Oct 24, 2014

Dijkstra's Algo

$d'(w) = \min_{\substack{u \in R \\ e=(u,w) \in E}} \{ d(u) + \ell_e$

1. $R \leftarrow \{s\}$, $d(s) = 0$ $O(1)$
2. While (**) $\exists$ a $x \notin R$ but $\exists\, u \in R$ & $(u,x) \in E$

$\leq n$ iterations →

Pick w that $\min d'(x)$ among such x ↑ $O(n)$ if d' in an array

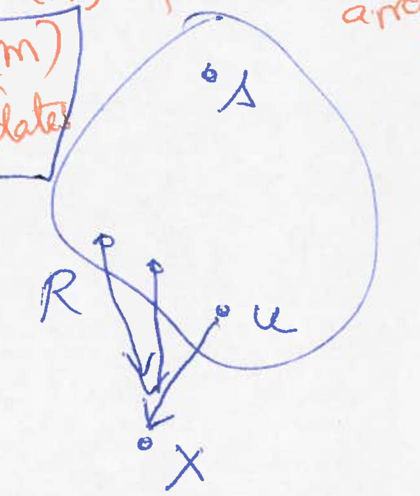
Add w to R $\}$ $O(1)$ ↑ $O(m)$ for updates

$d(w) \leftarrow d(w')$.

$O(m+n)$

$\Rightarrow O(n^2+mn)$ overall

better bound possible →



Correctness: P_u be the s-u path in Dijkstra's tree

(**) Checking condn & finding w can be done together.

Lemma: $\forall u$, P_u is ~~the~~ a shortest s-u path ($d(u)$ is correct)

$\Rightarrow$ Dijkstra's algo is correct.

Pf of Lemma: Argue for every iteration of algo

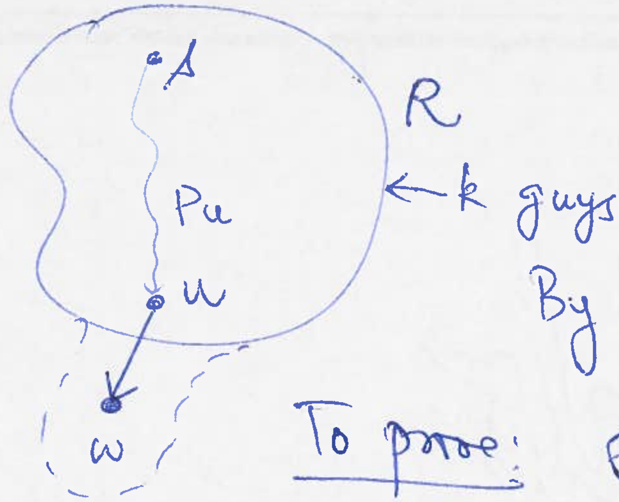
$\forall u \in R$, P_u is ~~the~~ a shortest s-u path & $d(u)$ is correct. = length of any shortest s-u path.

$\Rightarrow$ (true at end.)

Induction on $|R|$

Base case: $|R| = 1$ (s) R $P_s = s$ ✓ $d(s) = 0$ ✓

Inductive hypotheses: True for $|R| = k \geq 1$

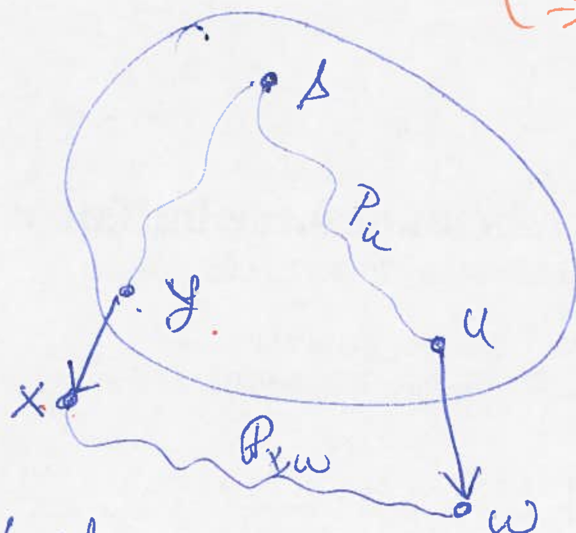


$$d(w) = d'(w) = d(u) + l_{u,w}$$

By I.H. $\exists$ P_u is shortest s-u path.

<u>To prove:</u> $P_w = P_u, (u,w)$ is shortest s-w path.

($\Rightarrow$ after $(k+1)^{th}$ iteration $d(w)$ is correct)



For contradiction assume $\exists$ s-w path P'_w s.t $l(P'_w) < l(P_w)$

P'_w starts inside of R & ends up outside.

$$P'_w = P_y, (y,x), P_{x,w}$$

$$l(P'_w) = \underbrace{\underbrace{l(P_y)}_{d''(y)} + l_{(y,x)}}_{} + \underbrace{l(P_{x,w})}_{\geq 0 \text{ as } l_e \geq 0}$$

$\geq d'(x) + 0$

$\geq d'(w)$ [as w chosen over x]

$= l(P_w) \longrightarrow$ a contradiction!