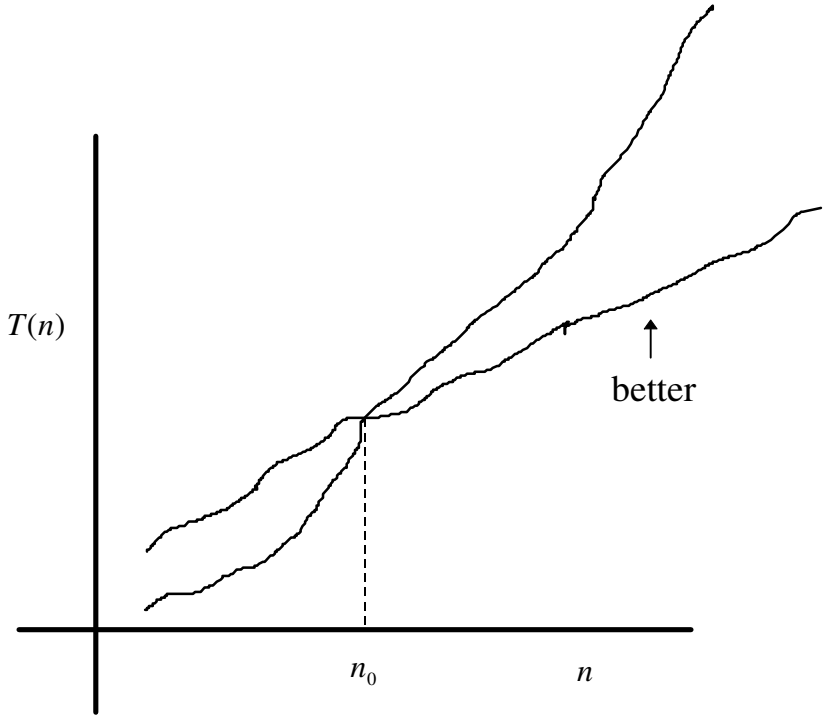
Asymptotic Analysis

1. Ignore machine-dependent constants.
2. Look at *growth* of $T(n)$ as $n \to \infty$.
3. Θ notation:
   - drop low-order terms
   - ignore leading constants
   - e.g., $3n^3 + 10n^2 + n + 17$ is $\Theta(n^3)$
4. As n gets large, a $\Theta(n^3)$ time algorithm beats a $\Theta(n^2)$ time algorithm.



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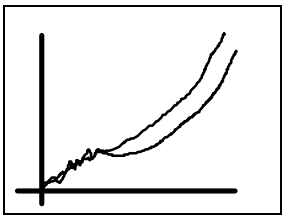
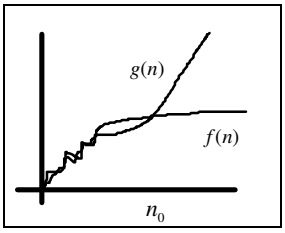
Asymptotic Notation

Suppose f and g are positive functions of n, then

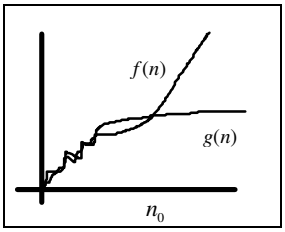
1. $f(n)$ is $\Theta(g(n))$ iff there exists constants $c_1, c_2,$ and n_0 such that $c_1 g(n) \le f(n) \le c_2 g(n)$ whenever $n \ge n_0$.



2. $f(n)$ is $O(g(n))$ iff there exists constants c and n_0 such that $f(n) \le cg(n)$ whenever $n \ge n_0$.



3. $f(n)$ is $\Omega(g(n))$ iff there exists constants c and n_0 such that $cg(n) \le f(n)$ whenever $n \ge n_0$.



- Use asymptotic notation to simplify analysis and capture *growth rate*.
- Want *simplest* and *best* function inside of $O, o, \Omega, \omega,$ and Θ.

EXAMPLE:

Given $f(t) = 5 + \sin(t)$ and $g(t) = 1,$ then

$5 + \sin(t) = \Theta(1)$

since

$4 * 1 \le 5 + \sin(t) \le 6 * 1$

Note (also) that: $f(t) = O(1), f(t) = \Omega(1),$ but **b**est is $f(t) = \Theta(1)$

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EXAMPLE: $\sum_{k=1}^{n} k^p = \Theta(n^{p+1}), p > 1$ a fixed constant.

Rightside:

$$\sum_{k=1}^{n} k^p \le n \times n^p = n^{p+1}$$

Note: n is total number of terms

Note: n^p is the largest term

Leftside:

$$\sum_{k=1}^{\lfloor n/2 \rfloor} k^p + \sum_{k=\lfloor n/2 \rfloor+1}^{n} k^p = \sum_{k=1}^{n} k^p$$

So, $\sum_{k=\lfloor n/2 \rfloor+1}^{n} k^p \le \sum_{k=1}^{n} k^p$

$$(n - \lfloor n/2 \rfloor)(\lfloor n/2 \rfloor + 1)^p \le \sum_{k=1}^{n} k^p$$

Note: $(n - \lfloor n/2 \rfloor)$ is the total number of terms

Note: $(\lfloor n/2 \rfloor + 1)^p$ is the smallest term

$(n/2)(n/2)^p \le \sum_{k=1}^{n} k^p$ since $n/2 \le n - \lfloor n/2 \rfloor$ and $n/2 \le \lfloor n/2 \rfloor + 1$

$$n^{p+1} / 2^{p+1} \le \sum_{k=1}^{n} k^p$$

Conclusion: $\therefore n^{p+1} / 2^{p+1} \le \sum_{k=1}^{n} k^p \le n^{p+1} \Rightarrow \sum_{k=1}^{n} k^p = \Theta(n^{p+1})$

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Asymptotic Analysis and Limits

To help determine the relationship between f and g, we can examine $\lim_{n\to\infty} \frac{f(n)}{g(n)} = L$

Possibilities:

1. $L = 0 \Rightarrow f = O(g)$ and $f \neq \Theta(g)$
2. $L = \infty \Rightarrow f = \Omega(g)$ and $f \neq \Theta(g)$
3. $L \neq 0$ is finite $\Rightarrow f = \Theta(g)$ (also $f = O(g)$ and $f = \Omega(g)$
4. no limit $\Rightarrow$ this approach is of no help (e.g., example with $\sin(t)$)

Example:

$$f(n) = \frac{n(n+1)}{2} = \Theta(n^2) = \Theta(g(n)) \text{ since}$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{n^2+n}{2n^2} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2} \to \frac{1}{2}$$

Example: If $P(n)$ is a polynomial of degree d, then $P(n) = \Theta(n^d)$.

Example: Compare n^{100} and 2^n. {Remember that $\frac{d}{dx} e^{f(x)} = e^{f(x)} f'(x)$.}

$$\lim_{n\to\infty} \frac{2^n}{n^{100}} = \lim_{n\to\infty} \frac{e^{\ln 2^n}}{n^{100}} \to \text{L' Hospital' s Rule } 100\times \to \lim_{n\to\infty} \frac{(\ln 2)^{100} 2^n}{100!} \to \infty$$

So, $n^{100} = O(2^n)$ or $2^n = \Omega(n^{100})$, **b**ut $n^{100} \neq \Theta(2^n)$

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Logarithms (changing bases)

Example:

$$f(n) = \log_e(n), g(n) = n$$

$$\lim_{n\to\infty} \frac{n}{\log_e n} = \lim_{n\to\infty} \frac{1}{1/n} = \lim_{n\to\infty} n = \infty$$

$\therefore \quad \log_e n = O(n)$ since $n > \log_e n$ for large n

Notation: $\log_e x$ is written as ln x
$\log_2 x$ is written as $\lg x$
$\log_{10} x$ is written as $\log x$

Fact: $\log_b x = \log_b a \log_a x$

Conclusion: Since $\log_b a$ is a constant, it means that $\log_b x = \Theta(\log_a x)$, which tells us that the base of a logarithm is irrelevant inside the asymptotic notation.

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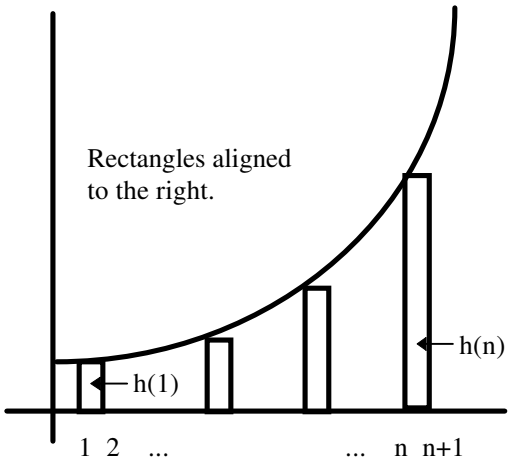
Summations vs. Integrals

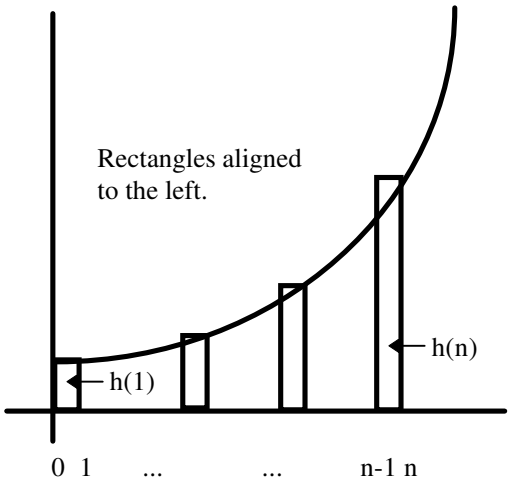
Frequent Situation: Given $f(n)$ as the sum of simpler functions dependent on n.

So, $f(n) = h(1) + h(2) + \ldots + h(n)$

E.g., $f(n) = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} = \Theta(n^2)$

Consider $f(n) = \sum_{i=1}^{n} h(i)$, h is nondecreasing





$$\sum_{i=1}^{n} h(i) \le \int_{1}^{n+1} h(t)dt \qquad \int_{0}^{n} h(t)dt \le \sum_{i=1}^{n} h(i)$$

$$\text{So, } \int_{0}^{n} h(t)dt \le \sum_{i=1}^{n} h(i) \le \int_{1}^{n+1} h(t)dt$$

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Example: Prove that $\int_0^n t\,dt \le \sum_{i=1}^{n} i \le \int_1^{n+1} t\,dt$

$$\frac{t^2}{2}\Big|_0^n \le \sum_{i=1}^{n} i \le \frac{t^2}{2}\Big|_1^{n+1}$$

$$\frac{n^2}{2} \le \sum_{i=1}^{n} i \le \frac{(n+1)^2}{2} - \frac{1}{2}$$

$$\le \frac{(n+1)^2}{2} = \frac{1}{2}n^2 + n + \frac{1}{2}$$

$$\le \frac{1}{2}n^2 + n^2 + n^2, \text{ for } n \ge 1$$

$$\therefore \ \frac{1}{2}n^2 \le \sum_{i=1}^{n} i \le \frac{5}{2}n^2$$

So, $f(n) = \sum_{i=1}^{n} i = \Theta(n^2)$

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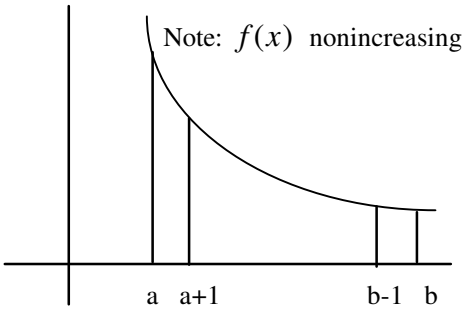
Example: Find the asymptotic complexity of $\sum_{i=1}^{n} \frac{1}{k}$



$$\sum_{k=a+1}^{b} f(k) \le \int_{a}^{b} f(x)dx \le \sum_{k=a}^{b-1} f(k)$$

1. $$\int_{1}^{n+1} \frac{1}{x}dx \le \sum_{k=1}^{n} f(k) \Rightarrow \int_{1}^{n+1} \frac{1}{x}dx = \ln x\Big|_{1}^{n+1} = \ln(n+1) - \ln 1 = \ln(n+1) \le \sum_{k=1}^{n} f(k)$$

2. $$\sum_{k=1}^{n} \frac{1}{k} \le \int_{0}^{n} \frac{1}{x} = \ln x\Big|_{0}^{n} = \infty \Rightarrow \text{ NO GOOD!}$$

$$\sum_{k=2}^{n} \frac{1}{k} \le \int_{1}^{n} \frac{1}{x}dx = \ln n \Rightarrow 1 + \sum_{k=2}^{n} \frac{1}{k} = \sum_{k=1}^{n} \frac{1}{k} \le \ln n + 1$$

So, $\ln n \le \ln(n+1) \le \sum_{k=1}^{n} f(k) \le \ln n + 1 \le 2\ln n,$ for n large enough.

Therefore, $\sum_{k=1}^{n} \frac{1}{k} = \Theta(\ln n)$.

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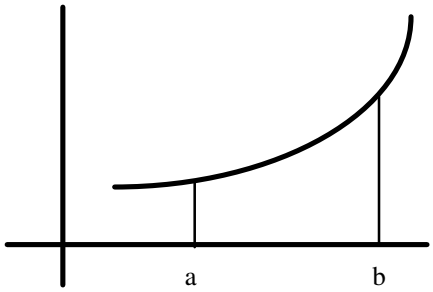
Example: an alternative solution to $\sum_{k=1}^{n} k^p = \Theta(n^{p+1})$

$f(x) = x^p$

$f'(x) = px^{p-1} > 0 \Rightarrow$ increasing function



$$\sum_{k=a}^{b-1} f(k) \le \int_a^b f(x)\,dx \le \sum_{k=a+1}^{b} f(k)$$

Left Right

Left:

$$\sum_{k=1}^{n} k^p \le \int_1^{n+1} x^p\,dx = \frac{x^{p+1}}{p+1}\Big|_1^{n+1}$$

$$= ((n+1)^{p+1} - 1)/(p+1)$$

$$\le (n+1)^{p+1}/(p+1)$$

$$\le (2n)^{p+1}/(p+1) \text{ since } n+1 \le 2n$$

$$= \frac{2^{p+1} n^{p+1}}{p+1}$$

Right: $\int_0^n x^p\,dx \le \sum_{k=1}^{n} k^p \Rightarrow \frac{1}{p+1} n^{p+1} \le \sum_{k=1}^{n} k^p$

Conclusion:

$$\therefore \quad \frac{1}{p+1} n^{p+1} \le \sum_{k=1}^{n} k^p \le \frac{2^{p+1}}{p+1} n^{p+1}$$

$$\Rightarrow \quad \sum_{k=1}^{n} k^p = \Theta(n^{p+1})$$

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