

HOMEWORK 1

Due Friday, September 18, 2009 at the beginning of class

For general homework policies and our suggestions, please see the accompanying document. For this homework, it might help if you read the subsection “Extensions” in Section 1.1 of the textbook.

Do not turn the first problem in and no collaboration is allowed on the second problem.

1. **(Do NOT turn this problem in)** This problem is just to get you start thinking about proofs and maybe brush up some previous proof techniques that you might have seen in other courses.

In the Sep 9 lecture, I claimed that the total number of perfect matchings when you have n men and n women (and as we assumed earlier, everyone is interested in everyone else of the other sex) is $n!$. (Recall $n! = n \times (n - 1) \times (n - 2) \times \dots \times 2 \times 1$.) Prove this claim.

2. **(You must work on this problem on your own: NO collaboration is allowed)** (10 + 10 = 20 points) Exercise 1 and 2 in Chapter 1.
3. (35 points) Exercise 4 in Chapter 1.
4. (45 points) This problem is inspired by a question raised by Tom in class. As Tom pointed out, the stable marriage problem does not handles divorces. This is because we assume everyone is interested in everyone else of the opposite sex and we assume that the preferences *do not change*.

In this problem, we will see the effect of changes in preferences in the outcome of the Gale-Shapley algorithm (for this problem you can assume the version of the Gale-Shapley algorithm that we did in class where the women do all the proposing).

Given an instance of the stable marriage problem (i.e. set of men M and the set of women W along with their preference lists: L_m and L_w for every $m \in M$ and $w \in W$ respectively), call a man $m \in M$ a *home-wrecker* if the following property holds. There exists an L'_m such that if m changes his preference list to L'_m (from L_m) then the Gale-Shapley algorithm matches everyone to someone else. In other words, let S_{orig} be the stable marriage output by the Gale-Shapley algorithm for the original input and S_{new} be the stable marriage output by the Gale-Shapley algorithm for the new instance of the problem where m 's preference list is replaced by L'_m (but everyone else has the same preference list as before). Then $S_{\text{orig}} \cap S_{\text{new}} = \emptyset$.

For every integer $n \geq 2$ prove the following: There exists an instance of the stable marriage problem with n men and n women such that there is a man who is a home-wrecker.

(Note: To get full credit you must present an example for every $n \geq 2$, that is, you have to present a “family” of examples. Further, your proof argument should work for every value of $n \geq 2$.)