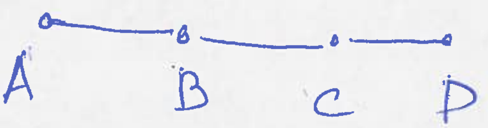
Sep 19, 2014

$G = (V, E)$ by default → undirected.

Def: A path is a sequence of vertices $u_1, \dots, u_k$ s.t.

$\forall i \quad u_i \in V \;\&\; \forall i < k \quad (u_i, u_{i+1}) \in E$

a u_1–u_k path



D, C, B, A ✓

A, B, C, D ✓

A, B, C, D, C ✓

A, C ✗

X, Y, Z ✓

Z, Y, X ✗

Note: Vertices can be repeated.

Def: A simple path has no repeated vertex.

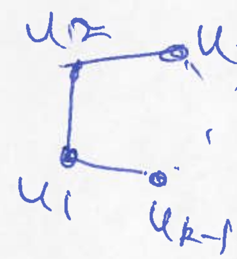
Def: Length of a path is # edges in the path.

Note: Length of a simple path $\leq |V| - 1 \; (= n-1)$

Cycles

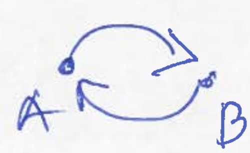
Def: A cycle is path $u_1, \dots u_{k-1}, u_k = u_1$

→ $u_1, \dots, u_{k-1}$ are distinct



Directed graphs: $k \geq 3$

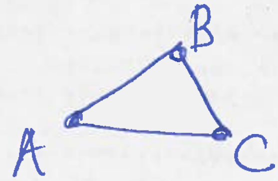
~~A, B~~ ✗



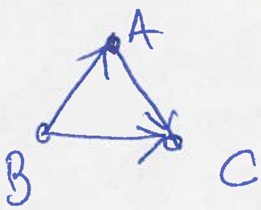
A, B, A ✓

Undirected graphs: $k \geq 4$

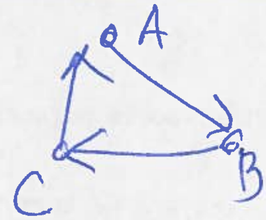
A, B, A ✗



A, B, C, A ✓



A, B, C, A ✗

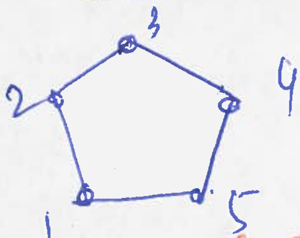


A, B, C, A

Def: u & w $\in V$ are connected if $\exists$ u–w path.

Base case: $\forall\, u \in V$, u is connected to u.

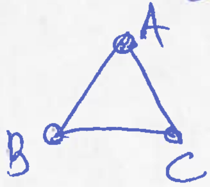
distance b/w u & w is the length of the shortest u–w path.



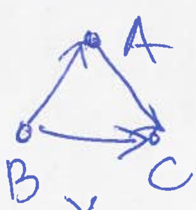
$\text{dist}(1, 5) = \min\{1, 4\} = 1.$

(undirected)

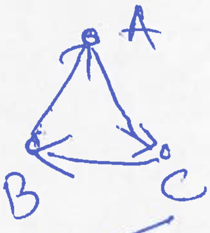
Def: G is connected if $\forall\ u, w \in V$, u & w are connected.



✓



✗ as C & A are not connected



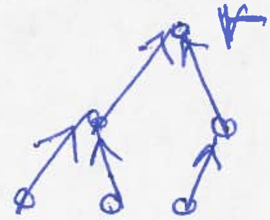
✓ strongly connected iff $\forall\ u, w$ $\exists$ u–w path

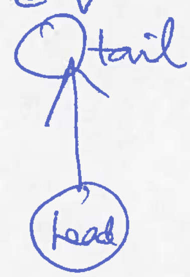
Def A (undirected) graph $T = (V, E)$ is a tree if
(i) T is connected
(ii) T has no cycle.

THM: If T is a tree on n vertices, it has exactly $n-1$ edges.

Pf idea: root tree $T = (V, E)$ at $r \in V$

Direct edge from child to parent





Argue:
(1) Every edge is outgoing for exactly one vertex. a unique vertex.
(2) r has no outgoing egde
$\Rightarrow |E| = |V \setminus \{r\}| = n-1$

Pf details: Pick a vertex $r \in V$ & root T at r.

u (tail) closer to r

w (head) further from r

Ex: Every $(u, w) \in E$ exactly one of u or w is closer to r than the other

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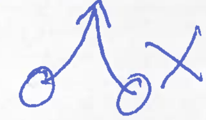
Claim 1: r ~~has no~~ is not the head of any directed edge.

Ex **Claim 2:** Every (directed) edge has a unique head.



Claim 3: Every non-root $u \in V$ is the head of some directed edge

Claim 4: Every $u \in V$ is the head of ≤ 1 edge.

Claim 1–4 $\Rightarrow$ 1-to-1 correspondence
b/w E & $V \setminus \{r\}$
$\Rightarrow |E| = |V \setminus \{r\}| = n-1$