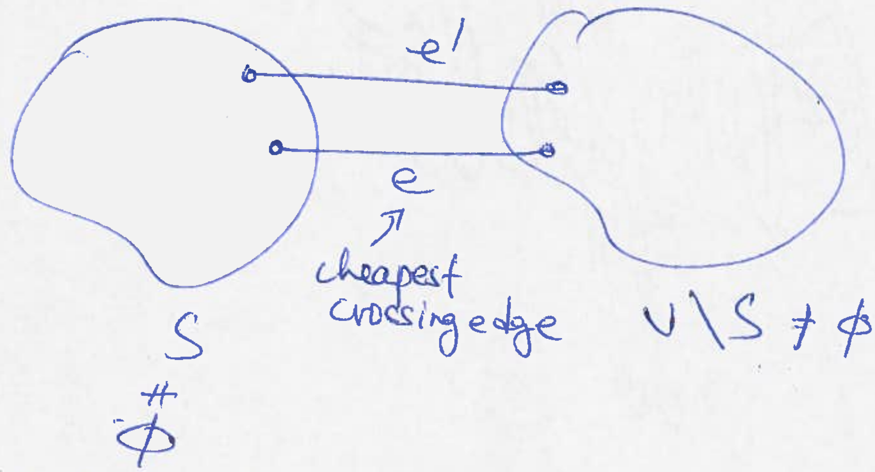
Oct 29, 2014

CUT PROPERTY LEMMA

$\boxed{\text{Assume all } c_e\text{'s are distinct}}$ ✓ (graph G)

$\forall\ \emptyset \neq S \neq V$

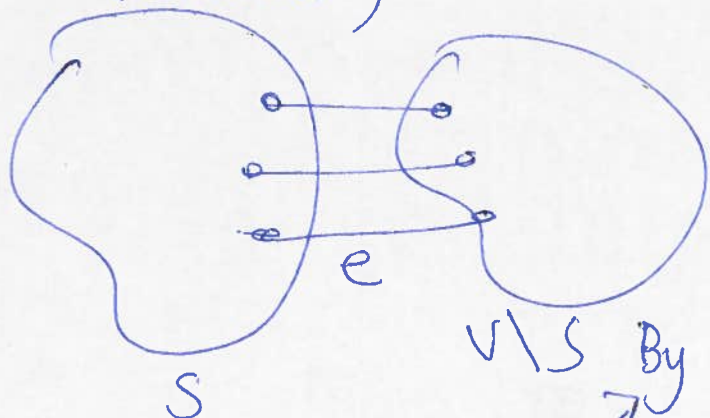


Let e be the cheapest crossing edge.

$\Rightarrow$ e is in ALL MSTs of G.

THM: Prim's algo is correct.

Consider the set S in any iteration of Prim's algo (before it terminates)



e be cheapest crossing edge.

By defn of Prim's algo, it picks edge e.

By cut property lemma, this is a "safe" choice.

$S \neq \emptyset$ (because $s \in S$)

$S \neq V$ (as algo has not terminated)

$(S, T) \leftarrow$ spanning tree on S

$\Rightarrow$ Are we done?

Need to argue: final output (V, T) is a spanning tree.

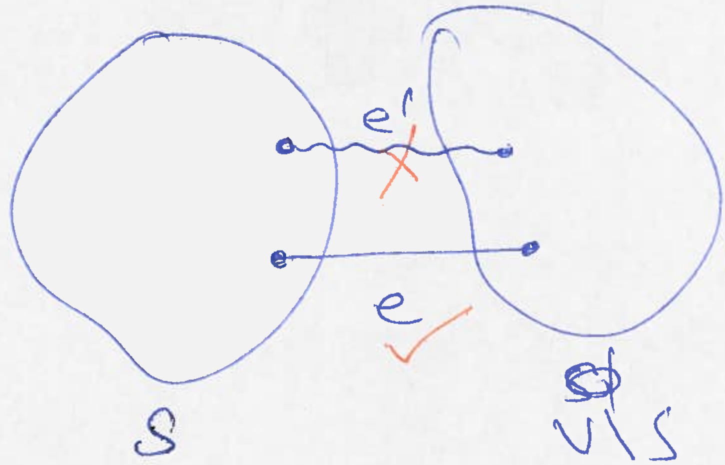
EX: Induction on $|S|$

Pf of Cut Property Lemma

By contradiction

Assume $\exists$ $\phi \neq S \neq V$ & $\exists$ an MST T (of

s.t.



e is cheapest crossing edge

BUT

$e \notin T$

Since T is an MST $\Rightarrow$ $\exists$ crossing edge $e' \neq e$ s.t. $e' \in T$

$T' \leftarrow (T \setminus \{e'\}) \cup \{e\}$

$c(T') = c(T) \underbrace{- c_{e'} + c_e}_{< 0}$ [but $c_e < c_{e'}$

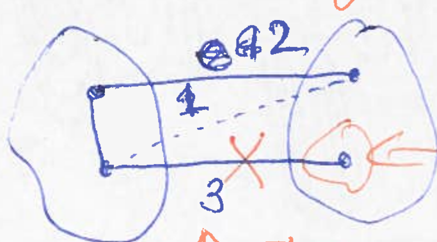
$< c(T)$ $\Rightarrow$ cheaper MST than T

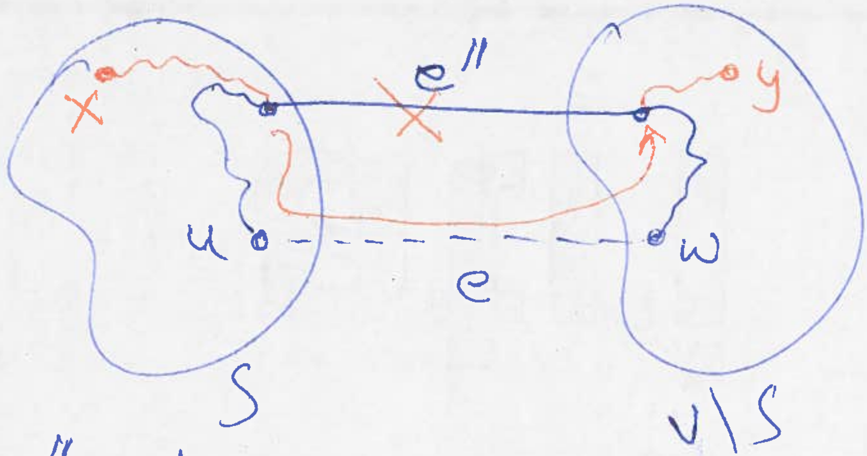
(not possible!)

$\Rightarrow$ contradiction.

have to argue T' is a spanning tree

$n = 4$





Since T is a spanning tree
$\exists$ a u-w path in T

Let e'' be a crossing edge in u-w path.

$T'' \leftarrow T \setminus \{e''\} \cup \{e\}$

As before $c(T'') = c(T) - c_{e''} + c_e < c(T)$

Argue: T'' is a spanning tree $\Rightarrow$ T is not an MST (contradiction)

$\Uparrow$ T'' is connected.

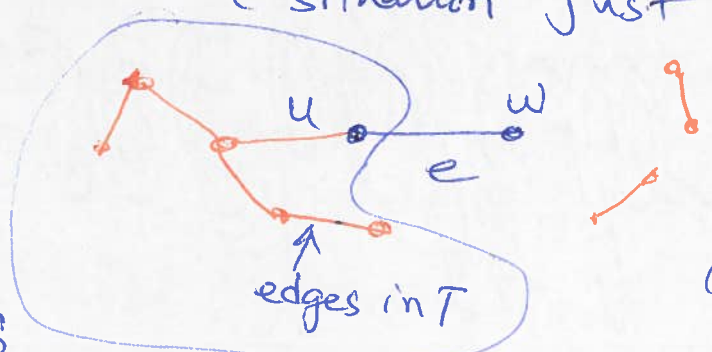
Consider $x, y \in V$: → Case 1: x, y connected in T via a path that does not have e'' in it. ✓

Case 2: (x, y) still connected in T'' via e.

Correctness of Kruskal's algo

(order by incr c_e ... ; add if edge e does not create a cycle)

→ Consider the situation just before e is added



S : set of all vertices connected to u via edges in (current) T

Claim 1: $S \neq \emptyset$ $u \in S$ ✓

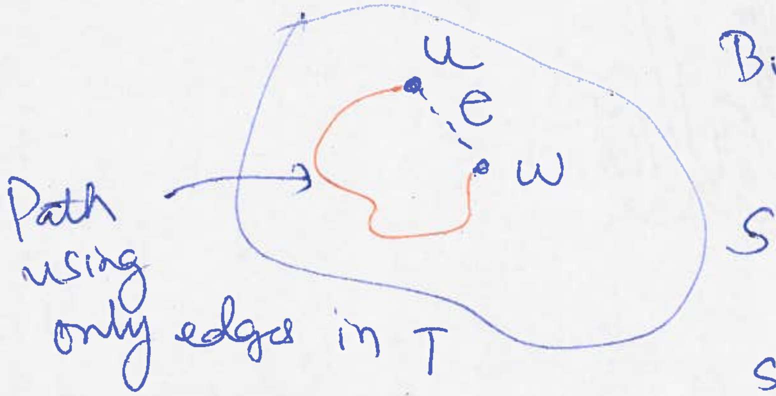
Claim 2: $S \neq V$

Claim 3: e is the cheapest crossing edge for $(S, V \setminus S)$

Claim 1+2+3 $\Rightarrow$ adding edge e was "safe"

Pf of Claim 2: Argue $w \notin S \Rightarrow S \neq V$

For contradiction assume $w \in S$



But if e is added to T $\Rightarrow$ $T \cup \{e\}$ has a cycle. (A contradiction since algo will not add e to T)

Lemma: e is the first crossing edge for $(S, V \setminus S)$ considered by K's algo.

$\Rightarrow$ Claim 3.