APPENDIX: PROPOSITIONS AND PROOFS

Recall that the two-dimensional mapping of a N-dimensional data point is defined by real and imaginary components of:

$$\mathcal{F}_1(\mathbf{x}[n]) = \sum_{n=0}^{N-1} x[n]\mathbf{W}_N^n = \sum_{n=0}^{N-1} x[n]e^{-i2\pi n/N}. \tag{1}$$

where $\mathbf{W}_N = e^{-i2\pi/N}$ is called twiddle factor, $2\pi/N$ is the base frequency.

Lemma 1 (Conjugate) *For any two complex numbers z and w, (1) $\overline{z+w}$ = $\overline{z}$ + $\overline{w}$, (2) $\overline{z\,w}$ = $\overline{z}\,\overline{w}$, (3) $\overline{\overline{z}} = z$.*

Lemma 2 (Square Expanding)

$$\left(\sum_{n=0}^{N-1} a_n\right)^2 = \sum_{n=0}^{N-1} a_n^2 + 2\sum_{k=1}^{N-1}\sum_{t=0}^{N-k-1} a_t\, a_{t+k}.$$

Lemma 3 (Cancellation) *Let $j \in \mathbb{N}$, then $\sum_{n=0}^{N-1} e^{-i2\pi jn/N} = \sum_{n=0}^{N-1} \cos(2\pi jn/N) = \sum_{n=0}^{N-1} \sin(2\pi jn/N) = \mathbf{0}$.*

Proof:

$$\sum_{n=0}^{N-1} e^{-i2\pi jn/N} = \frac{e^{-i2\pi jN/N}-1}{e^{-i2\pi/N}-1} = \frac{1-1}{e^{-i2\pi/N}-1} = \mathbf{0}.$$

then apply $e^{i\theta} = \cos\theta + i\sin\theta$, we get

$$\sum_{n=0}^{N-1} e^{-i2\pi jn/N} = \sum_{n=0}^{N-1} \cos(2\pi jn/N) - i\sum_{n=0}^{N-1} \sin(2\pi jn/N) = 0 + 0i.$$

Lemma 4 (Homomorphism) *FFHP is homomorphic. $\mathcal{F}_1(a\,\mathbf{x}[n] + b\,\mathbf{y}[n]) = a\,\mathcal{F}_1(\mathbf{x}[n]) + b\,\mathcal{F}_1(\mathbf{y}[n])$.*

From the formula in Eq. (1), we get

$$\begin{aligned}\mathcal{F}_1(a\,\mathbf{x}[n] + b\,\mathbf{y}[n]) &= \sum_{n=0}^{N-1}(a\,x[n] + b\,y[n])e^{-i2\pi n/N} = a\sum_{n=0}^{N-1} x[n]e^{-i2\pi n/N} + b\sum_{n=0}^{N-1} y[n]e^{-i2\pi n/N}\\ &= a\,\mathcal{F}_1(\mathbf{x}[n]) + b\,\mathcal{F}_1(\mathbf{y}[n]).\end{aligned}$$

□

Proposition 1 (Cancellation) *An N -dimensional point with equal dimension values will be mapped onto the origin. If $\mathbf{x}[n] = [a,\ldots,a]$, then $\mathcal{F}_1(\mathbf{x}[n]) = \mathbf{0}$.*

Proof: From the formula in Eq. (1), we get

$$\mathcal{F}_1(\mathbf{x}[n]) = \sum_{n=0}^{N-1} a\,e^{-i2\pi n/N} = a\sum_{n=0}^{N-1} e^{-i2\pi n/N}.$$

By Lemma 3, let $j = 1$. $\mathcal{F}_1(\mathbf{x}[n]) = \mathbf{0}$. □

Proposition 2 (Addition by a Constant) *Two N-dimensional points with addition of a constant to each dimension value will be mapped onto the same point. If $\mathbf{y}[n] = \mathbf{x}[n] + a$, then $\mathcal{F}_1(\mathbf{y}[n]) = \mathcal{F}_1(\mathbf{x}[n])$.*

Proof: From the formula in Eq. (1), we get

$$\begin{aligned}\mathcal{F}_1(\mathbf{y}[n]) &= \sum_{n=0}^{N-1}(x[n]+a)e^{-i2\pi n/N} = \sum_{n=0}^{N-1}(x[n]\,e^{-i2\pi n/N} + a\,e^{-i2\pi n/N})\\ &= \sum_{n=0}^{N-1}x[n]\,e^{-i2\pi n/N} + a\sum_{n=0}^{N-1}e^{-i2\pi n/N} = \mathcal{F}_1(\mathbf{x}[n]) + a\sum_{n=0}^{N-1}e^{-i2\pi n/N}\end{aligned}$$

By Proposition 1, the second summation is $\mathbf{0}$. □

Proposition 3 (Scaling by a Constant) *Two N-dimensional points with scaling of a constant to each dimension value will be mapped onto two points on a line through the origin. If $\mathbf{y}[n] = a\,\mathbf{x}[n]$, then $\mathcal{F}_1(\mathbf{y}[n]) = a\,\mathcal{F}_1(\mathbf{x}[n])$.*

Proof: From the formula in Eq. (1), we get

$$\mathcal{F}_1(\mathbf{y}[n]) = \sum_{n=0}^{N-1}a\,x[n]e^{-i2\pi n/N} = a\sum_{n=0}^{N-1}x[n]e^{-i2\pi n/N} = a\,\mathcal{F}_1(\mathbf{x}[n])$$

□

Proposition 4 (Time Shifting) *Two N-dimensional points with constant time shifting will be mapped onto a circle concentric to the unit circle. The angle between two images is $d2\pi/N$. If $\mathbf{y}[n] = \mathbf{x}[n-d]$, then $\mathcal{F}_1(\mathbf{y}[n]) = \mathcal{F}_1(\mathbf{x}[n])\mathbf{W}_N^d$.*

Proof: Assume $0 \le n < N$, let $l = n - d$, then $n = l + d$. When $n = 0$, $l = -d$ and when $n = N-1$, $l = N-1-d$. From the formula in Eq. (1), we get

$$\begin{aligned}\mathcal{F}_1(\mathbf{y}[n]) &= \mathcal{F}_1(\mathbf{x}[n-d]) = \sum_{l=-d}^{N-1-d}x[l]e^{-i2\pi(l+d)/N}\\ &= \sum_{l=-d}^{N-1-d}x[l]e^{-i2\pi l/N}e^{-i2\pi d/N} = \mathbf{W}_N^d\sum_{l=-d}^{N-1-d}x[l]e^{-i2\pi l/N}\end{aligned}$$

However, $e^{i2\pi n/N} = e^{i2\pi(n+N)/N}$ and $x[n] = x[n+N]$,

$$\sum_{l=-d}^{N-1-d}x[l]e^{-i2\pi l/N} = \sum_{l=-d}^{-1}x[l+N]\,e^{-i2\pi(l+N)/N} + \sum_{l=0}^{N-1-d}x[l]\,e^{-i2\pi l/N}$$

Let $t = l + N$ for the first summation and $t = l$ for the second summation, we get

$$\sum_{l=-d}^{N-1-d}x[l]e^{-i2\pi l/N} = \sum_{t=N-d}^{N-1}x[t]e^{-i2\pi t/N} + \sum_{t=0}^{N-1-d}x[t]e^{-i2\pi t/N} = \sum_{t=0}^{N-1}x[t]e^{-i2\pi t/N} = \mathcal{F}_1(\mathbf{x}[n])$$

Therefore, $\mathcal{F}_1(\mathbf{y}[n]) = \mathcal{F}_1(\mathbf{x}[n])\mathbf{W}_N^d$. □

Proposition 5 (Line) *Under FFHP, an N-dimensional line will be mapped onto a two dimensional.*

Proof: A N-dimensional line l through point P and parallel to a N-vector Δ can be expressed as $P + t\Delta$ where $-\infty \leq t \leq \infty$. Let Q and R are two different points on l, then $Q = P + \alpha\Delta$ and $R = P + \beta\Delta$, for some $\alpha, \beta \in \mathbb{R}$. Let the corresponding signals for P, Q, R, and Δ be $\mathbf{p}[n]$, $\mathbf{q}[n]$, $\mathbf{r}[n]$, and $\delta[n]$. By Lemma 4, $\mathcal{F}_1(\mathbf{q}[n]) = \mathcal{F}_1(\mathbf{p}[n] + \alpha\delta[n])) = \mathcal{F}_1(\mathbf{p}[n]) + \alpha\mathcal{F}_1(\delta[n])$ and $\mathcal{F}_1(\mathbf{r}[n]) = \mathcal{F}_1(\mathbf{p}[n]) + \beta\mathcal{F}_1(\delta[n])$. Compare the definition of a line above, $\mathcal{F}_1(\mathbf{q}[n])$ and $\mathcal{F}_1(\mathbf{r}[n])$ are two points on a two dimensional line through $\mathcal{F}_1(\mathbf{p}[n])$ and parallel to the vector $\mathcal{F}_1(\delta[n])$. □

Definition 1 (Mean, Autocovariance, Variance, Autocorrelation Coefficient) *The mean of a signal $\mathbf{x}[n]$ is defined as $\widehat{x} = \sum_{n=0}^{N-1} x[n]/N$. The k-th sample autocovariance coefficient of a signal $\mathbf{x}[n]$ is defined as $g_k = \sum_{n=0}^{N-1-k} (x[n] - \widehat{x})(x[n+k] - \widehat{x})/N$. g_0 is called the variance of $\mathbf{x}[n]$. The k-th sample autocorrelation coefficient is defined as $r_k = g_k/g_0$.*

Proposition 6 (Fundamental Distance) *Let $\mathbf{w}[n] = \mathbf{x}[n] - \mathbf{y}[n]$, be the difference between $\mathbf{x}[n]$ and $\mathbf{y}[n]$. The distance between $\mathcal{F}_1(\mathbf{x}[n])$ and $\mathcal{F}_1(\mathbf{y}[n])$ is*

$$\|\mathcal{F}_1(\mathbf{w}[n])\|^2 = g_0 N \left(1 + 2\sum_{k=1}^{N-1} r_k \cos(2\pi k/N)\right).$$

Proof: By Lemma 4, the distance between $\mathcal{F}_1(\mathbf{y}[n])$ and $\mathcal{F}_1(\mathbf{x}[n])$ is $\|\mathcal{F}_1(\mathbf{w}[n])\|$. From Eq. (1), we get

$$\|\mathcal{F}_1(\mathbf{w}[n])\| = \|\sum_{n=0}^{N-1} w[n]e^{-i2\pi n/N}\| = \|\sum_{n=0}^{N-1} w[n]\cos(2\pi n/N) - iw[n]\sin(2\pi n/N)\|$$

Let $\omega = 2\pi/N$, by Lemma 3, we have $\sum_{n=0}^{N-1}\cos(n\omega) = \sum_{n=0}^{N-1}\sin(n\omega) = 0$. Now add a term $\widehat{w}$, the mean of $\mathbf{w}[n]$,

$$\begin{aligned}\|\mathcal{F}_1(\mathbf{w}[n])\|^2 &= \left(\sum_{n=0}^{N-1} w[n]\cos(n\omega)\right)^2 + \left(\sum_{n=0}^{N-1} w[n]\sin(n\omega)\right)^2 \\ &= \left(\sum_{n=0}^{N-1} (w[n] - \widehat{w})\cos(n\omega)\right)^2 + \left(\sum_{n=0}^{N-1} (w[n] - \widehat{w})\sin(n\omega)\right)^2\end{aligned}$$

Expending each squaring terms by Lemma 2, we get

$$\sum_{n=0}^{N-1} (w[n] - \widehat{w})^2(\cos^2(n\omega) + \sin^2(n\omega)) + 2\sum_{k=1}^{N-1}\sum_{t=0}^{N-1-k} [(w[t] - \widehat{w})(w[t+k] - \widehat{w})\Delta]$$

where $\Delta = \cos(t\omega)\cos((t+k)\omega) + \sin(t\omega)\sin((t+k)\omega)$. By trigonometry identity $\cos\theta\cos\phi + \sin\theta\sin\phi = \cos(\phi - \theta)$, $\Delta = \cos(k\omega)$. Now we have

$$\begin{aligned}\|\mathcal{F}_1(\mathbf{w}[n])\|^2 &= \sum_{n=0}^{N-1} (w[n] - \widehat{w})^2 + 2\sum_{k=1}^{N-1}\sum_{t=0}^{N-1-k} [(w[t] - \widehat{w})(w[t+k] - \widehat{w})\cos(k\omega)] \\ &= N(g_0 + 2\sum_{k=1}^{N-1} g_k\cos(k\omega)) = g_0 N \left(1 + 2\sum_{k=1}^{N-1} r_k\cos(2\pi k/N)\right)\end{aligned}$$

□

Definition 2 (Twiddle Power Index) *For an N-point signal, the k-th harmonic twiddle power index (HTPI in short) is a sequence of N time indices chosen from $0, \ldots, N-1$. It corresponds to the order that a particular time point being mapped on to the consecutive powers of twiddle factor $\mathbf{W}_N^0$,. . . , $\mathbf{W}_N^{N-1}$, by the k-th harmonic.*

Example 1 *For a 5-point signal, the first harmonic twiddle power index is $[0, 1, 2, 3, 4]$. The second HTPI is $[0, 3, 1, 4, 2]$. The third HTPI is $[0, 2, 4, 1, 3]$. Take a closer look at the second HTPI. Since $\mathcal{F}_2(\mathbf{x}[n]) = \sum_{n=0}^{N-1} x[n]\mathbf{W}_N^{2k}$, we have $\mathcal{F}_2(\mathbf{x}[n]) = x[0]\mathbf{W}_5^0 + x[1]\mathbf{W}_5^2 + x[2]\mathbf{W}_5^4 + x[3]\mathbf{W}_5^1 + x[4]\mathbf{W}_5^3 = x[0]\mathbf{W}_5^0 + x[3]\mathbf{W}_5^1 + x[1]\mathbf{W}_5^2 + x[4]\mathbf{W}_5^3 + x[2]\mathbf{W}_5^4$.*

Proposition 7 (Harmonic Equivalence) *A k-th harmonic of a signal ($k > 1$) is equivalent to the first harmonic of the the origin signal being reordered by the k-th harmonic twiddle power index.*

Proof: By definition of harmonic and twiddle power index. □