Shortest Path

I/p: Directed graph $G = (V, E)$ $\forall e \in E$, $c_e \in \mathbb{R}$ (c_e can be < 0)

but G has no $-$ve cycle.

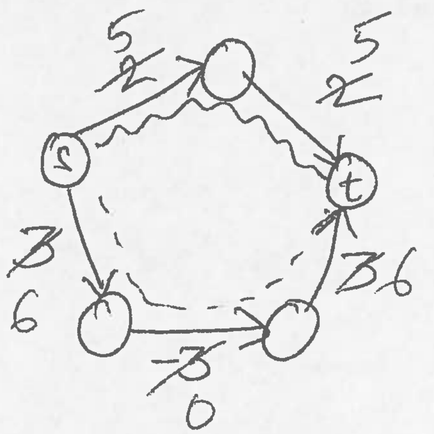
$t \in V$

O/p: Shortest paths from every $s \in V$ to t.

$\rightsquigarrow$ Dijkstra doesn't work.

$\rightsquigarrow$ Add a large number M s.t $c_e + M \geq 0$ $\forall e \in E$ & run ~~Dikj~~ Dijkstra's on i/p w/ cost

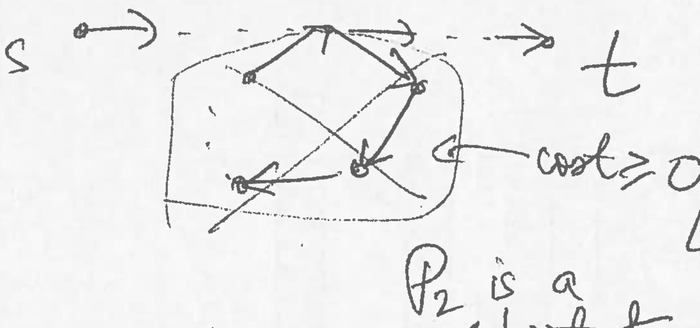
$$c'_e = c_e + M$$



Bellman-Ford

PROP: ~~If~~ G has no $-$ve cycle, then $\exists$ a shortest s-t path that is simpl.

Pf:



$P_1 = P_{11}, C, P_{12}$

$P_2 = P_{11}, P_{12}$

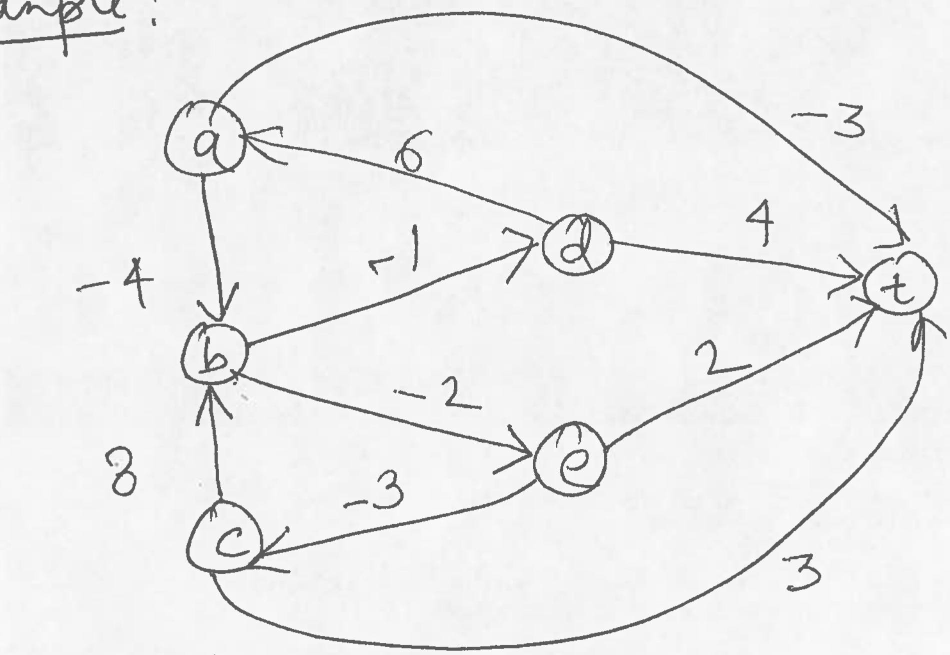
$\Downarrow$ cost$(P_2) \leq$ cost(P_1) as cost$(C) \geq 0$

P_2 is a shortest path.

$\Rightarrow$ only need to consider paths of length $n-1$.

For now: $\forall\, s \in V$, only want to compute shortest dist $d(s,t)$.

Example:



$OPT(v,i)$

$v \in V$

$0 \le i \le n-1$

→ length of shortest v-t path with $\le i$ edges.

$OPT(d,0)=\infty$, $OPT(d,1)=4$, $\boxed{OPT(d,2)=3}$

$\boxed{OPT(d,3)=3}$, $OPT(d,4)=2$, $OPT(d,5)=0$

$OPT(d,6)=0$, $OPT(d,7)=0$

Claim: $OPT(v,i) \ge OPT(v,i+1)$.

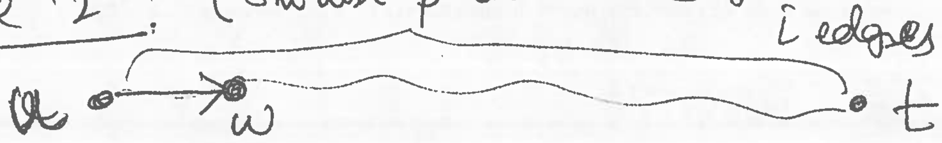
$d(s,t) = OPT(s,n-1)$.

$$\boxed{OPT(v,i) = \min\left(OPT(v,i-1),\ \min_{(v,w)\in E} c_{v,w} + OPT(w,i-1)\right) \quad v\in V,\ 1 \le i \le n-1}$$

Case 1: Shortest path $\le i$ edges uses $\le i-1$ edges.

$OPT(v,i) = OPT(v,i-1)$

Case "2": (shortest path $\le i$ edges uses exactly i edges)



$OPT(v,i) = c_{v,w} + OPT(w,i-1)$