Arbitrary Reference in Mathematical Reasoning

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Mathematicians use very often in their reasoning expressions of the kind "let *a* be an arbitrary object of the universe of discourse", for instance "let *a* be an arbitrary real number". Observe that there is no link between the letter "*a*" and the number which it is supposed to be indicating. Taking such expressions at face value, mathematical reasoning would seem to presuppose, at least ideally, the possibility of indicating *any* object of the universe of discourse, even when, as in the case of real numbers, not every object has a name in the language. Within the intuitionistic conception of mathematics, such a presupposition is made quite explicit by the doctrine according to which an object exists only as a mental construction of an *ideal mathematician*, so that any object is capable of being exhibited, and therefore indicated, by acquaintance with it. The ideal mathematician can refer to any object, in virtue of his direct access to his own mental constructions. In contrast, within the *classical* (*realist*) conception of mathematics, it would seem, from the literature, that the mathematical treatment of a domain of objects by no means requires the ideal possibility of individual reference to every object of the domain. It seems to be a widespread opinion that, once the objective existence of, e.g., the real numbers has been accepted, the general theory of real numbers is developable, by using the device of quantification, without any need that each number be capable of individual reference.

On the contrary, we claim that the *ideal possibility* of referring individually to any object of the universe of discourse is essential even in the realist perspective. We will call this claim *TAR* (*thesis of arbitrary reference*).

Some objections to *TAR*

There are a number of arguments in the literature, which seem to be in disagreement with our thesis. The most obvious is perhaps due to a misunderstanding of the notion of arbitrariness. One can argue that considering an arbitrary number is nothing but a way of speaking, which by no means involves the possibility of actually singling out such a number, since, for the very same arbitrariness, it is irrelevant which number one is speaking of. Indeed, when reasoning about an arbitrary number, there is no need to know it. Furthermore, the argument goes, the ignorance of which number one is referring to has the desired effect to grant generality to the reasoning: what is provable for a completely unknown number holds necessarily for all numbers. However, the lack of information about *a* cannot avoid the *assumption* that the letter "*a*" designates a precise number: lacking that assumption, it would make no sense talking about *a*, not even to say that it is unknown.

Some remarks on the notion of *possibility* involved by *TAR* are in order. Such a possibility is not to be understood as the ability of the speaker to exhibit or describe the arbitrary object he is referring to. Of course we don't need such ability in order to speak of an arbitrary number or of an arbitrary (possibly extraterrestrial) living being. What is needed is merely to *imagine* that the object in question has been in some way fixed. The verb "to fix" often recurs in the course of informal mathematical talk. For instance, the well-known ε-δ-definition of limit sounds: "however a positive number ε has been fixed, you can find a positive number δ such that . . .". In such a context you don't have to worry about *how* ε has been fixed, but you must *imagine* that *in some way* it has been fixed and that it may be *any* positive number. We want to hold that locutions of this kind are not to be regarded as a mere way of speaking, but that they play an essential role in mathematical rea-

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soning. The use in natural language of the indefinite article "a" may erroneously suggest that, in order to talk about an arbitrary object, there is no need to think of it as well-determined. You can realize that this suggestion is deceptive by reflecting on the use of pronouns, which do refer to a well-determined object, even when this has been introduced by means of the indefinite article. Consider, e.g., the talk:" Take *an* arbitrary real number. . . . Suppose *it* is irrational. . . ." Which number does the pronoun "it" refer to? Of course the correct answer is not "to any real number", but "to *the* number under consideration". In this answer the *definite* article is used just for referring to *the* number introduced by the *indefinite* article. The puzzle "How can the definite article be appropriate, since, as a matter of fact, no number has been fixed?" has, we maintain, the answer: "To consider an arbitrary number means to *imagine* that a number has been fixed. *Imagination* is all is required for this kind of reference". Arbitrary reference rests on our ability of imagining that an object of the universe of discourse has been fixed.

The importance of imagination in the platonist conception of mathematics has been emphasized by Bernays in his famous paper "On platonism in mathematics":

> The value of platonistically inspired mathematical conceptions is that they furnish models of abstract imagination. These stand out by their simplicity and logical strength. They form representations which extrapolate from certain regions of experience and intuition. (Bernays, 1935)

If, as we believe, *TAR* is correct, it is of remarkable interest for the philosophy of mathematics. It poses the problem of supplying a more definite content to the act of imagining involved by arbitrary reference, as a constituent of mathematical realism. Before addressing this problem, we want to discuss some further possible objections to our thesis.

An argument against *TAR* may contend that, though arbitrary reference occurs in informal reasoning, it is not essential, since it may be avoided by the use of quantifiers, which do not refer individually to any object of the quantification domain. This argument rests on the confusion between the locutions "any" and "each": talking about *any* object may seem to amount to talking about *each* object. This is not the case, however. Russell was clearly aware of the difference:

> The general enunciation tells us something about (say) all triangles, while the particular enunciation takes one triangle and asserts the same thing of this one triangle. But the triangle taken is *any* triangle, not some one special triangle; and thus, although, throughout the proof, only one triangle is dealt with, yet the proof retains its generality. If we say: "Let *ABC* be a triangle, then the sides *AB* and *AC* are together greater than the side *BC*", we are saying something about *one* triangle, not about *all* triangles; but the one triangle concerned is absolutely ambiguous, and our statement consequently is also absolutely ambiguous. We do not affirm any one definite proposition, but an undetermined one of all the propositions resulting from supposing *ABC* to be this or that triangle. This notion of ambiguous assertion is very important, and it is vital not to confound an ambiguous assertion with the definite assertion that the same thing holds in *all* cases.
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> The distinction between (1) asserting any value of a propositional function and (2) asserting that the function is always true is present throughout mathematics, as it is in Euclid's distinction of general and particular enunciations. In any chain of mathematical reasoning, the objects whose properties are being investigated are the arguments to any value of some propositional function. . . . For this reason, when *any* value of a propositional function is asserted, the argument . . . is called a *real* variable, whereas, when a function is said to be *always* true, or to be not always true, the argument is called an *apparent* variable. . . .
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> If ϕx is a propositional function, we will denote by "$(x)\cdot\phi x$" the proposition "ϕx is always true". . . . Then the distinction between the assertion of all values and the assertion of any is the distinction between (1) asserting $(x)\cdot\phi x$ and (2) asserting ϕx where x is undetermined. The latter differs from the former in that it cannot be treated as one determinate proposition.
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> The distinction between asserting ϕx and asserting $(x)\cdot\phi x$ was, I believe, first emphasized by Frege (1893, p. 31). His reason for introducing the distinction explicitly was the same which had caused it to be present in the practice of mathematicians, namely, that deduction can only be effected with *real* variables, not with apparent variables. In the case of Euclid's proofs, this is evident: we need (say) some one triangle *ABC* to reason about, though it does not matter what triangle it is. The triangle *ABC* is a *real* variable; and although it is *any* triangle, it remains the *same* triangle throughout the argument. But in the general enunciation the triangle is an apparent variable. If we adhere to the apparent variable, we cannot perform any deduction, and this is why in all proofs real variables have to be used. (Russell, 1908)

In today's formal logic Russell's distinction between *real* and *apparent* variables is faithfully reproduced, with a sheer change in terminology, by the well-known distinction between *free* and *quantified* variables. Singular reference plays an essential role in quantification theory. This fact is made quite perspicuous by the meaning of the quantification rules in natural deduction. According to the elimination rule for the existential quantifier, in order to derive a conclusion A from an existential assumption $\exists x\, P(x)$, one has to assume $P(a)$ (where a is a fresh free variable) and derive A from $P(a)$(with the due restrictions). This rule is justifiable only if it is granted that, under the existential assumption, one can consider an arbitrary object a such that $P(a)$. A similar observation holds for the introduction rule of the universal quantifier. The soundness of classical natural deduction rests therefore on the hidden assumption that every object of the domain is capable of being the referent of some act of reference. Thus formal logic does justice to the informal locution "let a be an object such that $P(a)$". What the textbooks of logic fail to tell us is what the act of referring to an arbitrary object consists in. About this issue the formal theory is silent and, as far as we know, the philosophical remarks in the literature seem to be somewhat confusing. Even Russell's passage, quoted above, though enlightening the need of reasoning about *single* arbitrary objects, seems to give a rather misleading explanation of the nature of arbitrary reference. To say that *any* triangle is not some one special triangle, but that it is absolutely ambiguous, might erroneously suggest that the triangle concerned is a strange object enjoying the strange property of being absolutely ambiguous. But, of course, an ontology of ambiguous objects would be far from desirable. Alternatively, Russell's explanation might suggest that what is ambiguous is the act of referring to any arbitrary triangle, in the sense that it is undetermined which triangle it refers to. Indeed Frege, after rejecting the first alternative, seems to hold the second:

> [Mr. E. Czuber] . . . defines a variable as an indefinite number. But are there indefinite numbers? Must numbers be divided into definite and indefinite? Are there indefinite men? Must not every object be definite? 'But is not the number n indefinite?' I am not acquainted with the number n. 'n' is not the proper name of any number, definite or indefinite. Nevertheless, we do sometimes say 'the number n'. How is this possible? Such an expression must be considered in a context. Let us take an example. 'If the number n is even, then $\cos n\pi = 1$'. Here only the whole has a sense, not the antecedent by itself nor the consequent by itself. The question whether the number n is even cannot be answered; no more can the question whether $\cos n\pi = 1$. For an answer to be given, 'n' would have to be the proper name of a number, and in that case this would necessarily be a definite one. We write the letter n in order to achieve generality. This presupposes that, if we replace it by the name of a number, both antecedent and consequent receive a sense.
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> Of course we may speak of indefiniteness here; but here the word 'indefinite' is not an adjective of 'number', but 'indefinitely' is an adverb, e. g., of the verb 'to indicate'. We cannot say that 'n' designates an indefinite number, but we *can* say that it indicates numbers indefinitely. (Frege, 1904)

But, besides being quite obscure what an act of *indicating indefinitely* is, such indefiniteness would be incompatible with the essential fact, clearly stressed by Russell, that, in the whole course of the proof, the referent is always the same. This means that the letter "a" indicates the same object in all its occurrences in the proof. If the referents of such occurrences were not well-determined objects, it would be meaningless to say that they are the same. Perhaps Frege believed this objection to be superseded by his doctrine of functions as *unsaturated entities*. In "Function and Concept" he says:

> . . . people who use the word 'function' ordinarily have in mind expressions in which a number is just indicated indefinitely by the letter x, e.g. '$2x^3 + x$' . . . [But] x must not be considered as belonging to the function; this letter only serves to indicate the kind of supplementation that is needed; it enables one to recognize the places where the sign for the argument must go in. (Frege, 1891)

It seems that, according to Frege, 'to indicate *indefinitely*' really means 'not to indicate anything at all'. The variables occurring in a functional expression do not denote anything; they are mere placeholders marking the gaps to be filled by individual names. According to this view, a reasoning about an arbitrary object x may be regarded as a schema of reasoning, i.e. as a function which maps every object into the reasoning obtained from the *schema* by replacing 'x' with a name of such object. It is plain, however, that such a schema can have all the desired instances only if every object is capable of being singled out and named. Thus regarding 'x' as a schematic letter is of no help to avoid *TAR*.

We think therefore that the ambiguity shown by Russell is to be understood in a purely *epistemic* sense. Referring to an arbitrary object a amounts to supposing that "a" designates an *unknown*, though *well-determined*, object. Being well-determined justifies the behavior of "a", in the course of the reasoning, as a name designating the same object in all its occurrences. On the other hand, being unknown guarantees that all that is established for it holds as well for any other object of the domain.

TAR as embodied in the logical concept of an object

The foregoing considerations show that arbitrary reference is essential for the proof theory of classical logic. Now, one can wonder if *TAR* is already implicit in the *semantics* of classical logic. Of course, the answer is certainly affirmative if one agrees that the meaning of logical constants is determined by the inference rules, since, as we have seen, arbitrary reference is involved in the quantification rules. We think, however, that such a thesis is not appropriate to mathematical realism. We assume therefore that the understanding of the semantic notions is prior to that of inference rules, which are justified *a posteriori* insofar as they are *recognized* as *truth-preserving*. Now, most working mathematicians do agree that the inference rules are truth preserving. They are inclined to recognize as *intuitively* correct the metamathematical formal proof of the soundness theorem. In particular they feel the cogency of the argument that, after recognizing $P(a)$ for a certain object a, without any assumption about a, one can rightfully conclude $(x)P(x)$. But, since the proof of $P(a)$ clearly exploits the *nameability* of a, the generalization is justified only under the assumption that every object is *nameable*. The fact that no mathematician feels the need of making this assumption explicit seems to suggest that the possibility in principle of referring to *any* object individually is implicit in the very same general concept of object.

The involvement of *TAR* is not made explicit by Tarski's definition of truth either. In fact, this rests on the definition of satisfaction of a formula, relative to an *assignment* of *arbitrary* members of the universe of discourse to the free variables, an assignment being understood as a *set-theoretical function*. So it may seem that the task of assigning arbitrary objects to the free variables is accomplished by a set-theoretical function. But that is illusory, since the problem of referring to an arbitrary object of the given domain shifts to that of referring to an arbitrary function from variables to objects. Thus, in order to avoid a regress *ad infinitum*, one must take, at some stage, arbitrary reference as primitive.

The commitment to *TAR* is quite evident in Hintikka's game theoretical semantics for first-order logic [see, e.g. (Hintikka, 1996)]. Here the meaning of logical constants is explained in terms of *choice acts*. With every sentence of a first-order language, Hintikka associates a game between two ideal players, the *verifier* and the *falsifier*, who are trying, respectively, to verify and to falsify the sentence. He then defines the truth and the falsity of a sentence as the existence of a *winning strategy*, respectively, for the verifier and for the falsifier. The game rules are defined in terms of arbitrary choices of individuals by the players and introduction of names for the chosen individuals. The definition proceeds by induction on the complexity of the sentence. In particular, the clause for the existential quantifier is the following:

> *A play relative to* $\exists xS(x)$ *starts with a choice of an individual b by the verifier. Then the plays continues as for* $S(b)$.
> (*The clause relative to the universal quantifier is similar, with the choice made by the falsifier*).

Hintikka observes that the name "b" does not necessarily belong to the given language but that, since the length of a sentence is finite, any play requires only finitely many new names. Hintikka takes for granted the ideal possibility of choosing any individual and giving it a name. He stresses the constructive flavor of his semantics, arising, in his view, from the fact that the meaning of logical constants is grounded on the notion of *action*. He proves the soundness of classical logic for his semantics and concludes that, in spite of the intuitionistic tenets, classical logic is constructively justified. Indeed Hintikka's game rules are perfectly intelligible from the intuitionistic viewpoint. However, his proof of soundness rests on a tacit *realistic* attitude concerning the existence of a winning strategy: once the game rules have been established, he regards as a well-determined objective fact the existence or non existence of a winning strategy for the verifier or for the falsifier. So Hintikka's proof that for every game there is a winning strategy for one of the two players fails intuitionistically. This explains why Hintikka's semantics turns out to be equivalent to the usual Tarskian semantics. Hintikka's "constructivism", based on the *action of choosing*, has nothing to do with intuitionistic *antirealism*.

We think, however, that Hintikka's philosophical perspective is coherent and that it supplies a faithful analysis of the usual mathematical reasoning. It makes explicit a "constructive" aspect hidden even behind the classical conception of mathematics. In particular it seems to regard the possibility in principle of choosing any individual as implicit in the logical concept of object.

The idea of a *choice act* seems to provide an appropriate framework for understanding the notion of arbitrary reference. What is the content of the assumption that an object has been arbitrarily assigned to letter "a"? Since, as a matter of fact, neither the mathematician who makes the assumption nor any other real human being has assigned any object to "*a*", the assumption must concern an *imaginary assignment*. The precise content of such an assumption might seem to be irrelevant, since no mathematical reasoning about *a* needs taking into account the way the supposed act of assigning has been done. Any talk about *a* does exploit, however, the counterfactual *possibility* of such an assignment. Therefore, a careful analysis of what is implicitly assumed in mathematical reasoning must face the problem of explaining how to understand the possibility at issue. How can we imagine that *a* has been fixed? At first glance, one could imagine that every object is capable of being characterized by some property (possibly non-expressible in the formal language), so that it would be fixed by means of a definite description. But then the describing property should meet the condition of being satisfied by a unique object; and this condition involves a quantification over all objects. Now, since, as we saw, the game-theoretical explanation of the meaning of quantifiers rests on the assumption that any object can be chosen, one can recognize that an object can be fixed by means of a characterizing property only under the assumption that it can be chosen. This suggests that arbitrary reference is more primitive than reference by description. We will pursue the idea that arbitrary reference is a sort of *direct reference* based on an imaginary *choice act*.

The ideal agent

Let us imagine that we have *direct access* to an *ideal agent*, who in turn has *direct access* to every object: he can *choose* any object at will (here we are identifying ourselves with the *working* mathematician carrying on the mathematical reasoning). We can explain the locution "Let *a* be an arbitrary object" as follows: we ask the agent to choose an object at his will (without communicating us anything about the chosen object) and call it "*a*". It is clear that the adjective "arbitrary" does not concern the nature of the chosen object, but the freedom of the choice act. Accordingly we will assume the following *choice act principle*:

CAP. *Every object of the universe of discourse is capable of being chosen by the ideal agent.*

Of course, *CAP* faces the problem of providing an account of what the act of choosing a mathematical object consists in. What does it mean to choose an infinite entity such as a real number or a set? It is hard to give a general answer, since any answer depends essentially on how the entities in question are conceived. *CAP* is to be seen as a constraint, which must be taken into account by any conception of mathematical objects. The structuralist development of mathematics has shown that mathematical theories do not determine the specific nature of the entities they are talking about. Therefore, given any mathematical theory, the possibility is left of searching for models built up from objects whose accessibility to the ideal agent is perspicuous. For first-order arithmetic a suitable model is Hilbert's model of numerals thought of as finite strings of strokes. These are, in Charles Parsons' words, quasi-concrete objects, i.e. types of spatio-temporal objects, the accessibility to which requires only a minimal idealization of the agent, needed for dealing with the mathematical infinite. All that the agent is expected to be able to do is to write down any finite (however long) string of strokes. It is worth noticing that, at least from a logical point of view, this idealization is also sufficient for interpreting any mathematical theory. For, as is well-known, it follows from the Löwenheim-Skolem theorem that every consistent theory is interpretable in first-order arithmetic. Although arithmetical models are, in general, far from being the "intended" model of the given theory (say set theory), it is surprising that, as soon as one has accepted the idealization of natural numbers, he can interpret within his framework any mathematical talk. Among other things, arithmetical models assure that, as soon as a theory is consistent, *CAP* is certainly satisfiable. Later we will show how to use *CAP* for justifying the constraint of predicativity for Russell's intensional logic and for defending and developing Boolos' interpretation of second-order logic based on plural quantification. For the moment we want to further explain some general aspects of *CAP*.

Our ideal agent, unlike the Brouwerian idealized mathematician, has no other job than that of performing arbitrary acts of choice. He is not expected to have the capacity of restricting his choices to objects satisfying some required condition. The inferential step from $\exists x F(x)$ to "let *a* be an arbitrary object such that $F(a)$" is justified by referring to a *completely arbitrary* choice of the agent, calling "*a*" the chosen object and *assuming* $F(a)$. Though this assumption may certainly be false, any logical consequence *B* we can draw from it must be true under the hypothesis $\exists x F(x)$ (provided that *a* does not occur in *B*). For, the existential hypothesis and *CAP* assure that the agent *could* have chosen (though unconsciously) an object satisfying *F*; and since we don't know anything about *a*, even in that counterfactual case our reasoning would be correct. And as the truth or falsity of *B* is quite independent of the effective choice of the agent, *B* is true anyway (under the existential assumption).

It may be instructive to compare *CAP* with the celebrated set-theoretical axiom of choice. *CAP* doesn't assert the existence of any mathematical object; it explains the meaning of free variables and justifies their inferential role. In contrast, the axiom of choice states that, given any set α of non empty pairwise disjoint sets, *there is* a set β (call it the *choice set*) sharing a unique element with each member of α. From a logical point of view, no act of choice is involved in the understanding of the content of this axiom (besides that, with which we are concerned, implicit in the general concept of an object). It is a purely existential statement expressible in the language of set theory: as all sets, a choice set exists, in a realistic perspective, quite independently of any human action. On the other hand, the act of choice seems to constitute the intuitive ground for the existence of a choice set. It is usually agreed that the axiom serves the purpose of granting the existence of a choice set even when its elements are not singled out by any propositional function. The existence of such a set seems to be intuitively justified by thinking of its members as *arbitrarily chosen*. This aspect was just the main source of the well-known dogged opposition to the axiom: it was charged of introducing into mathematics indefinable sets (a set being *definable* if it is the extension of a propositional function without parameters). It may be puzzling that, if α is finite, the axiom of choice is not needed for the existence of β, even when it is indefinable. In particular, given any single non empty set α, no axiom of choice is needed in order to guarantee the existence of a subset β of α with a unique member. The reason is to be found in a hidden application of *CAP*, implicit in quantification theory. For, the proof runs as follows: *let a be a member of* α; by the pairing axiom there is a set β whose unique member is *a*. It is clear that this kind of reasoning (formalizable in axiomatic set theory) is correct only under the hidden assumption that any object of the universe of sets is capable of being chosen.

The role of choice in mathematics, contrary to a widespread belief, is far from being restricted to the use of the axiom of choice; it is pervasive of the whole mathematics and logic. The axiom of choice seems to exploit the idea of choice in a more problematic way, since it involves the possibility of a *simultaneous choice* of infinitely many objects. Later we will argue, however, that this possibility is already implicit in the usual notion of a set as constituted by its members (in contrast with the logical notion of a class as extension of a property).

One may object, against the need of *CAP*, that the mathematical language can be understood by *direct extrapolation* from the ordinary talk about concrete objects (for which reference is not problematic), so that, in particular, arbitrary reference to mathematical objects would be immediately intelligible without any need of further explanations. According to this objection, the familiar understanding of a talk about any man, any horse or any pencil would make a talk about any real number immediately meaningful. This opinion seems to be shared by Shapiro. He notes the elusiveness of reference in mathematics, but doesn't seem to find it very problematic:

> Probably the most baffling, and intriguing, semantic notion is that of *reference*. The underlying philosophical issue is sometimes called the " 'fido'-fido problem". How does a term come to denote a particular object? What is the nature of the relationship between a singular term ("Fido") and the object that it denotes (Fido), if it denotes anything? Notice that model theory, by itself, has virtually nothing to say on this issue. In textbook developments of model theory, reference is taken as an unexplicated *primitive*. It is simply *stipulated* that an "interpretation" includes a function from the individual constants to the domain of discourse. This is a mere shell of the reference relation.
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> . . . As far as the model-theoretic scheme goes, it does not matter how this reference is to be accomplished or whether it can be accomplished in accordance with some theory or other. There is nothing problematic in the abstract consideration of models whose domains are beyond all causal contact. As far as model theory goes, reference can be *any* function between the singular terms of the language and the ontology. . . .
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> It is fair to say that when it comes to mathematics and theories of other *abstracta*, realism in ontology often falters over reference (about as much as it falters over epistemology). If we assume that ordinary languages are understood and if we accept the premise that model theory captures the structure of ordinary interpreted languages, then we can do better. There is, of course, no consensus on how reference to ordinary physical objects is accomplished. The theories are legion. I do presume, however, that reference to proverbial medium-sized physical objects is accomplished. . . . Understanding how to use ordinary language involves an understanding, at some level, of reference (however it works). (Shapiro, 1997)

Here Shapiro seems to hold that the comprehension of the notion of reference, acquired from the use of natural language, is sufficient for understanding the reference to the objects of any abstract mathematical structure. Our reply is that the understanding of the general notion of reference, "*however it works*", rests necessarily on the presupposition that *in some way* (though it doesn't matter which) it *must* work. Therefore the question: is there any way of referring to an arbitrary real number? Shapiro is an upholder of the so-called *realism ante rem*, according to which mathematical objects are conceived of as *positions* in *abstract structures*, whose existence is prior to their possible specific instances. But, aside from the difficulty of a non-metaphorical understanding of what such positions are, if they are to be treated as objects of quantification, one cannot avoid the question: what does it mean to single out an arbitrary position?

Arbitrary reference and impredicativity

In order to better enlighten the reasons that may have obscured the importance of *TAR* and *CAP*, it is worthwhile to discuss certain observations by Ramsey and Gödel concerning Russell's ramified type theory. Ramsey criticizes the doctrine of the *Principia Mathematica* according to which every class is defined by a propositional function. He observes that, since it is impossible to list all members of an infinite lass, there is no evidence that, in general, such a class is definable by a propositional function. He continues:

> To this it will be answered that a class can only be given by enumeration of its members, in which case it must be finite, or by giving a propositional function which defines it. So that we cannot be in any way concerned with infinite classes or aggregates, if such there be, which are not defined by propositional functions. But this argument contains a common mistake, for it supposes that, because we cannot consider a thing individually, we can have no concern with it at all. Thus, though an infinite indefinable class cannot be mentioned by itself, it is nevertheless involved in any statement beginning 'All classes' or 'There is a class such that', and if indefinable classes are excluded the meaning of all such statements will be fundamentally altered. (Ramsey, 1925)

Clearly Ramsey doesn't take into account the problem of arbitrary reference, of which, as we saw, Russell was aware. We want to suggest that one of Russell's reason for adopting the logicist notion of class as extension of a propositional function arises from the question: how can one choose an infinite class? Russell's answer was: through the choice of a propositional function. Russell's option seems to be justified by the consideration that propositional functions, because of their *intensional* nature, are, at least in principle, directly accessible to the human mind, whereas sets, understood as entities built up by their members, are not. An alternative option is that of fixing a set through a *simultaneous choice* of its elements. This will be considered later.

Ramsey's argument has been resumed by Gödel in his paper "Russell's mathematical logic". Gödel criticizes Russell's *vicious circle principle*, according to which no totality can contain members definable *only* in terms of the totality itself. Gödel observes that classical mathematics does not respect such a principle; and since classical mathematics can be reconstructed on the basis of *Principia*, this work itself cannot respect that principle either "if 'definable' means 'definable within the system' and no methods of defining outside the system (or outside other systems of classical mathematics) are known except such as involve still more comprehensive totalities than those occurring in the systems". He adds:

> I would consider this rather as a proof that the vicious circle principle is false than that classical mathematics is false, and this is indeed plausible also on its own account. For, first of all one may, on good grounds, deny that reference to a totality necessarily implies reference to all single elements of it or, in other words, that "all" means the same as an infinite logical conjunction. (Gödel, 1944)

Then Gödel observes that, even if "all" were intended as an infinite conjunction, the vicious circle principle would be tenable only within a constructive perspective:

> In this case [i.e. if the entities in question are constructed by us] there must clearly exist a definition (namely the description of the construction) which does not refer to a totality to which the object defined belongs, because the construction of a thing can certainly not be based on a totality of things to which the thing to be constructed itself belongs. If, however, it is a question of objects that exist independently of our constructions, there is nothing in the least absurd in the existence of totalities containing members, which can be described (i.e., uniquely characterized) only by reference to this totality. [ibid.]

In a note he points out that "an object a is said to be described by a propositional function $\phi(x)$ if $\phi(x)$ is true for $x = a$ and no other object".

Certainly by "definable" Russell doesn't mean "definable in the system". For, it is well known that those predicative propositional functions whose existence is granted by the axiom of *reducibility* fail, in general, to be definable in the object language of type theory: the range of the variables include functions indefinable in the object language. But, as we saw, the values of quantified variables must be capable of singular reference. It would therefore be circular to accept as range of quantification a universe with some members definable *only* by quantifying over the universe itself. Gödel's remarks don't take into account the sense of "definition" as ideal singular reference. By contrast, *TAR* supplies a plausible justification of the vicious circle principle, provided that "definable" is interpreted as "capable of singular reference". For, as we saw, quantification over a universe U presupposes the possibility of arbitrary reference to any member of U. So an act of arbitrary reference cannot involve in turn, on pain of vicious circularity, quantification over U. Precisely, we can restate the vicious circle principle as follows:

(*VCP*) *No universe of discourse can contain a member such that the agent can refer to it only by means of quantification over the universe itself.*

VCP leads to the rejection of the *impredicative comprehension principle* of second-order (and higher-order) logic

(*CP*) $\exists F \forall x (F(x) \leftrightarrow A(x))$,

provided second-order entities are understood, à la Russell, as *intensional* entities.

For, let F be the property expressed by the propositional function $A(x)$ (where x is a free individual variable). Because of the intensionality of F, there is no access to it but through its linguistic expression. In other words, a choice of F can be understood only as the thought of the formula $A(x)$ with its intended interpretation. A second-order quantification occurring in $A(x)$ would be therefore a violation of *VCP*.

For instance, take for $A(x)$ the propositional function

(*) $\forall G\ G(x)$.

The property F, whose existence is assured by *CP*, is, according to the intensional interpretation, the property of enjoying every property. There is no way of grasping this property without using a quantification over all properties. But that presupposes, as we saw, the *a priori* possibility of choosing *any* value of G. It follows that the property at issue cannot be a value of second-order variables. So *CAP* supplies an explanation of why one cannot take the universe of all individual properties as the range of second-order variables.

Besides, *VCP* is compatible with Russell's *reducibility axiom*. For, one can imagine that the agent has direct access to a certain universe of *primitive* properties (possibly non expressible in the formal language). If second-order variables are restricted to such properties, *CP* can be accepted, without any circularity, as a *richness* assumption: it says intuitively that the universe of primitive properties is so rich that every *extension* of a second-order propositional function is the extension of some primitive property.

Thus our analysis supplies a new reason for adopting a ramified hierarchy, when dealing with intensional properties and relations.

Plural reference vs. sets

Plural quantification is a reinterpretation of second-order monadic logic, proposed by Boolos (1984), (1985). In Boolos' perspective second-order monadic logic is ontologically innocent: contrary to the most accredited view, it doesn't entail any commitment to classes or to properties but only to individuals. According to Boolos, second-order quantification differs from first-order quantification only in that it refers to individuals *plurally*, while the latter refers to individuals *singularly*.

Boolos' view, though very attractive, is highly controversial. It has met the criticism of several philosophers of mathematics [see (Resnik, 1988) and (Parsons, 1990)]. Quine's old claim that second-order logic is "set theory in disguise" doesn't seem to have lost its advocates.

We want to show how the theory of arbitrary reference can throw new light on the theory of plural quantification.

Let us examine Resnik's criticism of Boolos' proposal.

Boolos argues that Quine's slogan "to be is to be the value of a variable" does not entail that the value of a second-order variable must be a set (or a property) of individuals. The slogan is compatible, Boolos claims, with the *plural interpretation*, according to which the value of such a variable is a *manifold* of individuals. To the purpose, he restates the Tarskian truth definition for second-order logic by modifying the notion of *assignment*.

Precisely, given a domain D of individuals, he defines as an assignment any binary relation R between variables and individuals which correlates a unique individual with every first-order variable, while it is subject to no constraint for second-order variables. So R may correlate a second-order variable with no, one or (possibly infinitely) many individuals. The satisfiability relation is inductively defined as usual, with the following clauses for atomic formulas and second-order existential quantification:

1) R satisfies the atomic formula Fx iff the correlate of x is one of the correlates of F;
2) R satisfies $\exists FA$ iff there is a relation R', differing from R at most for the correlates of F, such that R' satisfies A.
   (The universal quantifier is defined in terms of the existential one).

Truth is then defined as usual in terms of satisfaction. So the *set* of the correlates of F is not involved in the definition of truth.

This makes the notion of plural quantification precise and shows how it yields an alternative semantics for second-order logic. This semantics turns out to be equivalent to the usual one, according to which the values of second-order variables are all sets of individuals. And since the notion of *value of a variable* can be made precise only by the definition of assignment, the proposed reformulation shows that Quine’s slogan does not commit second-order logic to any entities but individuals.

It is clear, however, that Boolos’ device is, in itself, inadequate for the conclusion that plural quantification does not implicitly involve the notion of class. The problem is simply turned into the following: does the new definition of assignment presuppose the notion of set of individuals? The answer is certainly affirmative, of course, if relations are understood set-theoretically. But a relation can in turn be understood in terms of *plural reference* to certain ordered pairs (taking for granted the notion of ordered pair). So the definition of assignment becomes: certain ordered pairs R are an assignment if their first components are variables, their second components are individuals and every first-order variable occurs in exactly one of the R’s. However, the use of plural reference in the metalanguage begs the crucial question, whether plural reference involves surreptitiously the notion of set. Boolos is aware of this difficulty and doesn’t attempt to convince the opponents that plural reference is free of any commitment to sets. He only remarks that who is inclined to see plural reference as a genuine alternative to classes will certainly appreciate the possibility of recovering within his view the Tarskian definition of truth.

Indeed, several authors have raised some doubts about the alleged ontological innocence of plural reference. Resnik observes that the use of plural reference in natural language is ambiguous and that, at least in certain contexts, there is no evidence that it is free of any commitment to classes. The locution “there are some objects such that . . .” sometimes simply means “there is at least an object such that . . .”, so that it is expressible in first-order language. Sometimes, however, it has a meaning which one can hardly make explicit without invoking the notion of class. For instance, the famous Geach-Kaplan’s example “some critics admire only one another” is paraphrased by Boolos as “there are some critics such that each of them admires a critic only if the latter is one of them different from the former”. This proposition, not formalizable in first-order language, seems, according to Resnik, hardly interpretable without resorting to classes. How could we understand “one of them” without referring to a certain class and agreeing that the referent of “one” belongs to it? In general, while, according to Boolos, the use of plural reference in natural language would testify to the ontological innocence of second-order logic, according to Resnik the use of second-order logic for formalizing those plural references non expressible in first-order logic would bring to light certain ontological commitments hidden behind natural language. A similar criticism has been made by Parsons, although he attributes to Boolos the merit of throwing new light on the old notion of manifold:

> Boolos has not, in my view, made a convincing case for the claim that his interpretation of second-order logic is ontologically noncommittal. The great interest of his reading, in my view, is that he breathes new life into the older conception of pluralities or multiplicities. As a source of second-order logical forms, the plural and plural quantification are rightly distinguished from what was so much emphasized by Frege, predication and, more generally, expressions with argument places. In particular, if it is the idea of generalization of predicate places that we appeal to in making sense of second-order logic, then the most natural interpretations will be relative substitutional or by semantic ascent, and these will not license impredicative comprehension, and it is

> hard to see how that will be justified. But if one views examples such as Boolos's as involving 'pluralities', they are more like sets as understood in set theory in that no definition by a predicate is indicated, so that one need not expect them to be definable at all. Thus no obstacle to the acceptance of impredicative comprehension is removed.
>
> An advocate of Boolos' interpretation in an eliminative structuralist setting could grant my claims about ontological commitment, but then take a position analogous to the Fregean: second-order variables indeed have pluralities as their values, but these are not objects. It does not seem to me to have the same intuitive force as Frege's position, since there is no analogue to the regress argument that can be made if one views the reference of a predicate as an object. There will still be, just as with Frege's concepts, the irresistible temptation to talk of pluralities as if they were objects, as we have already noted above. The only gain this interpretation offers over the Fregean is a more convincing motivation of impredicativity. (Parsons, 1990)

Certainly the use of plural reference in natural language doesn't guarantee, in itself, its ontological innocence. Plural reference to individuals often seems nothing but a sloppy reference to a class of individuals. The attempts to paraphrase the language of classes by using locutions of natural language avoiding explicit reference to singular values of second-order variables cannot dispel the doubt that classes are only concealed. We believe, however, that the theory of arbitrary reference can support the claim that the role of classes as referents of second-order variables is inessential.

Parsons observes, in the quoted passage, that, though Boolos' interpretation does not reach the goal of ridding second-order logic of any commitment to classes, nevertheless it gives new evidence that classes can be thought of as pluralities in the set-theoretical sense, in contrast with classes in the logicist sense as extensions of predicates. This interpretation would have, according to Parsons, the merit of justifying the *impredicative* comprehension principle. For, a set as a plurality of individuals exists quite independently of any description of its members and, therefore, describing it by quantifying over all sets by no means yields any circularity.

We want to argue, however, that, in virtue of the doctrine of arbitrary reference, the very same notion of a set as constituted by its members rests on the notion of plural reference, so that the latter turns out to be more fundamental than the former.

Assume that second-order variables range over sets of individuals. According to *CAP*, every such set must be capable of being chosen by the agent. The problem arises how to conceive the act of choosing such a set (taking for granted the accessibility to any individual). Now, all we know about sets is that they are entities determined by their members. Although we regard a set as a single object, we lack any insight about its individuality. Once the logicist notion of a class as extension of a concept has been rejected, one has no longer any intuition of what should keep together the members of a set. This fact has been clearly pointed out by Black in his famous paper "The elusiveness of sets":

> . . . Cantor's formula, stripped to essential, runs quite simply: "A set is an assembly into a whole of (well-defined) objects". Here, the phrase "assembly into a whole" certainly suggests that something is *to be done* to the elements, in order for the "whole" or "the unified thing", which is the set to result. But *what* is to be done, if not merely thinking about, the set? . . . What kind of unification is in point? . . . The truth is that once the elements of a set have been identified, *nothing* need or can be done to produce the corresponding set. (Black, 1971)

But then it seems that there is no other way of access to a set than through its members. So a choice of a set must consist in the choice of its members. Now, the choice of infinitely many individuals may be thought of either as an infinite process of choosing a single individual at a time or as a simultaneous choice of all the individuals in question. The first alternative would allow the choice of only countably many individuals, whose totality would be undetermined (an infinite process of choices being forever *in fieri*). In this perspective a set could be thought of as a well-determined entity only by identifying it *intensionally* with the process itself of choosing its members. But the introduction of entities with an undetermined extension would be highly problematic (as it is the case for intuitionistic *lawless sequences*) and incompatible with the extensional conception of a set. So we are led to the second alternative of the *simultaneous* choice. The idea is expressed by Bernays in the already mentioned essay:

> [Platonism] abstracts from the possibility of giving definitions of sets, sequences, and functions. These notions are used in a "quasi-combinatorial" sense, by which I mean: in the sense of analogy of the infinite to the finite . . . we imagine functions engendered by an infinity of independent determinations which assign to each integer an integer, and we reason about the totality of these functions.
>
> In the same way, one views a set of integers as the result of infinitely many independent acts deciding for each number whether it should be included or excluded. (Bernays, 1935)

Now, the simultaneous choice of certain individuals is precisely what serves the purpose of *plurally* referring to such individuals. It follows that the arbitrary reference implicit in second-order quantification involves the same choice acts, whether second-order variables range singularly over sets or plurally over individuals.

In the plural interpretation the locution "let A be arbitrary individuals" means "choose at will some individuals simultaneously and call them 'A' ". In the set-theoretical interpretation the locution "let A be an arbitrary set of individuals" means "choose at will some individuals simultaneously and call 'A' their set". At this point it is plain that sets are inessential. The alleged role of sets of collecting individuals turns out to be illusory: what selects the members of a set is not the set itself but the *act* of choosing them simultaneously. Thus the arbitrary reference to certain individuals by no means presupposes the existence of their set; it merely presupposes the act of choosing them simultaneously. So the *ontological* innocence of plural reference is vindicated.

We can conclude that the plural interpretation of second-order logic is less ontologically committal than the set-theoretical one: both involve the same acts of choice, but the plural interpretation does not involve any second-order entities (acts, unlike sets, being no entities). The doubts raised by Resnik and Parsons are therefore superseded.

This conclusion does not entail, however, that second-order logic, in the plural interpretation, is no more problematic than first-order logic. Certainly it is, but not for ontological reasons. What is more problematic is the conception of *simultaneous* choice of (possibly infinitely) many individuals, compared with that of choice of a *single* individual involved in first-order logic. The question arises: given a suitable idealization of *singular* choice (depending on the nature of the individuals we are dealing with), how can we idealize a *simultaneous* choice? Though he doesn't explicitly talk about choices, Black suggests that one can easily conceive the act of indicating several things at once:

> The notion of "plural" or simultaneous reference to several things at once is really not at all mysterious. Just as I can point to a single thing, I can point to two things at once, using two hands, if necessary; pointing to two things at once need be no more perplexing than touching two things at once. Of course it would be a mistake to think that the rules for "multiple pointing" follow automatically from the rules for pointing proper; but the requisite conventions are almost too obvious to need specification. The rules for "plural reference" are no harder to elaborate. (Black, 1971)

Let's try to propose a suitable ideal picture of a *simultaneous choice*. Imagine that, instead of a unique agent, infinitely many agents are available. More precisely, imagine a *leader* agent at the head of a team of subagents, one for every individual of the universe of discourse. When the leader orders "choose!", each subagent shows *ad libitum* one of the signs 0, 1, say by lifting a shovel with the signs printed each in one of its faces. Relative to a simultaneous choice, an individual is *designated* if the corresponding agent shows 1. So a simultaneous choice plays the role of the characteristic function of the set of the designated individuals. In contrast, a *singular choice* simply consists in a choice of a single individual by the leader. Again such individual is said to be *designated* by the singular choice.

Now, if one accepts the ontology of sets of individuals, then he can regard a simultaneous choice as a device for arbitrarily referring to sets. But what is important is that, once this device has been introduced, sets, understood as genuine entities, become quite inessential for interpreting second-order monadic logic. In fact, second-order truth can be directly defined in terms of choices as follows.

Let ϕ be a second-order monadic formula whose free first-order variables are among $x_1, \ldots, x_m$ and free second-order variables among $X_1, \ldots, X_n$. Consider for each variable x_i a singular choice x_i^* ($i = 1, \ldots, m$) and for each variable X_j a simultaneous choice X_j^* ($j = 1, \ldots, n$). We will inductively define the truth value of ϕ relative to the choices $x_1^*, \ldots, x_m^*; X_1^*, \ldots, X_n^*$. We will state only the clauses for atomic formulas and for second-order quantifiers, the others being as usual:

1) if $\phi \equiv X_j\, x_i$, it is true if the individual designated by choice x_i^* is designated by choice X_j^*;
2) if $\phi \equiv \forall Y\psi$, it is true if, however a plural choice Y^* is performed, ψ is true relative to choices $x_1^*, \ldots, x_m^*, X_1^*, \ldots, X_n^*, Y^*$;
3) if $\phi \equiv \exists Y\psi$, it is true if it is possible to perform a plural choice Y^* in such a way that ψ turns out to be true relative to choices $x_1^*, \ldots, x_m^*, X_1^*, \ldots, X_n^*, Y^*$.

Observe that, while Boolos' truth definition explains plural quantification in the object language by assuming plural quantification in the metalanguage, the present approach explains singular and plural quantification in the object language assuming the notion of quantification over choices in the metalanguage. Such explanation avoids any circularity and its importance rests on the fact that choices are not objects but *acts*. Any talk which reifies acts treating them as objects is to be paraphrasable, in principle, so to avoid any reification. In particular, quantification over acts is to be understood in a purely potential sense. Clause 2) does not quantify over a mysterious realm of all acts of choice; what it requires for the truth of ϕ is that, if the leader orders a simultaneous choice relative to variable Y, then, *independently* of what each subagent chooses, ψ turns out to be true. Besides, such independence is to be thought of as an *objective fact*, which obtains or not quite independently of the knowledge, even on the part of the leader, of which is the case. Similarly for the *possibility* involved at clause 3): the possibility that the subagents make their choices so as to verify ψ is an objective fact which obtains or not quite independently of their knowledge. This guarantees the validity of classical logic. Accordingly, one could take only one of the two quantifiers as primitive and define the other in terms of that. The truth of $\forall Y\psi$ can be understood as the *impossibility* of a simultaneous choice falsifying ψ; the truth of $\exists Y\psi$ as denying that ψ turns out to be false, *independently* of what the subagents choose. We prefer, however, to take both quantifiers as primitive, since none of them seems more elementary than the other; we believe that each of them can help to clarify the other.

So far our semantics has been concerned with *monadic* second-order logic. Boolos' treatment extends plural quantification to *full* second-order logic by taking the notion of ordered pair as primitive. Lewis has proposed a codification of a pairing function by combining plural reference with mereology. Within our framework, even ordered pairs can be introduced by means of simultaneous choices, as follows.

Call a *binary choice* an act consisting in the choice by every subagent of two (not necessarily distinct) individuals in a certain order. A binary choice is a *pairing choice* if, for all individuals x, y, a unique subagent chooses them orderwise. We will assume the *possibility* of a pairing choice (the notion of possibility being explained as above) and we will speak of (*ordered*) *pairs* understanding the reference to such a choice. In this way our semantics extends to full (poliadic) second-order logic.

As we have already observed, our notion of choice can be viewed as an extension to the plural case of that used by Hintikka in his game-theoretical semantics. In fact, you could further stress the analogy with Hintikka's semantics by reformulating our semantics game-theoretically. For, with every sentence of second-order logic you can associate a game played by two teams, the team of *verifiers* and that of *falsifiers*. Each team consists of a leader and of one player for every individual. A move of a team consists of a singular choice by the leader or of a simultaneous choice by his players. The game rules are then defined as in Hintikka's theory, the moves relative to second-order quantifiers being simultaneous choices. The truth of a sentence is defined as the existence of a *winning strategy* for the team of the verifiers. The problem of the ontological commitment to sets shifts to that to strategies. Of course, if these are understood as set-theoretical functions, no step forward has been taken. But a winning strategy can be understood, without any reification, in terms of the notions of possibility and independence explained above. To say that there is a winning strategy for a team means that this team *can* win *any* play, quite *independently* of the moves of the opposing team.

To prefer the formulation à la Tarski or that à la Hintikka is a matter of taste. Both exploit the same primitives: *choice acts*, *possibility* and *independence*.

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