Approximating Submodular Functions Everywhere

Nick Harvey

University of Waterloo
Department of Combinatorics & Optimization

October 4th, 2009

Joint work with M. Goemans, S. Iwata and V. Mirrokni

Motivating Example

- Microsoft Office consists of a set A of products.
  e.g., $A = \{\text{Word}, \text{Excel}, \text{Outlook}, \text{PowerPoint}, \ldots\}$.
- The typical consumer has a utility function $f : 2^A \to \mathbb{R}$
- Want to learn f without asking consumer too many questions
  (Perhaps useful in pricing different bundles of Office?)

Motivating Example

- Microsoft Office consists of a set A of products.
  e.g., $A = \{\text{Word}, \text{Excel}, \text{Outlook}, \text{PowerPoint}, ...\}$.
- The typical consumer has a utility function $f : 2^A \rightarrow \mathbb{R}$
- Want to learn f without asking consumer too many questions
  (Perhaps useful in pricing different bundles of Office?)
- Some assumption on f is required
  Submodularity is natural assumption

Submodular Functions

- Definition
  $f : 2^{[n]} \to \mathbb{R}$ is submodular if, for all $A, B \subseteq [n]$:
  $$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$
  Equivalent definition
  f is submodular if, for all $A \subseteq B$ and $i \notin B$:
  $$f(A \cup \{i\}) - f(A) \geq f(B \cup \{i\}) - f(B)$$
  - Discrete analogue of convex functions [Lovász '83]
  - Arise in combinatorial optimization, probability, economics (diminishing returns), geometry, etc.
- Fundamental Examples
  Rank function of a matroid, cut function of a graph, ...

Optimizing Submodular Functions
(Given Oracle Access)

Minimization

- Can solve $\min_S f(S)$ with polynomially many oracle calls [GLS], [Schrijver '01], [Iwata, Fleischer, Fujishige '01], ...

  Example: Given matroids $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$

$$\max\{|I| : I \in \mathcal{I}_1 \cap \mathcal{I}_2\} = \min\{r_1(S) + r_2(E \setminus S) : S \subseteq E\}$$

Maximization

- Can approximate $\max_S f(S)$ to within $2/5$, assuming $f \geq 0$. [Feige, Mirrokni, Vondrák '07]

Approximating Submodular Functions Everywhere

Definition

$f : 2^{[n]} \to \mathbb{R}$ is monotone if, for all $A \subseteq B \subseteq [n]$:

$$f(A) \leq f(B)$$

Problem

Given a monotone, submodular f, construct using $\mathrm{poly}(n)$ *oracle queries a function $\hat{f}$ such that:*

$$\hat{f}(S) \leq f(S) \leq \alpha(n) \cdot \hat{f}(S) \qquad \forall\, S \subseteq [n]$$

Definition

$f : 2^{[n]} \to \mathbb{R}$ is monotone if, for all $A \subseteq B \subseteq [n]$:

$$f(A) \leq f(B)$$

Problem

Given a monotone, submodular f, construct using poly(n) *oracle queries a function $\hat{f}$ such that:*

$$\hat{f}(S) \leq f(S) \leq \alpha(n) \cdot \hat{f}(S) \qquad \forall\, S \subseteq [n]$$

Approximation Quality

- How small can we make $\alpha(n)$?
- $\alpha(n) = n$ is trivial

Problem

Given a monotone, submodular f, construct using poly(n) *oracle queries a function $\hat{f}$ such that:*

$$\hat{f}(S) \leq f(S) \leq \alpha(n) \cdot \hat{f}(S) \qquad \forall\, S \subseteq [n]$$

Our Positive Result

A deterministic algorithm that constructs $\hat{f}(S) = \sqrt{\sum_{i \in S} c_i}$ with

- $\alpha(n) = \sqrt{n+1}$ for matroid rank functions f, or
- $\alpha(n) = O(\sqrt{n} \log n)$ for general monotone submodular f

Also, $\hat{f}$ is submodular: $\hat{f}(S) = \sqrt{\sum_{i \in S} c_i}$ for some scalars c_i.

Our Positive Result

A deterministic algorithm that constructs $\hat{f}(S) = \sqrt{\sum_{i \in S} c_i}$ with

- $\alpha(n) = \sqrt{n+1}$ for matroid rank functions f, or
- $\alpha(n) = O(\sqrt{n} \log n)$ for general monotone submodular f

Our Negative Result

With polynomially many oracle calls, $\alpha(n) = \Omega(\sqrt{n} / \log n)$ (even for randomized algs)

Problem (Svitkina and Fleischer '08)

Given submodular functions $f_i : 2^V \to \mathbb{R}$ *for* $i \in [k]$, *partition* V *into* $V_1, \cdots, V_k$ *to*

$$\min_{V_1,\ldots,V_k} \max_i f_i(V_i)$$

For $f_i(S) = \sum_{j \in S} c_{i,j}$, this is scheduling on unrelated machines. [Lenstra, Shmoys, Tardos '90]

Problem (Svitkina and Fleischer '08)

Given submodular functions $f_i : 2^V \to \mathbb{R}$ *for* $i \in [k]$*,*
partition V *into* $V_1, \cdots, V_k$ *to*

$$\min_{V_1,\ldots,V_k} \max_i \; f_i(V_i)$$

For $f_i(S) = \sum_{j \in S} c_{i,j}$, this is scheduling on unrelated machines. [Lenstra, Shmoys, Tardos '90]

Our solution

Approximate f_i by $\hat{f}_i(S) = \sqrt{\sum_{j \in S} c_{i,j}}$ for each i. Then solve

$$\min_{V_1,\ldots,V_k} \max_i \; \hat{f}_i^{\,2}(V_i)$$

using Lenstra, Shmoys, Tardos. Get $O(\sqrt{n} \log n)$-approx solution.

Problem (Golovin '05, Khot and Ponnuswami '07)

Given submodular functions $f_i : 2^V \to \mathbb{R}$ *for* $i \in [k]$*,*
partition V *into* $V_1, \cdots, V_k$ *to*

$$\max_{V_1,\ldots,V_k} \min_i \; f_i(V_i)$$

For $f_i(S) = \sum_{j \in S} c_{i,j}$, this is Santa Claus problem.
There is a $\tilde{O}(\sqrt{k})$-approximation algorithm [Asadpour-Saberi '07].

Immediately get $\tilde{O}(\sqrt{n}\, k^{1/4})$-approximate solution.

Polymatroid

Definition

Given submodular f, polymatroid

$$P_f = \left\{ x \in \mathbb{R}^n_+ : \sum_{i \in S} x_i \leq f(S) \text{ for all } S \subseteq [n] \right\}$$

A few properties [Edmonds '70]:

- Can optimize over P_f with greedy algorithm
- Separation problem for P_f is submodular fctn minimization
- For monotone f, can reconstruct f:

$$f(S) = \max_{x \in P_f} \langle 1_S, x \rangle$$

Our Approach: Geometric Relaxation

We know:

$$f(S) = \max_{x \in P_f} \langle 1_S, x \rangle$$

Suppose that:

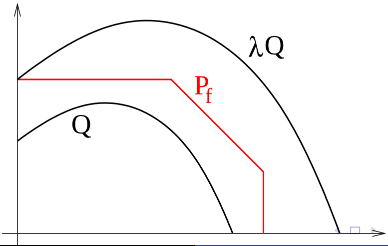
$$Q \subseteq P_f \subseteq \lambda Q$$

Then:

$$\hat{f}(S) \leq f(S) \leq \lambda \hat{f}(S)$$

where

$$\hat{f}(S) = \max_{x \in Q} \langle 1_S, x \rangle$$



John's Theorem [1948]

Maximum Volume Ellipsoids

Definition

A convex body K is centrally symmetric if $x \in K \iff -x \in K$.

John's Theorem [1948]

Maximum Volume Ellipsoids

Definition

A convex body K is centrally symmetric if
$x \in K \iff -x \in K$.

Definition

An ellipsoid E is an α-ellipsoidal approximation of K if
$E \subseteq K \subseteq \alpha \cdot E$.

John's Theorem [1948]

Maximum Volume Ellipsoids

Definition

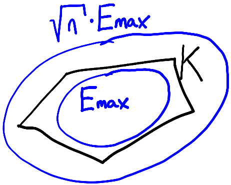
A convex body K is centrally symmetric if
$x \in K \iff -x \in K$.

Definition

An ellipsoid E is an α-ellipsoidal approximation of K if
$E \subseteq K \subseteq \alpha \cdot E$.

Theorem

Let K be a centrally symmetric convex body in $\mathbb{R}^n$. Let E_{max} (or John ellipsoid) be maximum volume ellipsoid contained in K. Then $K \subseteq \sqrt{n} \cdot E_{max}$.



John's Theorem [1948]

Maximum Volume Ellipsoids

Definition

A convex body K is centrally symmetric if
$x \in K \iff -x \in K$.

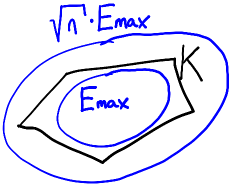
Definition

An ellipsoid E is an α-ellipsoidal approximation of K if
$E \subseteq K \subseteq \alpha \cdot E$.

Theorem

Let K be a centrally symmetric convex body in $\mathbb{R}^n$. Let E_{max} (or John ellipsoid) be maximum volume ellipsoid contained in K. Then $K \subseteq \sqrt{n} \cdot E_{max}$.

Algorithmically?



Ellipsoids Basics

Definition

- An ellipsoid is

$$E(A) = \{x \in \mathbb{R}^n : x^T A x \leq 1\}$$

where $A \succ 0$ is positive definite matrix.

Handy notation

- Write $\|x\|_A = \sqrt{x^T A x}$. Then

$$E(A) = \{x \in \mathbb{R}^n : \|x\|_A \leq 1\}$$

Optimizing over ellipsoids

- $\max_{x \in E(A)} \langle c, x \rangle = \|c\|_{A^{-1}}$

Algorithms for Ellipsoidal Approximations

Explicitly Given Polytopes

- Can find E_{max} in P-time (up to ϵ) if explicitly given as $K = \{x : Ax \leq b\}$
  [Grötschel, Lovász and Schrijver '88], [Nesterov, Nemirovski '89], [Khachiyan, Todd '93], ...

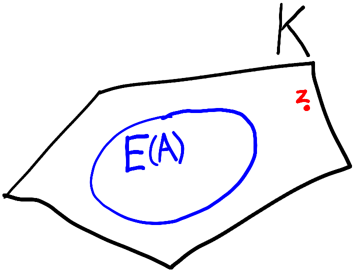
Polytopes given by Separation Oracle

- only $n+1$-ellipsoidal approximation for convex bodies given by weak separation oracle [Grötschel, Lovász and Schrijver '88]
- No (randomized) $n^{1-\epsilon}$-ellipsoidal approximation [J. Soto '08]

Finding Larger and Larger Inscribed Ellipsoids

Informal Statement

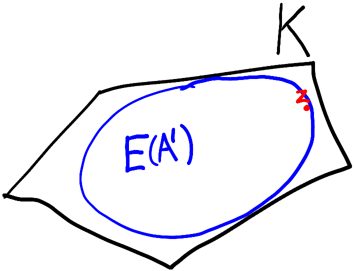
- We have $A \succ 0$ such that $E(A) \subseteq K$.
- Suppose we find $z \in K$ but z far outside of $E(A)$.
- Then should be able to find $A' \succ 0$ such that
  - $E(A') \subseteq K$
  - $\text{vol}\, E(A') > \text{vol}\, E(A)$



Finding Larger and Larger Inscribed Ellipsoids

Informal Statement

- We have $A \succ 0$ such that $E(A) \subseteq K$.
- Suppose we find $z \in K$ but z far outside of $E(A)$.
- Then should be able to find $A' \succ 0$ such that
  - $E(A') \subseteq K$
  - $\operatorname{vol} E(A') > \operatorname{vol} E(A)$



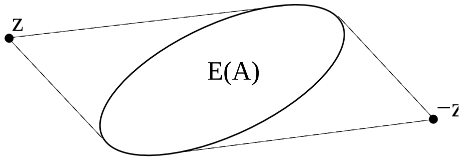
Theorem

If $A \succ 0$ *and* $z \in \mathbb{R}^n$ *with* $d = \|z\|_A^2 \geq n$ *then* $E(A')$ *is max volume ellipsoid inscribed in* $\operatorname{conv}\{E(A), z, -z\}$ *where*

$$A' = \frac{n}{d}\frac{d-1}{n-1}A + \frac{n}{d^2}\left(1 - \frac{d-1}{n-1}\right) Azz^T A$$

Moreover, $\operatorname{vol} E(A') = k_n(d) \cdot \operatorname{vol} E(A)$ *where*

$$k_n(d) = \sqrt{\left(\frac{d}{n}\right)^n \left(\frac{n-1}{d-1}\right)^{n-1}}$$



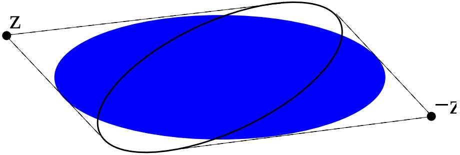
Theorem

If $A \succ 0$ *and* $z \in \mathbb{R}^n$ *with* $d = \|z\|_A^2 \geq n$ *then* $E(A')$ *is max volume ellipsoid inscribed in* $\mathrm{conv}\{E(A), z, -z\}$ *where*

$$A' = \frac{n}{d}\frac{d-1}{n-1}A + \frac{n}{d^2}\left(1 - \frac{d-1}{n-1}\right) Azz^T A$$

Moreover, $\mathrm{vol}\, E(A') = k_n(d) \cdot \mathrm{vol}\, E(A)$ *where*

$$k_n(d) = \sqrt{\left(\frac{d}{n}\right)^n \left(\frac{n-1}{d-1}\right)^{n-1}}$$



$\operatorname{vol} E(A') = k_n(d) \cdot \operatorname{vol} E(A)$ where

$$k_n(d) = \sqrt{\left(\frac{d}{n}\right)^n \left(\frac{n-1}{d-1}\right)^{n-1}}$$

Remarks

- $k_n(d) > 1$ for $d > n$ proves John's theorem
- Significant volume increase for $d \geq n+1$:
  $k_n(n+1) = 1 + \Theta(1/n^2)$
- Polar statement previously known [Todd '82]
  A' gives formula for minimum volume ellipsoid containing

$$E(A) \cap \{\, x \,:\, -b \leq \langle c, x \rangle \leq b \,\}$$

Review of Plan

- Given monotone, submodular f, make $n^{O(1)}$ queries, construct $\hat{f}$ s.t.

$$\hat{f}(S) \leq f(S) \leq \tilde{O}(\sqrt{n}) \cdot \hat{f}(S) \qquad \forall\, S \subseteq V.$$

Review of Plan

- Given monotone, submodular f, make $n^{O(1)}$ queries, construct $\hat{f}$ s.t.

$$\hat{f}(S) \leq f(S) \leq \tilde{O}(\sqrt{n}) \cdot \hat{f}(S) \qquad \forall\, S \subseteq V.$$

- Can reconstruct f from the polymatroid

$$P_f = \left\{ x \in \mathbb{R}^n_+ : \textstyle\sum_{i \in S} x_i \leq f(S) \qquad \forall\, S \subseteq [n] \right\}$$

by $f(S) = \max_{x \in P_f} \langle 1_S, x \rangle$.

Review of Plan

- Given monotone, submodular f, make $n^{O(1)}$ queries, construct $\hat{f}$ s.t.
$$\hat{f}(S) \leq f(S) \leq \tilde{O}(\sqrt{n}) \cdot \hat{f}(S) \qquad \forall\, S \subseteq V.$$
- Can reconstruct f from the polymatroid
$$P_f = \left\{ x \in \mathbb{R}^n_+ : \textstyle\sum_{i \in S} x_i \leq f(S) \qquad \forall\, S \subseteq [n] \right\}$$
by $f(S) = \max_{x \in P_f} \langle 1_S, x \rangle$.
- Make P_f centrally symmetric by reflections:
$$S(P_f) = \{\, x \;:\; (|x_1|, |x_2|, \cdots, |x_n|) \in P_f \,\}$$

Review of Plan

- Given monotone, submodular f, make $n^{O(1)}$ queries, construct $\hat{f}$ s.t.

$$\hat{f}(S) \leq f(S) \leq \tilde{O}(\sqrt{n}) \cdot \hat{f}(S) \qquad \forall\, S \subseteq V.$$

- Can reconstruct f from the polymatroid

$$P_f = \left\{ x \in \mathbb{R}^n_+ : \textstyle\sum_{i \in S} x_i \leq f(S) \qquad \forall\, S \subseteq [n] \right\}$$

by $f(S) = \max_{x \in P_f} \langle 1_S, x \rangle$.

- Make P_f centrally symmetric by reflections:

$$S(P_f) = \{\, x \;:\; (|x_1|, |x_2|, \cdots, |x_n|) \in P_f \,\}$$

- Max volume ellipsoid E_{max} has

$$E_{max} \subseteq S(P_f) \subseteq \sqrt{n} \cdot E_{max}.$$

Take $\hat{f}(S) = \max_{x \in E_{max}} \langle 1_S, x \rangle$.

Review of Plan

- Given monotone, submodular f, make $n^{O(1)}$ queries, construct $\hat{f}$ s.t.
$$\hat{f}(S) \leq f(S) \leq \tilde{O}(\sqrt{n}) \cdot \hat{f}(S) \qquad \forall\, S \subseteq V.$$
- Can reconstruct f from the polymatroid
$$P_f = \left\{ x \in \mathbb{R}^n_+ : \textstyle\sum_{i \in S} x_i \leq f(S) \qquad \forall\, S \subseteq [n] \right\}$$
by $f(S) = \max_{x \in P_f} \langle 1_S, x \rangle$.
- Make P_f centrally symmetric by reflections:
$$S(P_f) = \{\, x \;:\; (|x_1|, |x_2|, \cdots, |x_n|) \in P_f \,\}$$
- Max volume ellipsoid E_{max} has
$$E_{max} \subseteq S(P_f) \subseteq \sqrt{n} \cdot E_{max}.$$
Take $\hat{f}(S) = \max_{x \in E_{max}} \langle 1_S, x \rangle$.
- Compute ellipsoids $E_1, E_2, \ldots$ in $S(P_f)$ that converge to E_{max}.

Review of Plan

- Given monotone, submodular f, make $n^{O(1)}$ queries, construct $\hat{f}$ s.t.
$$\hat{f}(S) \leq f(S) \leq \tilde{O}(\sqrt{n}) \cdot \hat{f}(S) \qquad \forall\, S \subseteq V.$$
- Can reconstruct f from the polymatroid
$$P_f = \left\{ x \in \mathbb{R}^n_+ : \sum_{i \in S} x_i \leq f(S) \qquad \forall\, S \subseteq [n] \right\}$$
by $f(S) = \max_{x \in P_f} \langle 1_S, x \rangle$.
- Make P_f centrally symmetric by reflections:
$$S(P_f) = \{\, x \ :\ (|x_1|, |x_2|, \cdots, |x_n|) \in P_f \,\}$$
- Max volume ellipsoid E_{max} has
$$E_{max} \subseteq S(P_f) \subseteq \sqrt{n} \cdot E_{max}.$$
Take $\hat{f}(S) = \max_{x \in E_{max}} \langle 1_S, x \rangle$.
- Compute ellipsoids $E_1, E_2, \ldots$ in $S(P_f)$ that converge to E_{max}.
Given $E_i = E(A_i)$, need $z \in S(P_f)$ with $\|z\|_{A_i} \geq \sqrt{n+1}$.
  - If $\exists z$, can compute E_{i+1} of larger volume.
  - If $\nexists z$, then $E_i \approx E_{max}$.

- Ellipsoidal Norm Maximization

  Given $A \succ 0$ and well-bounded convex body K by separation oracle.
  (So $B(r) \subseteq K \subseteq B(R)$ where $B(d)$ is ball of radius d.)
  Solve

$$\max_{x \in K} \|x\|_A$$

- **Ellipsoidal Norm Maximization**

  Given $A \succ 0$ and well-bounded convex body K by separation oracle. (So $B(r) \subseteq K \subseteq B(R)$ where $B(d)$ is ball of radius d.) Solve

  $$\max_{x \in K} \|x\|_A$$

- **Bad News**

  Ellipsoidal Norm Maximization NP-complete for $S(P_f)$ and P_f. (Even if f is a graphic matroid rank function.)

- Ellipsoidal Norm Maximization

  Given $A \succ 0$ and well-bounded convex body K by separation oracle.
  (So $B(r) \subseteq K \subseteq B(R)$ where $B(d)$ is ball of radius d.)
  Solve

  $$\max_{x \in K} \|x\|_A$$

- Bad News

  Ellipsoidal Norm Maximization NP-complete for $S(P_f)$ and P_f.
  (Even if f is a graphic matroid rank function.)

- Approximations are good enough

  P-time α-approx. algorithm for Ellipsoidal Norm Maximization
  $\implies$ P-time $\alpha\sqrt{n+1}$-ellipsoidal approximation for K
  (in $O(n^3 \log(R/r))$ iterations)

Our Task

Given $A \succ 0$, and f find $\max_{x \in S(P_f)} \|x\|_A$.

Our Task

Given $A \succ 0$, and f find $\max_{x \in S(P_f)} \|x\|_A$.

Observation: Symmetry Helps

$S(P_f)$ invariant under axis-aligned reflections.

(Diagonal $\{\pm 1\}$ matrices.)

$\implies$ same is true for E_{max}

$\implies$ $E_{max} = E(D)$ where D is diagonal.

Our Task

Given diagonal $D \succ 0$, and f find

$$\max_{x \in S(P_f)} \|x\|_D$$

Equivalently,

$$\begin{array}{ll} \max & \sum_i d_i x_i^2 \\ \text{s.t.} & x \in P_f \end{array}$$

- Maximizing convex function over convex set $\Rightarrow$ max attained at vertex.

Our Task

Given diagonal $D \succ 0$, and f find

$$\begin{array}{ll} \max & \sum_i d_i x_i^2 \\ \text{s.t.} & x \in P_f \end{array}$$

- Maximizing convex function over convex set $\Rightarrow$ max attained at vertex.

Matroid Case

If f is matroid rank function
$\implies$ vertices in $\{0,1\}^n \implies x_i^2 = x_i$.
Our task is

$$\begin{array}{ll} \max & \sum_i d_i x_i \\ \text{s.t.} & x \in P_f \end{array}$$

This is the max weight base problem, solvable by greedy algorithm.

Our Task

Given diagonal $D \succ 0$, and f find

$$\begin{array}{ll} \max & \sum_i d_i x_i^2 \\ \text{s.t.} & x \in P_f \end{array}$$

- Maximizing convex function over convex set
  $\Rightarrow$ max attained at vertex.

General Monotone Submodular Case

More complicated: uses approximate maximization of submodular function [Nemhauser, Wolsey, Fischer '78], etc.
Can find $O(\log n)$-approximate maximum.

Summary of Algorithm

Theorem

In P-time, construct a (submodular) function $\hat{f}(S) = \sqrt{\sum_{i \in S} c_i}$ *with*

- $\alpha(n) = \sqrt{n+1}$ *for matroid rank functions* f*, or*
- $\alpha(n) = O(\sqrt{n} \log n)$ *for general monotone submodular* f*.*

The algorithm is deterministic.

$\Omega(\sqrt{n}/\log n)$ Lower Bound

Theorem

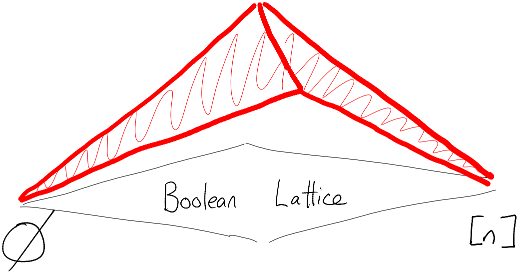
With poly(n) *queries, cannot approximate* f *better than* $\dfrac{\sqrt{n}}{\log n}$.
Even for randomized algs, and even if f *is matroid rank function.*

$\Omega(\sqrt{n}/\log n)$ Lower Bound

Informal Idea

Theorem

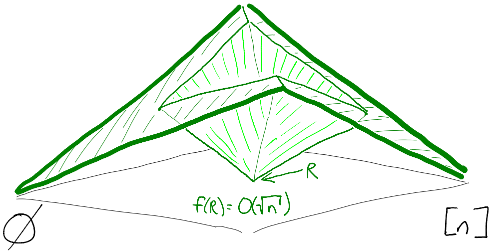
With poly(n) *queries, cannot approximate* f *better than* $\frac{\sqrt{n}}{\log n}$.
Even for randomized algs, and even if f *is matroid rank function.*



$f(S) = \min\{|S|, n-|S|\}$

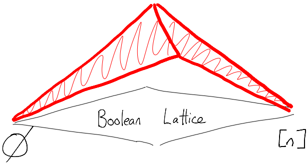
Theorem

With poly(n) *queries, cannot approximate f better than* $\dfrac{\sqrt{n}}{\log n}$.
Even for randomized algs, and even if f is matroid rank function.

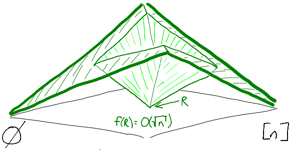


$$f'(S) = \min\left\{ |S|,\ n-|S|,\ \frac{n}{2} + |S \cap \bar{R}| - |S \cap R| + O(\sqrt{n}) \right\}$$

Discrepancy Argument



$f(S) = \min\{|S|, n-|S|\}$

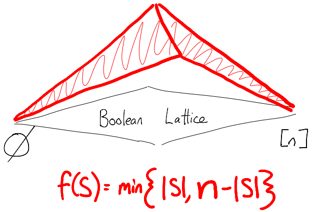


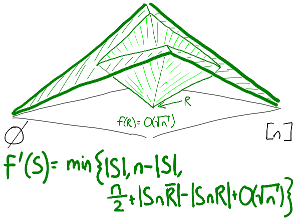
$f'(S) = \min\{|S|, n-|S|, \frac{n}{2} + |S \cap \bar{R}| - |S \cap R| + O(\sqrt{n})\}$

Algorithm performs queries $S_1, \ldots, S_k$.
A query S_i distinguishes f from f' iff

$$|S_i \cap R| - |S_i \cap \bar{R}| > O(\sqrt{n})$$

Discrepancy Argument





Algorithm performs queries $S_1, \ldots, S_k$.
A query S_i distinguishes f from f' iff

$$|S_i \cap R| - |S_i \cap \bar{R}| > O(\sqrt{n})$$

Standard discrepancy argument: For uniformly random R,

$$||S_i \cap R| - |S_i \cap \bar{R}|| \leq \sqrt{2n \ln(2k)} \qquad \forall\, i$$

So algorithm fails to find random R.

PAC-like Model

Problem

Let $f : 2^{[n]} \to \mathbb{R}_+$ *be monotone and submodular. Let* D *be a distribution on* $2^{[n]}$*. Given* $\mathrm{poly}(n)$ *training samples* $S_1, S_2, \ldots$ *from* D*, labeled by* f*, construct a function* $\hat{f}$ *such that*

$$\hat{f}(S) \leq f(S) \leq \alpha(n) \cdot \hat{f}(S),$$

with prob. at least $1 - \epsilon$ *when* S *is a new test sample from* D*.*

PAC-like Model

Problem

Let $f : 2^{[n]} \to \mathbb{R}_+$ be monotone and submodular. Let D be a distribution on $2^{[n]}$. Given $\mathrm{poly}(n)$ training samples $S_1, S_2, \ldots$ from D, labeled by f, construct a function $\hat{f}$ such that

$$\hat{f}(S) \leq f(S) \leq \alpha(n) \cdot \hat{f}(S),$$

with prob. at least $1 - \epsilon$ when S is a new test sample from D.

Theorem (Balcan, H. 2009)

Can achieve approximation $\alpha(n) = \sqrt{n+1}$.
No algorithm can achieve approximation $\alpha(n) = \Omega(n^{1/3} / \log n)$.

Lower bound for PAC model

Problem

Find embedding $\rho : \{0,1\}^d \rightarrow \{0,1\}^n$ *and* $\alpha \ll \beta$ *such that, for every* **Boolean** *function* $f : \{0,1\}^d \rightarrow \{0,1\}$*, there is a* **submodular** *function* $\tilde{f} : \{0,1\}^n \rightarrow \mathbb{R}_+$ *satisfying*

$$\begin{aligned} f(S) = 0 \quad &\Longrightarrow \quad \tilde{f}(\rho(S)) = \alpha \\ f(S) = 1 \quad &\Longrightarrow \quad \tilde{f}(\rho(S)) = \beta \end{aligned}$$

Lower bound for PAC model

Problem

Find embedding $\rho : \{0,1\}^d \to \{0,1\}^n$ *and* $\alpha \ll \beta$ *such that, for every* **Boolean** *function* $f : \{0,1\}^d \to \{0,1\}$, *there is a* **submodular** *function* $\tilde{f} : \{0,1\}^n \to \mathbb{R}_+$ *satisfying*

$$
\begin{aligned}
f(S) = 0 \quad &\Longrightarrow \quad \tilde{f}(\rho(S)) = \alpha \\
f(S) = 1 \quad &\Longrightarrow \quad \tilde{f}(\rho(S)) = \beta
\end{aligned}
$$

Theorem (Balcan, H. 2009)

This is possible with $\alpha = \log^2 n$, $\beta = n^{1/3}$ *and* $d = \Theta(\log^2 n)$.

Summary

Problem

Given a monotone, submodular f, construct using $\text{poly}(n)$ *oracle queries a function $\hat{f}$ such that:*

$$\hat{f}(S) \leq f(S) \leq \alpha(n) \cdot \hat{f}(S) \qquad \forall\, S \subseteq [n]$$

Our Positive Result

A deterministic algorithm that constructs $\hat{f}(S) = \sqrt{\sum_{i \in S} c_i}$ with

- $\alpha(n) = \sqrt{n+1}$ for matroid rank functions f, or
- $\alpha(n) = O(\sqrt{n} \log n)$ for general monotone submodular f

Our Negative Result

With polynomially many oracle calls, $\alpha(n) = \Omega(\sqrt{n} / \log n)$ (even for randomized algs)