '=', whereas the indefinite article indicates the so-called predication sense of 'is', which we symbolize simply by juxtaposition of predicate letters and symbolic terms. The following sentence is an example.

If Alice is polyandrous and married to Alfred, then Alfred *is a* husband of Alice but it is not the case that he *is the* only husband of Alice .

On the basis of the scheme of abbreviation

F^1 : *a* is polyandrous
G^2 : *a* is married to *b*
H^2 : *a* is a husband of *b*
A^0 : Alice
B^0 : Alfred

the sentence is symbolized by

$$FA \wedge G(AB) \rightarrow H(BA) \wedge \sim \bigwedge x[H(xA) \rightarrow x = B] \quad .$$

EXERCISES, GROUP I

1. For each sentence of group A, find the sentence of group B that is a symbolization of it.

GROUP A

(1) Something differs from everything except itself.
(2) Everything is identical with something.
(3) Everything is distinct from something.
(4) Everything equal to a thing differs from everything not equal to that thing.
(5) Nothing differs from everything.
(6) Things equal to the same thing are equal to each other.
(7) Nothing differs from itself.

GROUP B

(a) $\sim \bigvee x \bigwedge y\ x \neq y$
(b) $\bigwedge x \bigvee y\ x = y$
(c) $\bigvee x \bigwedge y (y \neq x \rightarrow x \neq y)$
(d) $\bigwedge x \bigwedge y (\bigvee x [x = z \wedge y = z] \rightarrow x = y)$
(e) $\bigwedge x \bigvee y\ y \neq x$
(f) $\sim \bigvee x\ x \neq x$
(g) $\bigwedge x \bigwedge z [x = z \rightarrow \bigwedge y (y \neq z \rightarrow x \neq y)]$

EXERCISES, GROUP II

Symbolize each of the following sentences on the basis of the scheme of abbreviation that accompanies it. Exercises 2 and 3 are solved for illustration.

2. None but Alfred and his teacher are able to solve the problem. (F^2 : *a* is able to solve *b*; A^0 : Alfred; B^0 : the problem; C^1 : the teacher of *a*)

Two interpretations of No. 2 are acceptable. One interpretation would leave open the question whether Alfred and his teacher are able to solve the problem, and the other would answer that question in the affirmative. On the former interpretation, No. 2 passes into

$$\bigwedge x [F(xB) \rightarrow x = A \vee x = C(A)] \quad ;$$

on the latter, it passes into

$$\bigwedge x [F(xB) \leftrightarrow x = A \vee x = C(A)] \quad .$$

3. Alfred attended the conference and arrived at it before everyone else who attended it except Mary. (F^2 : *a* attended *b*; G^3 : *a* arrived at *b* before *c*; A^0 : Alfred; B^0 : the conference; C^0 : Mary)

Here the word 'else' emphasizes our implicit understanding of 'arrives at before', that is, that Alfred did not arrive at the conference before Alfred. And again both a weaker and a stronger interpretation are acceptable; thus No. 3 could pass into either

$$F(AB) \wedge F(CB) \wedge \bigwedge x [F(xB) \wedge x \neq A \wedge x \neq C \rightarrow G(ABx)]$$

or

$$F(AB) \wedge F(CB) \wedge \bigwedge x (F(xB) \rightarrow [x \neq A \wedge x \neq C \leftrightarrow G(ABx)]) \quad .$$

Note that

$$\bigwedge x [F(xB) \wedge G(xBA) \rightarrow x = C]$$

could not replace the third conjunct in the symbolization of No. 3. On the given scheme, it symbolizes

> If anyone who attended the conference arrived there before Alfred, it was Mary .

And this last sentence, in contrast to No. 3, would be true if everyone who attended the conference arrived at the same time.

4. Alfred is the only member of the class who can read Greek. (F^2 : *a* is a member of *b*; G^2 : *a* can read *b*; A^0 : Alfred; B^0 : the class; C^0 : Greek)

5. Greensleeves can jump farther than any other frog in Calaveras County. (F^2 : *a* can jump farther than *b*; G^1 : *a* is a frog; H^2 : *a* is in *b*; A^0 : Greensleeves; B^0 : Calaveras County)

6. Pergolesi was the most promising composer of his time. (A^0 : Pergolesi; F^2 : *a* was more promising than *b*; G^1 : *a* is a composer; H^2 : *a* was contemporary with *b*)

7. Anyone whose mother is the wife of a citizen, whose father is married to his or her mother, and whose father has no other children is the child of a citizen. (F^1 : *a* is a citizen; G^2 : *a* is married to *b*; H^2 : *a* is a child of *b*; A^1 : the mother of *a*; B^1 : the wife of *a*; C^1 : the father of *a*; D^1 : the child of *a*)

8. If the sister of Alfred is the wife of the brother-in-law of Alfred, then the father-in-law of Alfred is the father-in-law of the sister of Alfred. (A^0 : Alfred; B^1 : the wife of *a*; C^1 : the sister of *a*; D^1 : the brother-in-law of *a*; E^1 : the father-in-law of *a*)

9. If a sister of Alfred is the wife of a brother-in-law of Alfred, then the father-in-law of Alfred is the father-in-law of a sister of Alfred. (A^0 : Alfred; B^1 : the wife of *a*; C^1 : the father-in-law of *a*; F^2 : *a* is a sister of *b*; G^2 : *a* is a brother-in-law of *b*)

10. If a sister of Alfred is the wife of a brother-in-law of Alfred, then the father-in-law of Alfred is the father-in-law of that sister of Alfred. (A^0 : Alfred; B^1 : the wife of *a*; C^1 : the father-in-law of *a*; F^2 : *a* is a sister of *b*; G^2 : *a* is a brother-in-law of *b*)

11. Brothers and sisters I have none, but that man's father is my father's son. (A^0 : the speaker; B^0 : that man; C^1 : the father of *a*; D^1 : the son of *a*; F^2 : *a* is a brother of *b*; G^2 : *a* is a sister of *b*)

12. The product of distinct positive integers is less than their sum if and only if one of them is equal to its square. (A^2 : the product of *a* and *b*; B^2 : the sum of *a* and *b*; C^1 : the square of *a*; F^1 : *a* is a positive integer; G^2 : *a* is less than *b*)

3. Inference rules. Notions of bondage and freedom, as well as the notion of proper substitution of a term for a variable, are to be understood

3. Informal notational conventions. Once again we shall employ the conventions of chapter II for omitting parentheses and replacing some of them by brackets.

As in chapters IV and V, we shall usually omit superscripts from operation and predicate letters and employ the convention of inserting parentheses and brackets to avoid ambiguity. In the present context, however, the fact that our convention does not require parentheses or brackets to accompany a 1-place predicate letter could lead to ambiguity. For example, given our previous convention, the expression

(1) $A(\imath x Fx)$

could have as its official counterpart either

(2) $A^1 \imath x F^1 x$

or

(3) $A^2 \imath x F^0 x$.

To avoid such an ambiguity, without the cumbersome requirement that 1-place predicate letters be accompanied by parentheses or brackets, we shall never omit a superscript from a 0-place predicate letter and shall always understand expressions like (1) to have expressions like (2) as their official counterpart. Thus the best we can obtain as an informal counterpart to (3) is the expression

$$A(\imath x F^0 x) \quad .$$

We shall also continue the convention of writing, where ζ and η are symbolic terms,

$$\zeta \neq \eta$$

for

$$\sim \zeta = \eta \quad .$$

In formulating inference rules and criteria of bondage and freedom, we always have in mind official notation.

EXERCISES

5. For each term or formula (in unofficial notation) below, delete inserted parentheses and brackets and restore omitted superscripts in accordance with the conventions of this section.

(a) $F[A \imath x F(xA)]$
(b) $A[\imath x\, A(xy) = x]$
(c) $G \imath x G(Ax)$
(d) $A(\imath x F^0 x A)$
(e) $A(\imath x F x A)$

4. Translation and symbolization. An *abbreviation* is to be understood in the same way as in chapter IV (p. 209). The process of *literal translation into English on the basis of a given scheme of abbreviation* may now begin with either a symbolic term or a symbolic formula, and if successful will end with a term or formula of English. The process consists of the steps given in chapter IV (pp. 209–10), with one modification. Step (v) should now read as follows:

(*v*) *Eliminate sentential connectives, quantifier phrases, '=', and descriptive phrases in favor of the corresponding phrases of English, preserving all parentheses.*

An English term or formula ψ is said to be a *free translation* (or simply a *translation*) of a symbolic term or formula ϕ (and ϕ is said to be a *symbolization* of ψ) on the basis of a given scheme of abbreviation if ψ is a stylistic variant of the literal English translation of ϕ based on that scheme.

For example, on the basis of the scheme

F^1 : *a* is a doctor
G^2 : *a* is father of *b*
H^2 : *a* loves *b*
J^2 : *a* wrote *b*
K^2 : *a* is larger than *b*
A^0 : Socrates
B^0 : *Waverley* ,

the descriptive names

(1) $\imath x Fx$,
(2) $\imath x G(xA)$,
(3) $\imath y \wedge x H(xy)$,
(4) $\imath z\, z = \imath y J(yB)$,
(5) $\imath x \wedge y[y \neq x \rightarrow K(xy)]$

have the following respective literal English translations:

the object *x* such that *x* is a doctor ,
the object *x* such that *x* is father of Socrates ,
the object *y* such that, for each *x*, *x* loves *y* ,
the object *z* such that *z* is identical with the object *y* such that *y* wrote *Waverley* ,
the object *x* such that for each *y* (if it is not the case that *y* is identical with *x*, then *x* is larger than *y*) .

On the basis of the same scheme, the following terms of English are free translations of (1)–(5):

(6) the doctor ,

(7) the father of Socrates ,
what everyone loves ,
he who is identical with the one who wrote *Waverley* ,
(8) the largest thing .

Thus we regard 'the', 'what', 'he who', and 'the one who', in some of their uses, as stylistic variants of 'the object . . . such that'.

It should not be supposed that every occurrence of 'the' can be supplanted by a phrase of description. Consider, for example, the sentences

(9) The whale is a mammal ,
(10) The members of the club are Republicans ,
(11) The trespassers were prosecuted .

(9) asserts that every whale is a mammal, not that the object x such that x is a whale is a mammal; and attempts to express (10) and (11) with the help of phrases of description would lead to ungrammatical results. In (6) – (8), 'the' generates a name, but this is not the case in (9) – (11); even 'the whale' cannot reasonably be construed as designating in (9) a single object.

To find a symbolization of a given term or formula of English on the basis of a given scheme of abbreviation the reader will find it useful to proceed roughly as follows:

(1) *Introduce 'is identical with', phrases of description, phrases of quantity, and phrases of connection, the latter accompanied by parentheses and occurring canonically, in place of their stylistic variants.*

(2) *Reverse the steps leading from a symbolic term or formula to a literal English translation.*

For example, consider the scheme of abbreviation

F^2 : a is wife of b
G^2 : a is more salacious than b
H^1 : a is a woman
J^2 : a is mentioned by b
A^0 : Justinian
B^0 : Gibbon

and the sentence

The wife of Justinian is the most salacious woman mentioned by Gibbon .

In step (1) we might obtain

The object x such that x is wife of Justinian is identical with the object x such that ((x is a woman and x is mentioned by Gibbon) and for each y (if ((y is a woman and y is mentioned by Gibbon) and it is not the case that y is identical with x), then x is more salacious than y)) ,

and carrying through the successive parts of step (2) we obtain the symbolization

$$\imath x F(xA) = \imath x(Hx \wedge J(xB) \wedge \bigwedge y[Hy \wedge J(yB) \wedge y \neq x \rightarrow G(xy)]) \quad .$$

EXERCISES

6. Provide a literal and a free translation into English of the term

$$\imath x F[x \imath y F(yx)]$$

on the basis of the scheme

F^2 : a loves b .

Symbolize each of the following sentences on the basis of the scheme of abbreviation that accompanies it. Exercise 7 is solved for illustration.

7. The man who lives at the North Pole does not live there.

On the basis of the scheme of abbreviation

F^1 : a is man
G^2 : a lives at b
A : the North Pole ,

No. 7 becomes

$$\sim G(\imath x[Fx \wedge G(xA)]A) \quad .$$

8. The positive square root of 2 is a positive even prime. (F^1 : a is positive; G^2 : a is a square root of b; H^1 : a is even; J^1 : a is prime; A^0 : 2)

9. Mary loves the one who loves Mary. (F^2 : a loves b; A^0 : Mary)

10. If he who murdered Desdemona was murdered by Desdemona's murderer, then he committed suicide. (F^2 : a murdered b; G^1 : a committed suicide; A^0 : Desdemona)

11. The hardest problem on the examination was solved by no one. (F^1 : a is a problem on the examination; G^2 : a is harder than b; H^2 : a was solved by b)

12. If the author of *Waverley* is the author of *Ivanhoe*, then the author of *Ivanhoe* is an author of *Waverley*. (F^2 : a is an author of b; A^0 : *Waverley*; B^0 : *Ivanhoe*)

13. God is that than which nothing greater can be conceived. (A^0 : God; G^2 : a is greater than b; F^1 : a can be conceived)

14. The positive square root of the square of the even prime is irrational. (F^1 : a is positive; G^2 : a is a square root of b; H^1 : a is even; I^1 : a is prime; J^1 : a is irrational; A^1 : the square of a)

15. The smallest positive integer is that positive integer which

when multiplied by itself equals itself. (F^2 : *a* is less than *b*; G^1 : *a* is positive; H^1 : *a* is an integer; A^2 : the product of *a* and *b*)

16. There is no greatest positive integer. (F^2 : *a* is greater than *b*; G^1 : *a* is positive; H^1 : *a* is an integer)

5. Inference rules. For the logic of the descriptive operator we must adopt two new inference rules, the first of which is the following:

Proper descriptions (PD): $$\frac{\bigvee\beta\bigwedge\alpha(\phi_\alpha \leftrightarrow \alpha = \beta)}{\phi\iota\alpha\phi}$$

Here α and β are variables, ϕ_α is a symbolic formula in which β is not free, and

$$\phi\iota\alpha\phi$$

comes from ϕ_α by proper substitution of the term

$$\iota\alpha\phi$$

for α. PD corresponds to the principle that if a given condition is satisfied by one and only one object, then the object satisfying that condition satisfies it. PD leads, for example, from the sentence

$$\bigvee y\bigwedge x(Fx \leftrightarrow x = y)$$

(which could be translated as the assertion that exactly one thing wrote *Waverley*) to the sentence

$$F\iota xFx$$

(which would then assert that the author of *Waverley* wrote *Waverley*). A somewhat more complex application of PD is the inference from the formula

$$\bigvee y\bigwedge x(Fx \wedge G(xzw) \leftrightarrow x = y)$$

(which could be taken as asserting that there is exactly one integer between *z* and *w*) to the formula

$$F\iota x[Fx \wedge G(xzw)] \wedge G(\iota x[Fx \wedge G(xzw)]\, z\, w)$$

(which would then assert that the integer between *z* and *w* is an integer between *z* and *w*).

Upon first consideration of such examples, one might believe that any English sentence that is a translation of a conclusion of PD is true. But consideration of further examples would quickly dispel this belief. The sentence

> The man who lives at the center of the earth is a man living at the center of the earth

is false, for it implies the falsehood that some man lives at the center of

the earth. The assumption of propriety that constitutes the premise of PD is, however, sufficient to exclude such examples.

Because we seldom have occasion to use improper descriptive names (that is, symbolic terms corresponding to improper definite descriptions), we could develop a significant part of the logic of the descriptive operator on the basis of the single rule PD. There is, however, a strong reason for introducing along with PD a rule pertaining to improper descriptive terms. Without further rules we could not justify inferences that are intuitively plausible whether the descriptive terms are proper or not, such as that from '$F\iota xGx$' to either '$F\iota yGy$' or '$F\iota x \sim\sim Gx$'. There are other reasons for adding a rule concerned with improper descriptions, and these will appear in sections 6 and 7 of this chapter.

The relevant new rule has the following form:

Improper descriptions (ID): $$\frac{\sim\bigvee\beta\bigwedge\alpha(\phi_\alpha \leftrightarrow \alpha = \beta)}{\iota\alpha\phi = \iota\gamma\,\gamma \neq \gamma}$$

Here α, β, and γ are variables and ϕ_α is a symbolic formula in which β is not free.

The rule ID has no intuitive counterpart, simply because ordinary language shuns improper definite descriptions. The rule corresponds, however, to the resolution that every improper definite description is to designate the same object as the particular improper definite description

> the object *x* such that *x* differs from *x* ,

and this resolution accords with the Fregean decision made on page 308.

ID leads, for example, from the sentence

(1) $$\sim\bigvee y\bigwedge x(Fx \leftrightarrow x = y)$$

(which could be translated as the assertion that not exactly one object is presently king of France), to the sentence

(2) $$\iota xFx = \iota x\, x \neq x$$

(which would then assert that the present king of France is the object *x* such that *x* differs from *x*; or, less paradoxically, that the descriptive names 'the present king of France' and 'the object *x* such that *x* differs from *x*' designate the same object).

The descriptive name '$\iota x\, x \neq x$' that occurs in (2) could be read idiomatically 'the nonidentical thing'. Given this reading, it is essential to distinguish carefully between linguistic expressions and nonlinguistic objects. Clearly there is no unique nonidentical thing (see T334, p. 281), and it is just that necessary fact that enables the linguistic expression 'the nonidentical thing' to play its intended role in (2), that is, in the conclusion of ID. Since it is impossible for the descriptive name '$\iota x\, x \neq x$' to designate an object, no ambiguity can occur if some arbitrary object is chosen as its designation. And (2) then asserts, given (1), that the improper description