Nov 5, 2014

Multiplying 2 integers

1

Input: $a = a_{n-1}, \dots, a_0$ (MSB → a_{n-1}, LSB → a_0)

$b = b_{n-1}, \dots, a_0$

(Note: binary representation. Any constant base is OK.)

$$\text{Dec}(a) = \sum_{i=0}^{n-1} a_i \cdot 2^i$$

$a = 1101$ $\quad$ $\text{Dec}(a) = 13$

$b = 0011$ $\quad$ $\text{Dec}(b) = 3$

```
   1101
 x 0011
 ------
   1101
  1101
 0000
0000
 ------
 100111
```

Adding & generating n rows $O(n^2)$ — n rows

$c \longrightarrow 100111$ $\quad$ $\text{Dec}(c) = 39$

$a = a_{n-1} \dots, a_0$ $\qquad$ $a = 1101$

$\lceil \frac{n}{2} \rceil$

$a^0 = a_{\lceil \frac{n}{2} \rceil - 1}, \dots, a_0$ $\qquad$ $a^0 = 01$ $\quad$ $\text{Dec}(a^0) = 1$

$a^1 = a_{n-1}, \dots, a_{\lceil \frac{n}{2} \rceil}$ $\qquad$ $a^1 = 11$ $\quad$ $\text{Dec}(a^1) = 3$

$$\text{Dec}(a^0) = \sum_{i=0}^{\lceil \frac{n}{2} \rceil - 1} a_i \cdot 2^i$$

$$\text{Dec}(a^1) = \sum_{i=\lceil \frac{n}{2} \rceil}^{n-1} a_i \cdot 2^{i - \lceil \frac{n}{2} \rceil}$$

$\text{Dec}(01) = 0 \cdot 2 + 1 \cdot 1 = 1$

$\text{Dec}(11) = 2 \cdot 1 + 1 \cdot 1 = 3$

$$\mathrm{Dec}(a) = \mathrm{Dec}(a^1)\cdot 2^{\lceil \frac{n}{2} \rceil} + \mathrm{Dec}(a^0)$$

$$= \mathrm{Dec}(a^1)\cdot 2^{\lceil \frac{n}{2} \rceil} + \mathrm{Dec}(a^0)$$

$\mathrm{Dec}(1101) = 13$

$$\stackrel{?}{=} \mathrm{Dec}(11)\cdot 2^2 + \mathrm{Dec}(01)$$

$$= 3\cdot 4 + 1 = 13.$$

$$\mathrm{Dec}(a) = \sum_{i=0}^{n-1} a_i \cdot 2^i$$

$$= \sum_{i=\lceil \frac{n}{2} \rceil}^{n-1} a_i \cdot 2^i + \underbrace{\sum_{i=0}^{\lceil \frac{n}{2} \rceil - 1} a_i \cdot 2^i}_{= \mathrm{Dec}(a^0)}$$

$$\sum_{i=\lceil \frac{n}{2} \rceil}^{n-1} a_i \cdot 2^i = \underbrace{2^{\lceil \frac{n}{2} \rceil} \cdot \sum_{i=\lceil \frac{n}{2} \rceil}^{n-1} a_i \cdot 2^{i - \lceil \frac{n}{2} \rceil}}_{} = 2^{\lceil \frac{n}{2} \rceil} \cdot \mathrm{Dec}(a^1)$$

$$= \mathrm{Dec}(a^1)\cdot 2^{\lceil \frac{n}{2} \rceil} + \mathrm{Dec}(a^0)$$

$b = b_{n-1}, \ldots, b_0$ $\qquad 0011 \qquad \mathrm{Dec}(0011) = 3$

$b^1 = b_{n-1}, \ldots, b_{\lceil \frac{n}{2} \rceil}$ $\qquad b^1 = 00 \qquad \mathrm{Dec}(00) = 0$

$b^0 = b_{\lceil \frac{n}{2} \rceil - 1}, \ldots, b_0$ $\qquad b^0 = 11 \qquad \mathrm{Dec}(11) = 3$

$\mathrm{Dec}(b) = \mathrm{Dec}(b^1)\cdot 2^{\lceil \frac{n}{2} \rceil} + \mathrm{Dec}(b^0)$ $\qquad 3 = 0\cdot 2^2 + 3$

$$\underbrace{a\cdot b}_{n\text{-bit}} = \left(a^1\cdot 2^{\lceil \frac{n}{2}\rceil} + a^0\right)\cdot\left(b^1\cdot 2^{\lceil \frac{n}{2}\rceil} + b^0\right)$$

$$= \underbrace{a^1\cdot b^1}\cdot 2^{2\lceil \frac{n}{2}\rceil} + \left(\underbrace{a^1\cdot b^0} + \underbrace{a^0\cdot b^1}\right)\cdot 2^{\lceil \frac{n}{2}\rceil} + \underbrace{a^0\cdot b^0}_{\sim \frac{n}{2}\text{ bits}}$$

$13\cdot 3 = 39$

$$= 3\cdot 0\cdot 2^4 + (3\cdot 3 + 1\cdot 0)\cdot 2^2 + 1\cdot 3$$

$$= 0 + 9\cdot 4 + 3 = 39.$$

$$(a^1+a^0)(b^1+b^0) = a^1\cdot b^1 + a^1b^0 + a^0b^1 + a^0b^0$$

$$\Rightarrow \boxed{a^1b^0 + a^0b^1 = (a^1+a^0)\cdot(b^1+b^0) - a^1\cdot b^1 - a^0b^0}$$

$a^1 = 3$
$a^0 = 1$
$b^1 = 0$
$b^0 = 3$

LHS $= 3\cdot 3 + 1\cdot 0 = 9$

RHS $= (3+1)(0+3) - 3\cdot 0 - 1\cdot 3 = 12 - 3 = 9.$