CS531 HW1 Solution

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February 24, 1998

1. This problem can be first categorized into two parts roughly, the exponential and polynomial, where increase exponential functions > polynomial functions > constant > decrease exponential functions.

   - Increase exponential : $(3/2)^n, n!$
   - Polynomial : $n, n^{1/2}, \log n, \log(\log n), \log^2 n$
   - Constant : 4
   - Decrease exponential : $(1/3)^n$

   Over here, I will just point out some difficult ones

   (a) $n! > (3/2)^n$:
   $n! = n * (n-1) * ....1$
   $(3/2)^n = (3/2) * (3/2)...(3/2)$
   each of them has n entries but every entry in $n!$ is greater than $(3/2)$ except 1 (a constrant fraction) as $n \approx \infty$, so $n! > (3/2)^n$.

   (b) $\log n, \log\log n$ and $\log^2 n$: Thinking of $\log n = x$ the previous 3 entries will be $x, \log x$ and x^2, respectively. Therefore $\log^2 n > \log n > \log\log n$.

   (c) $\log^2 n$ and $n^{1/2}$: If you assign $n = 2^k$, then $\log^2 n = k^2$ and $n^{1/2} = 2^{k/2}$, so $n^{1/2} > \log^2$.

   Final sequence : $n! > (3/2)^n > n > n^{1/2} > \log^2 n > \log n > \log\log n > 4 > (1/3)^n$

2. (a) FALSE. Counterexample : $f(n) = n, g(n) = n^2$.

   (b) FALSE. Counterexample : $f(n) = n, g(n) = n^2$.

   (c) FALSE. Counterexample : $f(n) = 1/n$

   (d) TRUE. $f(n) = O(g(n) \Rightarrow f(n) \leq C \cdot g(n)$ for $n > N$ and C is a constant. Therefore $1/C \cdot f(n) \leq g(n)$ for $n > N$ and if we take $C' = 1/C$, we can get $g(n) = \Omega(f(n))$.

   (e) TRUE. If we assign $g(n) = o(f(n)) \Rightarrow 0 \leq g(n) < c \cdot f(n)$, which means $f(n) \leq g(n) + f(n) < (c+1)f(n) \Rightarrow f(n) \leq g(n) + f(n) \leq (c+1)f(n)$, So $f(n) + o(f(n)) = \Theta(f(n))$.

3. Over here I just prove the case of constants > 1, if they are not in this case, you can prove it in the similar way yourself.

   (a) $\log^k n = O(n^e)$ by applying L'Hospitals rule

   $$\lim_{n\to\infty} \frac{\log^k n}{n^e} = 0$$

   (b) $n^k = O(c^n)$: by applying L'Hospitals rule

   $$\lim_{n\to\infty} \frac{n^k}{c^n} = 0$$

(c) $2^{n/2} = O(2^n)$: by applying L'Hospitals rule

$$\lim_{n\to\infty} \frac{2^{n/2}}{2^n} = \lim_{n\to\infty} \frac{1}{2^{n/2}} = 0 \Rightarrow 2^{n/2} = O(2^n)$$

4. Take $N = 1$ and $C = 18$, we can get for every $n > N$, $17n^{1/6} \leq Cn^{1/5}$.
5. For monotonic decrease function $f(k)$, $\sum_{k=m}^{n} f(k)$ is bounded by

$$\int_{m}^{n+1} f(x)dx \leq \sum_{k=m}^{n} f(k) \leq \int_{m-1}^{n} f(x)dx$$

Lower bound : $\sum_{k=1}^{n}(1/k) \geq \int_{1}^{n+1} dx/x = \ln(n+1)$
Upper Bound : $\sum_{k=1}^{n}(1/k) = 1 + \sum_{k=2}^{n}(1/k) \leq 1 + \int_{1}^{n} dx/x = \ln n + 1$
Therefore

$$\sum_{k=1}^{n} 1/k = \Theta(\lg n)$$