Chapter 3: Order Notation

$\Theta, O, \Omega, o, \omega, \lfloor, \rfloor, \lceil, \rceil, \log, \lg, \ln, !, \sum, \prod$

Asymptotic Behavior

- Let $f(n)$ and $g(n)$ be nonnegative functions when $n \geq 0$.
- Define $f(n) \prec g(n)$
  (read "$f(n)$ is asymptotically less than $g(n)$")
  if there exists a positive constant k such that
  $f(n) < g(n)$ when $n \geq k$.
  (This definition is not in the book.)
- Examples:
  $2n + 100 \prec 3n - 100$ (consider $n \geq 200$)
  $1000n^2 \prec 2^n/1000$ (consider $n \geq 30$)
- $\Theta(g(n))$ includes all functions bounded by constant factors of $g(n)$.

Order Notation

- Theta-notation:
  $f(n) \in \Theta(g(n))$ if $c_1\, g(n) \prec f(n) \prec c_2\, g(n)$
  for some positive constants c_1 and c_2.
- Big-Oh-notation:
  $f(n) \in O(g(n))$ if $f(n) \prec c\, g(n)$
  for some positive constant c.
- Big-Omega-notation:
  $f(n) \in \Omega(g(n))$ if $c\, g(n) \prec f(n)$
  for some positive constant c.
- little-oh-notation:
  $f(n) \in o(g(n))$ if $f(n) \prec c\, g(n)$
  for all positive constants c.

Order Notation Continued

- little-omega-notation:
  $f(n) \in \omega(g(n))$ if $c\,g(n) \prec f(n)$
  for all positive constants c.

If $f(n)$ is …	is $f(n)$ also …? $o(g(n))$	$O(g(n))$	$\Theta(g(n))$	$\Omega(g(n))$	$\omega(g(n))$
$o(g(n))$					
$O(g(n))$					
$\Theta(g(n))$					
$\Omega(g(n))$					
$\omega(g(n))$					

More Notation

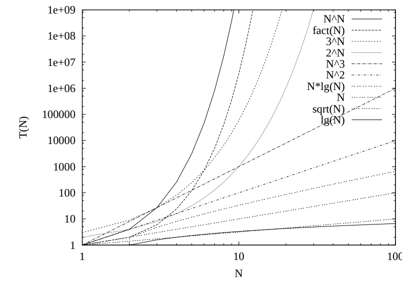
$f(n)$ is …	if …
monotonically increasing	$m \le n \rightarrow f(m) \le f(n)$
monotonically decreasing	$m \le n \rightarrow f(m) \ge f(n)$
strictly increasing	$m < n \rightarrow f(m) < f(n)$
strictly decreasing	$m < n \rightarrow f(m) > f(n)$

- $\lfloor x \rfloor$ is the floor of x. $\lceil x \rceil$ is the ceiling of x.
- $\sum_{i=0}^{d} a_i\, n^i$ is polynomial in n of degree d.
- a^n is exponential in n.
  $\log_a n$ is logarithmic in n.

More Notation Continued

- Also, $\log_a n = \dfrac{\log_b n}{\log_b a}$
  $\lg n = \log_2 n$
  $\ln n = \log_e n$ where $e = 2.71828\cdots$
- $n!$ is n factorial.
- Let $0 < a < 1 < b < c$.
  $\Theta(1) \prec \Theta(\lg n) \prec \Theta(n^a) \prec \Theta(n) \prec \Theta(n \lg n) \prec \Theta(n^b) \prec \Theta(n^c) \prec$
  $\Theta(b^n) \prec \Theta(c^n) \prec \Theta(n!) \prec \Theta(n^n)$

Log-Log Plot of Some Functions



Summation Notation

- $\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$
- $\sum_{i=1}^{n} c\, a_i = c \sum_{i=1}^{n} a_i$
- $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$
- If $a_1, a_2, \ldots, a_n$ is increasing, then:
  $\sum_{i=1}^{n} a_i$ is $O(n\, a_n)$
  $\sum_{i=1}^{n} a_i$ is $\Omega\left(n\, a_{n/2}\right)$

Some Miscellaneous Stuff

- $\prod_{i=1}^{n} a_i = a_1\, a_2 \cdots a_n$
- $\log_b \left(\prod_{i=1}^{n} a_i \right) = \sum_{i=1}^{n} (\log_b a_i)$
- Sets, Relations, Functions, Graphs, Trees
  See Appendix B