CS4/531
Spring 1997
Homework #2

(No credit will be given for unsupported answers.)

1. [CLR, 28.1-4] Prove that any sorting network on n inputs has depth at least $\lg n$.
2. [CLR, 28.1-5] Prove that the number of comparators in any sorting network is $\Omega(n \lg n)$.
3. [CLR, 28-1] A comparison network is a **transposition network** if each comparator connects adjacent lines, as in the alternative 8 element bitonic sorter given in class.
   a) Prove that any transposition network with n inputs that sorts has $\Omega(n^2)$ comparators.
   b) Prove that a transposition network with n inputs is a sorting network if and only if it sorts the sequence $\langle n, n-1, n-2, \ldots, 1 \rangle$. (Hint: use an induction argument.)
4. [Quinn, 10-10] Prove or disprove: All sequences containing fewer than four elements are bitonic sequences.
5. [Akl, 3.1] Consider a combinational circuit C whose processors are comparators. The circuit takes a sequence of numbers $\{x_1, x_2, \ldots, x_n\}$ as input and produces these same numbers as output in some order. Let the sequence $\{y_1, y_2, \ldots, y_n\}$ denote the circuit's output. Now suppose that F is a monotonically increasing function. Show that if C is given the sequence $\{F(x_1), F(x_2), \ldots, F(x_n)\}$ as input, it produces $\{F(y_1), F(y_2), \ldots, F(y_n)\}$ as output.
6. [Akl, 3.2] Let C be a comparison circuit as just defined in 5. When given an input sequence of n binary digits in random order, C produces an output sequence of 0's followed by 1's. That is, C sorts any permutation of 0's and 1's. Show that such a circuit can correctly sort any sequence of arbitrary numbers (i.e., not necessarily 0's and 1's). Note: this is known as the *0-1 sorting principle*.
7. Use the *0-1 sorting principle* to prove that the following odd-even merging network correctly merges sorted sequences $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$.

a) The odd-indexed elements of the two sequences, that is $\{x_1, x_3, ..., x_{n-1}\}$ and $\{y_1, y_3, ..., y_{n-1}\}$, are merged to produce a *sorted* sequence $\{u_1, u_2, ..., u_n\}$.

b) Simultaneously, the even-indexed elements of the two sequences, that is $\{x_2, x_4, ..., x_n\}$ and $\{y_2, y_4, ..., y_n\}$, are merged to produce a *sorted* sequence $\{v_1, v_2, ..., v_n\}$.

c) Finally, the output sequence $\{z_1, z_2, ..., z_{2n}\}$ is obtained from

$$z_1 = u_1, z_{2n} = v_n, z_{2i} = \min(u_{i+1}, v_i), z_{2i+1} = \max(u_{i+1}, v_i), \text{ for } i = 1,2,\ldots,n-1.$$