Oct 27, 2014 — Minimum Spanning Tree (MST)

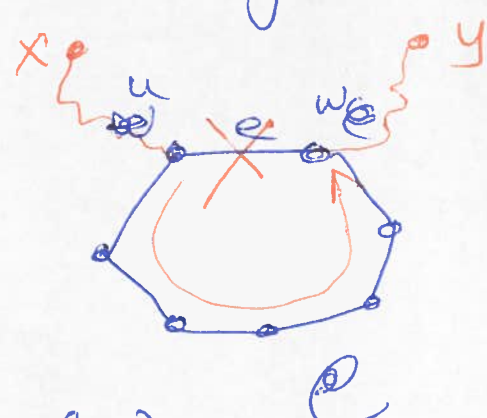
Input: $G = (V, E)$ $\quad \forall e \in E,\ c_e \geq 0$ (for simplicity)
(G is connected)

Output: (i) $E' \subseteq E$ s.t. $T = (V, E')$ is connected.
(ii) $c(T) = \sum_{e \in E'} c_e$ is min.

$\boxed{\text{Assume } \forall e,\ c_e > 0}$

Claim: Optimal $T = (V, E')$ is a tree. ($\Rightarrow$ MST)

Pf: By contradiction.
T is not a tree $\xRightarrow{\text{(as } T \text{ is connected)}}$ T has a cycle $\mathcal{C}$

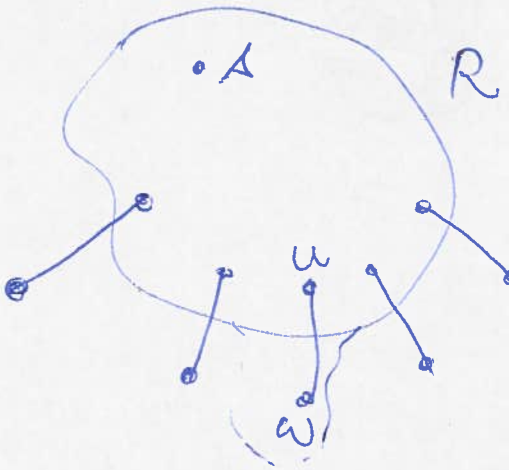


Pick any $e \in E'$ & remove it
$T' = (V, E' \setminus \{e\})$
$\rightarrow$ T' is connected
$\rightarrow$ $c(T') = c(T) - \underbrace{c_e}_{>0} < c(T)$ $\longrightarrow$ contradiction (T is not optimal).

Algorithms

Prim's ("Dijksta-lite")

Some start vertex s.



Some partial tree on R

→ Pick the cheapest "crossing" edge (u, w) $(u \in R, w \notin R)$

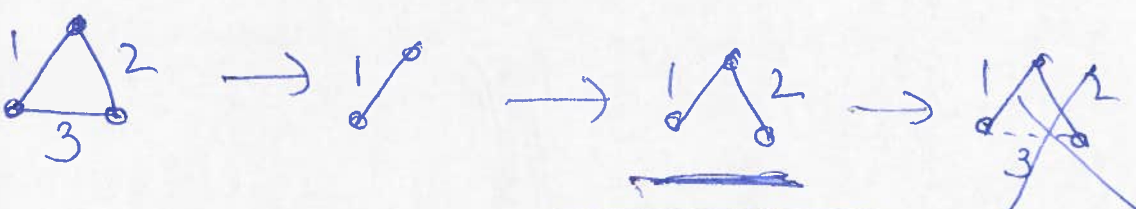
→ Add w to R

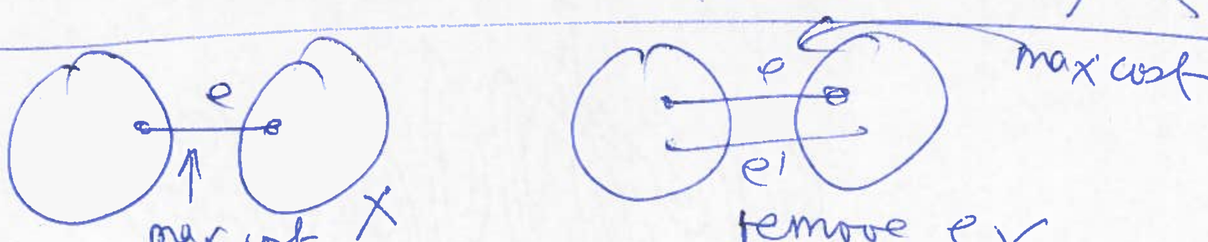
→ Stop when no more crossing edges $(R = V)$

Kruskal:

Sort c_e in increasing order & in this order:

Add e to current set E' unless adding it creates a cycle.





Reverse-Delete algo:

Sort c_e in decreasing order

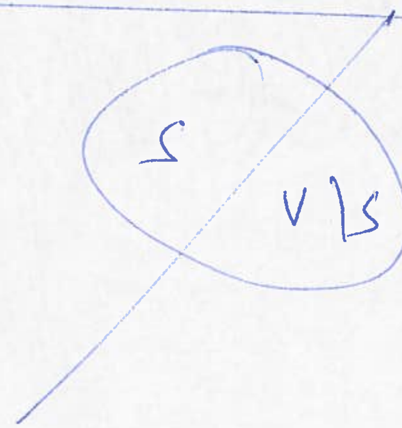
Remove edges in this order unless it disconnects the graph.

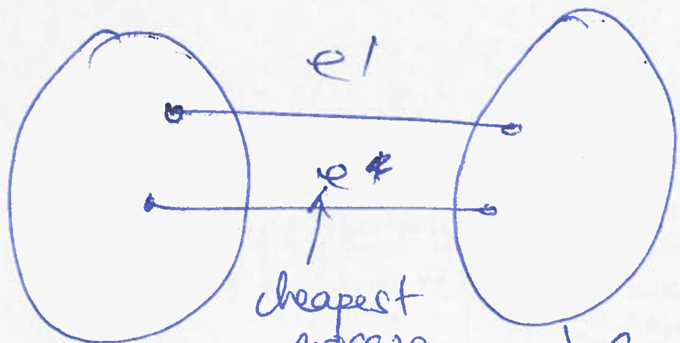
CUT PROPERTY LEMMA:

Assume all edge costs are distinct.
$c_e \neq c_{e'} \quad \forall\, e \neq e'$

$S \subseteq V$

a cut $\quad S,\ V \setminus S$





$S \neq \phi$ $\qquad V \setminus S = \phi$ $\quad (\Leftrightarrow S \neq V)$

Let e be the cheapest crossing edge (unique)

$\Rightarrow e$ is in <u>ALL</u> MSTs for G.