SOLVING A TRIDIAGONAL SYSTEM OF EQUATIONS USING MPI

Course: CSE 633 Parallel Algorithms

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CONTENTS

What is a Tridiagonal Matrix?

- A tridiagonal matrix has nonzero elements on the main diagonal, the first diagonal below this (sub-diagonal), and the first diagonal (super-diagonal) above the main diagonal only.
- An example of tridiagonal matrix is as follows:

```
[[2 8 0 0 0]
 [3 7 5 0 0]
 [0 3 6 5 0]
 [0 0 5 3 8]
 [0 0 0 2 8]]
```

- A tridiagonal matrix can be used to denote a system of linear equations of the form $AX = b$ as follows:

```
[[2 8 0 0 0] [['x1']     [[6]
 [3 7 5 0 0]  ['x2']      [9]
 [0 3 6 5 0]  ['x3']  =   [6]
 [0 0 5 3 8]  ['x4']      [1]
 [0 0 0 2 8]] ['x5']]     [1]]
```

- The equations for this system can be written as:
  - $2x_2 + 8x_3 = 6$
  - $3x_1 + 7x_2 + 5x_3 = 9$
  - $3x_2 + 6x_3 + 5x_4 = 6$
  - $5x_3 + 3x_4 + 8x_5 = 1$
  - $2x_4 + 8x_5 = 1$

Possible Solutions

- Such systems of linear equations can be solved using the traditional techniques such as:
  - Cramer's rule (Ideally *feasible* for n < 9)
  - Gauss Elimination (Okay for n < 300)

- For solving tridiagonal systems of equations, special techniques exist which utilize the properties of these matrices to reduce calculations.
  - Cyclic Reduction
  - Prefix Product

L, D, U Matrices

- The tridiagonal matrix is interpreted as:
  - L : Sub-diagonal (Green)
  - D: Main diagonal (Yellow)
  - U: Super-diagonal (Blue)

```
[[2 8 0 0 0] [['x1']     [[6]
 [3 7 5 0 0]  ['x2']      [9]
 [0 3 6 5 0]  ['x3']  =   [6]
 [0 0 5 3 8]  ['x4']      [1]
 [0 0 0 2 8]] ['x5']]     [1]]
      A           X          b
```

- Here, L_1 and U_5 are both identity matrices of size 1.

Prefix Operations

- Prefix operations gives us the intermediate results on performing a particular operation on data.
- For example, performing the prefix sum operation on the following data:

1	2	3	4	5	6	7	8	9	10
1	3	6	10	15	21	28	36	45	55

- This concept comes in handy when solving the problem at hand.
- The solution for the n^{th} can be calculated by multiplying the n^{th} prefix with the first one.

Parallel Prefix

- In a parallel setting, the parallel prefix operation takes O(logn) operations.
- In each step, the processors communicate with other processors which are apart from it in powers of two.
  - Step 1: 2^0 apart
  - Step 2: 2^1 apart
  - Step 3: 2^2 apart and so on.
- Each processor maintains 2 values, its local prefix and the total prefix, denoted by S^p and T^p in this case.
- Only T^p is exchanged by processors in all steps.
- If processor *k* receives the T^p_x value from processor *x*, then:
  - If rank(k) < rank(x):
    - $T^p_k = T^p_k . T^p_x$
  - If rank(k) > rank(x):
    - $T^p_k = T^p_k . T^p_x$
    - $S^p_k = S^p_k . T^p_x$

Parallel Prefix Example

Processor 1	Initial Data	1	2	3	4

Processor 2	Initial Data	5	6	7	8

Processor 3	Initial Data	9	10	11	12

Processor 4	Initial Data	13	14	15	16

Parallel Prefix Example

Processor 1					
	Initial Data	1	2	3	4
	Serial Prefix	1	3	6	10

Processor 2					
	Initial Data	5	6	7	8
	Serial Prefix	5	11	18	26

Processor 3					
	Initial Data	9	10	11	12
	Serial Prefix	9	19	30	42

Processor 4					
	Initial Data	13	14	15	16
	Serial Prefix	13	27	42	58

Parallel Prefix Example

Processor 1					
	Initial Data	1	2	3	4
	Serial Prefix	1	3	6	10
	Parallel Prefix			S^p_k	T^p_k
		Initialize S^p_k and T^p_k		10	10

Processor 2					
	Initial Data	5	6	7	8
	Serial Prefix	5	11	18	26
	Parallel Prefix			S^p_k	T^p_k
		Initialize S^p_k and T^p_k		26	26

Processor 3					
	Initial Data	9	10	11	12
	Serial Prefix	9	19	30	42
	Parallel Prefix			Spk	T^p_k
		Initialize S^p_k and T^p_k		42	42

Processor 4					
	Initial Data	13	14	15	16
	Serial Prefix	13	27	42	58
	Parallel Prefix			S^p_k	T^p_k
		Initialize S^p_k and T^p_k		58	58

Parallel Prefix Example

Processor 1					
	Initial Data	1	2	3	4
	Serial Prefix	1	3	6	10
	Parallel Prefix	Communication With		S^p_k	T^p_k
		Initialize S^p_k and T^p_k		10	10
		Processor 2		10	36

Processor 2					
	Initial Data	5	6	7	8
	Serial Prefix	5	11	18	26
	Parallel Prefix	Communication With		S^p_k	T^p_k
		Initialize S^p_k and T^p_k		26	26
		Processor 1		36	36

Processor 3					
	Initial Data	9	10	11	12
	Serial Prefix	9	19	30	42
	Parallel Prefix	Communication With		Spk	T^p_k
		Initialize S^p_k and T^p_k		42	42
		Processor 4		42	100

Processor 4					
	Initial Data	13	14	15	16
	Serial Prefix	13	27	42	58
	Parallel Prefix	Communication With		S^p_k	T^p_k
		Initialize S^p_k and T^p_k		58	58
		Processor 3		100	100

Parallel Prefix Example

Processor 1					
	Initial Data	1	2	3	4
	Serial Prefix	1	3	6	10
	Parallel Prefix	Communication With		S^p_k	T^p_k
		Initialize S^p_k and T^p_k		10	10
		Processor 2		10	36
		Processor 3		10	136

Processor 2					
	Initial Data	5	6	7	8
	Serial Prefix	5	11	18	26
	Parallel Prefix	Communication With		S^p_k	T^p_k
		Initialize S^p_k and T^p_k		26	26
		Processor 1		36	36
		Processor 4		36	136

Processor 3					
	Initial Data	9	10	11	12
	Serial Prefix	9	19	30	42
	Parallel Prefix	Communication With		Spk	T^p_k
		Initialize S^p_k and T^p_k		42	42
		Processor 4		42	100
		Processor 1		78	136

Processor 4					
	Initial Data	13	14	15	16
	Serial Prefix	13	27	42	58
	Parallel Prefix	Communication With		S^p_k	T^p_k
		Initialize S^p_k and T^p_k		58	58
		Processor 3		100	100
		Processor 2		136	136

Parallel Prefix Example

Processor 1					
	Initial Data	1	2	3	4
	Serial Prefix	1	3	6	10
	Parallel Prefix	Communication With		S^p_k	T^p_k
		Initialize S^p_k and T^p_k		10	10
		Processor 2		10	36
		Processor 3		10	136
	Final Data				

Processor 2					
	Initial Data	5	6	7	8
	Serial Prefix	5	11	18	26
	Parallel Prefix	Communication With		S^p_k	T^p_k
		Initialize S^p_k and T^p_k		26	26
		Processor 1		36	36
		Processor 4		36	136
	Final Data				

Processor 3					
	Initial Data	9	10	11	12
	Serial Prefix	9	19	30	42
	Parallel Prefix	Communication With		Spk	T^p_k
		Initialize S^p_k and T^p_k		42	42
		Processor 4		42	100
		Processor 1		78	136
	Final Data				

Processor 4					
	Initial Data	13	14	15	16
	Serial Prefix	13	27	42	58
	Parallel Prefix	Communication With		S^p_k	T^p_k
		Initialize S^p_k and T^p_k		58	58
		Processor 3		100	100
		Processor 2		136	136
	Final Data				

Parallel Prefix Example

Processor 1					
	Initial Data	1	2	3	4
	Serial Prefix	1	3	6	10
	Parallel Prefix	Communication With		S^p_k	T^p_k
		Initialize S^p_k and T^p_k		10	10
		Processor 2		10	36
		Processor 3		10	136
	Final Data	1	3	6	10

Processor 2					
	Initial Data	5	6	7	8
	Serial Prefix	5	11	18	26
	Parallel Prefix	Communication With		S^p_k	T^p_k
		Initialize S^p_k and T^p_k		26	26
		Processor 1		36	36
		Processor 4		36	136
	Final Data	15	21	28	36

Processor 3					
	Initial Data	9	10	11	12
	Serial Prefix	9	19	30	42
	Parallel Prefix	Communication With		Spk	T^p_k
		Initialize S^p_k and T^p_k		42	42
		Processor 4		42	100
		Processor 1		78	136
	Final Data	45	55	66	78

Processor 4					
	Initial Data	13	14	15	16
	Serial Prefix	13	27	42	58
	Parallel Prefix	Communication With		S^p_k	T^p_k
		Initialize S^p_k and T^p_k		58	58
		Processor 3		100	100
		Processor 2		136	136
	Final Data	91	105	120	136

Sequential Algorithm

- Get L, D, U, X, y matrices
- Compute U_i^{-1} f and the following:

$$\tilde{D}_i = -U_i^{-1} D_i$$
$$\tilde{L}_i = -U_i^{-1} L_i$$
$$\tilde{b}_i = U_i^{-1} b_i$$

- Construct B_i as

$$B_i = \begin{bmatrix} \tilde{D}_i & \tilde{L}_i & \tilde{b}_i \\ I & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Perform prefix product operation on the B_i matrices and store it in S where S_i which denotes the prefix product of i^{th} B matrix.

Sequential Algorithm

- After computing the prefix of each matrix in S^s, the solutions can be calculated using:

$$Y_{i+1} = B_i Y_i = B_i B_{i-1} Y_{i-1} = \cdots = B_i B_{i-1} B_{i-2} \cdots B_1 Y_1$$
$$= S_i Y_1 \quad \text{where } S_i = B_i B_{i-1} B_{i-2} \cdots B_1.$$
$$= \begin{bmatrix} x_{i+1} \\ x_i \\ 1 \end{bmatrix}$$

- Here, Y1 is calculated as follows: $Y_1 = [x_1\ x_0\ 1]^T$

  where x_0 is a zero matrix from boundary conditions and x_1 is

$$x_1 = -[S_N^{11}]^{-1} S_N^{13}$$

Sequential Example

- Suppose we have the following three equations we want to solve:

$$
\begin{aligned}
1x_1 + 1x_2 &= 6 \\
2x_1 + 7x_2 + 8x_3 &= 9 \\
3x_2 + 5x_3 &= 6
\end{aligned}
$$

Sequential Example

- Suppose we have the following three equations we want to solve:

$$
\begin{aligned}
1x_1 + 1x_2 &= 6 \\
2x_1 + 7x_2 + 8x_3 &= 9 \\
3x_2 + 5x_3 &= 6
\end{aligned}
$$

X	$[x_1, \quad x_2, \quad x_3]$
b	
L	
D	
U	

Sequential Example

- Suppose we have the following three equations we want to solve:

$$\begin{aligned} 1x_1 + 1x_2 &= 6 \\ 2x_1 + 7x_2 + 8x_3 &= 9 \\ 3x_2 + 5x_3 &= 6 \end{aligned}$$

X	$[x_1, \quad x_2, \quad x_3]$
b	[6, 9, 6]
L	[1, 2, 3]
D	[1, 7, 5]
U	[1, 8, 1]

Sequential Example

- Suppose we have the following three equations we want to solve:

$$1x_1 + 1x_2 = 6$$
$$2x_1 + 7x_2 + 8x_3 = 9$$
$$3x_2 + 5x_3 = 6$$

X	$[x_1,$	$x_2,$	$x_3]$
b	[6,	9,	6]
L	[1,	2,	3]
D	[1,	7,	5]
U	[1,	8,	1]
U^{-1}			

$$inv(1 \times 1\ matrix) = \frac{1}{value\ in\ 1 \times 1\ matrix}$$

Sequential Example

- Suppose we have the following three equations we want to solve:

$$1x_1 + 1x_2 = 6$$
$$2x_1 + 7x_2 + 8x_3 = 9$$
$$3x_2 + 5x_3 = 6$$

X	$[x_1,$	$x_2,$	$x_3]$
b	[6,	9,	6]
L	[1,	2,	3]
D	[1,	7,	5]
U	[1,	8,	1]
U^{-1}	[1,	0.125,	1]

$$inv(1 \times 1\ matrix) = \frac{1}{value\ in\ 1 \times 1\ matrix}$$

Sequential Example

- Suppose we have the following three equations we want to solve:

$$1x_1 + 1x_2 = 6$$
$$2x_1 + 7x_2 + 8x_3 = 9$$
$$3x_2 + 5x_3 = 6$$

X	$[x_1,$	$x_2,$	$x_3]$
b	[6,	9,	6]
L	[1,	2,	3]
D	[1,	7,	5]
U	[1,	8,	1]
U^{-1}	[1,	0.125,	1]
D~			

$$\tilde{D}_i = -U_i^{-1} D_i$$

Sequential Example

- Suppose we have the following three equations we want to solve:

$$
\begin{aligned}
1x_1 + 1x_2 &= 6 \\
2x_1 + 7x_2 + 8x_3 &= 9 \\
3x_2 + 5x_3 &= 6
\end{aligned}
$$

X	$[x_1,$	$x_2,$	$x_3]$
b	[6,	9,	6]
L	[1,	2,	3]
D	[1,	7,	5]
U	[1,	8,	1]
U^{-1}	[1,	0.125,	1]
D~	[-1,	-0.875,	-5]

$$\tilde{D}_i = -U_i^{-1} D_i$$

Sequential Example

- Suppose we have the following three equations we want to solve:

$$1x_1 + 1x_2 = 6$$
$$2x_1 + 7x_2 + 8x_3 = 9$$
$$3x_2 + 5x_3 = 6$$

X	$[x_1,$	$x_2,$	$x_3]$
b	[6,	9,	6]
L	[1,	2,	3]
D	[1,	7,	5]
U	[1,	8,	1]
U^{-1}	[1,	0.125,	1]
D˜	[-1,	-0.875,	-5]
L˜			

$$\tilde{L}_i = -U_i^{-1} L_i$$

Sequential Example

- Suppose we have the following three equations we want to solve:

$$
\begin{aligned}
1x_1 + 1x_2 &= 6 \\
2x_1 + 7x_2 + 8x_3 &= 9 \\
3x_2 + 5x_3 &= 6
\end{aligned}
$$

X	$[x_1,$	$x_2,$	$x_3]$
b	[6,	9,	6]
L	[1,	2,	3]
D	[1,	7,	5]
U	[1,	8,	1]
U^{-1}	[1,	0.125,	1]
D˜	[-1,	-0.875,	-5]
L˜	[-1,	-0.25,	-3]

$$\tilde{L}_i = -U_i^{-1} L_i$$

Sequential Example

- Suppose we have the following three equations we want to solve:

$$
\begin{aligned}
1x_1 + 1x_2 &= 6 \\
2x_1 + 7x_2 + 8x_3 &= 9 \\
3x_2 + 5x_3 &= 6
\end{aligned}
$$

X	$[x_1,$	$x_2,$	$x_3]$
b	[6,	9,	6]
L	[1,	2,	3]
D	[1,	7,	5]
U	[1,	8,	1]
U^{-1}	[1,	0.125,	1]
D˜	[-1,	-0.875,	-5]
L˜	[-1,	-0.25,	-3]
b˜			

$$\tilde{b}_i = U_i^{-1} b_i$$

Sequential Example

- Suppose we have the following three equations we want to solve:

$$
\begin{aligned}
1x_1 + 1x_2 &= 6 \\
2x_1 + 7x_2 + 8x_3 &= 9 \\
3x_2 + 5x_3 &= 6
\end{aligned}
$$

X	$[x_1,$	$x_2,$	$x_3]$
b	[6,	9,	6]
L	[1,	2,	3]
D	[1,	7,	5]
U	[1,	8,	1]
U^{-1}	[1,	0.125,	1]
D˜	[-1,	-0.875,	-5]
L˜	[-1,	-0.25,	-3]
b˜	[6,	1.125,	6]

$$\tilde{b}_i = U_i^{-1} b_i$$

Sequential Example

- Suppose we have the following three equations we want to solve:

$$1x_1 + 1x_2 = 6$$
$$2x_1 + 7x_2 + 8x_3 = 9$$
$$3x_2 + 5x_3 = 6$$

X	[x_1, x_2, x_3]
b	[6, 9, 6]
L	[1, 2, 3]
D	[1, 7, 5]
U	[1, 8, 1]
U^{-1}	[1, 0.125, 1]
D˜	[-1, -0.875, -5]
L˜	[-1, -0.25, -3]
b˜	[6, 1.125, 6]

$$\tilde{b}_i = U_i^{-1} b_i$$

Sequential Example

D˜	[-1, -0.875, -5]
L˜	[-1, -0.25, -3]
b˜	[6, 1.125, 6]

	i = 1	i = 2	i = 3

Sequential Example

D˜	[-1, -0.875, -5]
L˜	[-1, -0.25, -3]
b˜	[6, 1.125, 6]

	i = 1	i = 2	i = 3
B			

$$B_i = \begin{bmatrix} \tilde{D}_i & \tilde{L}_i & \tilde{b}_i \\ I & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Sequential Example

D˜	[-1, -0.875, -5]
L˜	[-1, -0.25, -3]
b˜	[6, 1.125, 6]

	i = 1	i = 2	i = 3
B	$\begin{bmatrix} -1 & -1 & 6 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -0.875 & -0.25 & 1.125 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -5 & -3 & 6 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$B_i = \begin{bmatrix} \tilde{D}_i & \tilde{L}_i & \tilde{b}_i \\ I & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Sequential Example

D˜	[-1, -0.875, -5]
L˜	[-1, -0.25, -3]
b˜	[6, 1.125, 6]

	i = 1	i = 2	i = 3
B	$\begin{bmatrix} -1 & -1 & 6 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -0.875 & -0.25 & 1.125 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -5 & -3 & 6 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
S	[ ]		

Sequential Example

D˜	[-1, -0.875, -5]
L˜	[-1, -0.25, -3]
b˜	[6, 1.125, 6]

	i = 1	i = 2	i = 3
B	$\begin{bmatrix} -1 & -1 & 6 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -0.875 & -0.25 & 1.125 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -5 & -3 & 6 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
S	$\left[\begin{bmatrix} -1 & -1 & 6 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \right]$		

Sequential Example

D˜	[-1, -0.875, -5]
L˜	[-1, -0.25, -3]
b˜	[6, 1.125, 6]

	i = 1	i = 2	i = 3
B	$\begin{bmatrix} -1 & -1 & 6 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -0.875 & -0.25 & 1.125 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -5 & -3 & 6 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
S	$\left[\begin{bmatrix} -1 & -1 & 6 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \right.$		$\left.\right]$

Sequential Example

D˜	[-1, -0.875, -5]
L˜	[-1, -0.25, -3]
b˜	[6, 1.125, 6]

	i = 1	i = 2	i = 3
B	$\begin{bmatrix} -1 & -1 & 6 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -0.875 & -0.25 & 1.125 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -5 & -3 & 6 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
S	$\left[\begin{bmatrix} -1 & -1 & 6 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0.625 & 0.875 & -4.125 \\ -1 & -1 & 6 \\ 0 & 0 & 1 \end{bmatrix}, \right]$		

Sequential Example

D˜	[-1, -0.875, -5]
L˜	[-1, -0.25, -3]
b˜	[6, 1.125, 6]

	i = 1	i = 2	i = 3
B	$\begin{bmatrix} -1 & -1 & 6 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -0.875 & -0.25 & 1.125 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -5 & -3 & 6 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
S	$\left[\begin{bmatrix} -1 & -1 & 6 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0.625 & 0.875 & -4.125 \\ -1 & -1 & 6 \\ 0 & 0 & 1 \end{bmatrix}, \right]$		

Sequential Example

D̃	[-1, -0.875, -5]
L̃	[-1, -0.25, -3]
b̃	[6, 1.125, 6]

	i = 1	i = 2	i = 3
B	$\begin{bmatrix} -1 & -1 & 6 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -0.875 & -0.25 & 1.125 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -5 & -3 & 6 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
S	$\left[\begin{bmatrix} -1 & -1 & 6 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0.625 & 0.875 & -4.125 \\ -1 & -1 & 6 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -0.125 & -1.375 & 8.625 \\ 0.625 & 0.875 & -4.125 \\ 0 & 0 & 1 \end{bmatrix}\right]$		

Sequential Example

D˜	[-1, -0.875, -5]
L˜	[-1, -0.25, -3]
b˜	[6, 1.125, 6]

	i = 1	i = 2	i = 3
B	$\begin{bmatrix} -1 & -1 & 6 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -0.875 & -0.25 & 1.125 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -5 & -3 & 6 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
S	$\left[\begin{bmatrix} -1 & -1 & 6 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0.625 & 0.875 & -4.125 \\ -1 & -1 & 6 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -0.125 & -1.375 & 8.625 \\ 0.625 & 0.875 & -4.125 \\ 0 & 0 & 1 \end{bmatrix}\right]$		
Y_1			

$$Y_1 = [x_1\ x_0\ 1]^T$$

Sequential Example

D˜	[-1, -0.875, -5]
L˜	[-1, -0.25, -3]
b˜	[6, 1.125, 6]

	i = 1	i = 2	i = 3
B	$\begin{bmatrix} -1 & -1 & 6 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -0.875 & -0.25 & 1.125 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -5 & -3 & 6 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
S	$\left[\begin{bmatrix} -1 & -1 & 6 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0.625 & 0.875 & -4.125 \\ -1 & -1 & 6 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -0.125 & -1.375 & 8.625 \\ 0.625 & 0.875 & -4.125 \\ 0 & 0 & 1 \end{bmatrix}\right]$		
x_1			
Y_1			

$$Y_1 = [x_1\ x_0\ 1]^T$$

$$x_1 = -[S_N^{11}]^{-1} S_N^{13}$$

Sequential Example

D˜	[-1, -0.875, -5]
L˜	[-1, -0.25, -3]
b˜	[6, 1.125, 6]

	i = 1	i = 2	i = 3
B	$\begin{bmatrix} -1 & -1 & 6 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -0.875 & -0.25 & 1.125 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -5 & -3 & 6 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
S	$\left[\begin{bmatrix} -1 & -1 & 6 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0.625 & 0.875 & -4.125 \\ -1 & -1 & 6 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -0.125 & -1.375 & 8.625 \\ 0.625 & 0.875 & -4.125 \\ 0 & 0 & 1 \end{bmatrix}\right]$		
x_1			
Y_1			

$$\begin{bmatrix} S_N^{11} & S_N^{12} & S_N^{13} \\ S_N^{21} & S_N^{22} & S_N^{23} \\ 0 & 0 & 1 \end{bmatrix}_N$$

$$Y_1 = [x_1\ x_0\ 1]^T$$

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Sequential Example

D˜	[-1, -0.875, -5]
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x_1	69		
Y_1			

$$\begin{bmatrix} S_N^{11} & S_N^{12} & S_N^{13} \\ S_N^{21} & S_N^{22} & S_N^{23} \\ 0 & 0 & 1 \end{bmatrix}_N$$

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Y_1	$[69, 0, 1]^T$		

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Sequential Example

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x_1	69		
Y_1	$[69, 0, 1]^T$		
Y			

$$Y_{i+1} = B_i Y_i = B_i B_{i-1} Y_{i-1} = \cdots = B_i B_{i-1} B_{i-2} \cdots B_1 Y_1$$
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B	$\begin{bmatrix} -1 & -1 & 6 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -0.875 & -0.25 & 1.125 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -5 & -3 & 6 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
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x_1	69		
Y_1	$[69, 0, 1]^T$		
Y	$[69, 0, 1]^T$		

$$Y_{i+1} = B_i Y_i = B_i B_{i-1} Y_{i-1} = \cdots = B_i B_{i-1} B_{i-2} \cdots B_1 Y_1$$
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B	$\begin{bmatrix} -1 & -1 & 6 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -0.875 & -0.25 & 1.125 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -5 & -3 & 6 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
S	$\left[\begin{bmatrix} -1 & -1 & 6 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0.625 & 0.875 & -4.125 \\ -1 & -1 & 6 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -0.125 & -1.375 & 8.625 \\ 0.625 & 0.875 & -4.125 \\ 0 & 0 & 1 \end{bmatrix}\right]$		
x_1	69		
Y_1	$[69, 0, 1]^T$		
Y	$[69, 0, 1]^T$		

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x_1	69		
Y_1	$[69, 0, 1]^T$		
Y	$[69, 0, 1]^T$		

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B	$\begin{bmatrix} -1 & -1 & 6 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -0.875 & -0.25 & 1.125 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -5 & -3 & 6 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
S	$\left[\begin{bmatrix} -1 & -1 & 6 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0.625 & 0.875 & -4.125 \\ -1 & -1 & 6 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -0.125 & -1.375 & 8.625 \\ 0.625 & 0.875 & -4.125 \\ 0 & 0 & 1 \end{bmatrix}\right]$		
x_1	69		
Y_1	$[69, 0, 1]^T$		
Y	$[69, 0, 1]^T$		

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	i = 1	i = 2	i = 3
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S	$\left[\begin{bmatrix} -1 & -1 & 6 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0.625 & 0.875 & -4.125 \\ -1 & -1 & 6 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -0.125 & -1.375 & 8.625 \\ 0.625 & 0.875 & -4.125 \\ 0 & 0 & 1 \end{bmatrix}\right]$		
x_1	69		
Y_1	$[69, 0, 1]^T$		
Y	$[69, 0, 1]^T$	$[-63, 69, 1]^T$	

$$Y_{i+1} = B_i Y_i = B_i B_{i-1} Y_{i-1} = \cdots = B_i B_{i-1} B_{i-2} \cdots B_1 Y_1$$
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Sequential Example

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L˜	[-1, -0.25, -3]
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	i = 1	i = 2	i = 3
B	$\begin{bmatrix} -1 & -1 & 6 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -0.875 & -0.25 & 1.125 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -5 & -3 & 6 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
S	$\left[\begin{bmatrix} -1 & -1 & 6 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0.625 & 0.875 & -4.125 \\ -1 & -1 & 6 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -0.125 & -1.375 & 8.625 \\ 0.625 & 0.875 & -4.125 \\ 0 & 0 & 1 \end{bmatrix}\right]$		
x_1	69		
Y_1	$[69, 0, 1]^T$		
Y	$[69, 0, 1]^T$	$[-63, 69, 1]^T$	

$$Y_{i+1} = B_i Y_i = B_i B_{i-1} Y_{i-1} = \cdots = B_i B_{i-1} B_{i-2} \cdots B_1 Y_1$$
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Sequential Example

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L˜	[-1, -0.25, -3]
b˜	[6, 1.125, 6]

	i = 1	i = 2	i = 3
B	$\begin{bmatrix} -1 & -1 & 6 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -0.875 & -0.25 & 1.125 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -5 & -3 & 6 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
S	$\left[\begin{bmatrix} -1 & -1 & 6 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0.625 & 0.875 & -4.125 \\ -1 & -1 & 6 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -0.125 & -1.375 & 8.625 \\ 0.625 & 0.875 & -4.125 \\ 0 & 0 & 1 \end{bmatrix}\right]$		
x_1	69		
Y_1	$[69, 0, 1]^T$		
Y	$[69, 0, 1]^T$	$[-63, 69, 1]^T$	

$$Y_{i+1} = B_i Y_i = B_i B_{i-1} Y_{i-1} = \cdots = B_i B_{i-1} B_{i-2} \cdots B_1 Y_1$$
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Sequential Example

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L˜	[-1, -0.25, -3]
b˜	[6, 1.125, 6]

	i = 1	i = 2	i = 3
B	$\begin{bmatrix} -1 & -1 & 6 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -0.875 & -0.25 & 1.125 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -5 & -3 & 6 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
S	$\left[\begin{bmatrix} -1 & -1 & 6 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0.625 & 0.875 & -4.125 \\ -1 & -1 & 6 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -0.125 & -1.375 & 8.625 \\ 0.625 & 0.875 & -4.125 \\ 0 & 0 & 1 \end{bmatrix}\right]$		
x_1	69		
Y_1	[69, 0, 1]^T		
Y	[69, 0, 1]^T	[-63, 69, 1]^T	

$$Y_{i+1} = B_i Y_i = B_i B_{i-1} Y_{i-1} = \cdots = B_i B_{i-1} B_{i-2} \cdots B_1 Y_1$$
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$$= \begin{bmatrix} x_{i+1} \\ x_i \\ 1 \end{bmatrix}$$

Sequential Example

D̃	[-1, -0.875, -5]
L̃	[-1, -0.25, -3]
b̃	[6, 1.125, 6]

	i = 1	i = 2	i = 3
B	$\begin{bmatrix} -1 & -1 & 6 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -0.875 & -0.25 & 1.125 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -5 & -3 & 6 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
S	$\left[\begin{bmatrix} -1 & -1 & 6 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0.625 & 0.875 & -4.125 \\ -1 & -1 & 6 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -0.125 & -1.375 & 8.625 \\ 0.625 & 0.875 & -4.125 \\ 0 & 0 & 1 \end{bmatrix}\right]$		
x_1	69		
Y_1	$[69, 0, 1]^T$		
Y	$[69, 0, 1]^T$	$[-63, 69, 1]^T$	$[39, -63, 1]^T$

$$Y_{i+1} = B_i Y_i = B_i B_{i-1} Y_{i-1} = \cdots = B_i B_{i-1} B_{i-2} \cdots B_1 Y_1$$
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$$= \begin{bmatrix} x_{i+1} \\ x_i \\ 1 \end{bmatrix}$$

Sequential Example

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L˜	[-1, -0.25, -3]
b˜	[6, 1.125, 6]

	i = 1	i = 2	i = 3
B	$\begin{bmatrix} -1 & -1 & 6 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -0.875 & -0.25 & 1.125 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -5 & -3 & 6 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
S	$\left[\begin{bmatrix} -1 & -1 & 6 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0.625 & 0.875 & -4.125 \\ -1 & -1 & 6 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -0.125 & -1.375 & 8.625 \\ 0.625 & 0.875 & -4.125 \\ 0 & 0 & 1 \end{bmatrix}\right]$		
x_1	69		
Y_1	$[69, 0, 1]^T$		
Y	$[69, 0, 1]^T$	$[-63, 69, 1]^T$	$[39, -63, 1]^T$
X			

$$Y_{i+1} = \begin{bmatrix} x_{i+1} \\ x_i \\ 1 \end{bmatrix}$$

Sequential Example

D˜	[-1, -0.875, -5]
L˜	[-1, -0.25, -3]
b˜	[6, 1.125, 6]

	i = 1	i = 2	i = 3
B	$\begin{bmatrix} -1 & -1 & 6 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -0.875 & -0.25 & 1.125 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -5 & -3 & 6 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
S	$\left[\begin{bmatrix} -1 & -1 & 6 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0.625 & 0.875 & -4.125 \\ -1 & -1 & 6 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -0.125 & -1.375 & 8.625 \\ 0.625 & 0.875 & -4.125 \\ 0 & 0 & 1 \end{bmatrix}\right]$		
x_1	69		
Y_1	$[69, 0, 1]^T$		
Y	$[69, 0, 1]^T$	$[-63, 69, 1]^T$	$[39, -63, 1]^T$
X			

$$Y_{i+1} = \begin{bmatrix} x_{i+1} \\ x_i \\ 1 \end{bmatrix}$$

Sequential Example

D˜	[-1, -0.875, -5]
L˜	[-1, -0.25, -3]
b˜	[6, 1.125, 6]

	i = 1	i = 2	i = 3
B	$\begin{bmatrix} -1 & -1 & 6 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -0.875 & -0.25 & 1.125 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -5 & -3 & 6 \\ ID & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
S	$\left[\begin{bmatrix} -1 & -1 & 6 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0.625 & 0.875 & -4.125 \\ -1 & -1 & 6 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -0.125 & -1.375 & 8.625 \\ 0.625 & 0.875 & -4.125 \\ 0 & 0 & 1 \end{bmatrix}\right]$		
x_1	69		
Y_1	$[69, 0, 1]^T$		
Y	$[69, 0, 1]^T$	$[-63, 69, 1]^T$	$[39, -63, 1]^T$
X	$x_1 = 69$	$x_2 = -63$	$x_3 = 39$

$$Y_{i+1} = \begin{bmatrix} x_{i+1} \\ x_i \\ 1 \end{bmatrix}$$

Parallelizing the Algorithm

- Assign N/P rows to each processor. Then in parallel:
- Compute U_i^{-1} f and the following:

$$\tilde{D}_i = -U_i^{-1} D_i$$
$$\tilde{L}_i = -U_i^{-1} L_i$$
$$\tilde{b}_i = U_i^{-1} b_i.$$

- Construct B_i as

$$B_i = \begin{bmatrix} \tilde{D}_i & \tilde{L}_i & \tilde{b}_i \\ I & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Perform prefix product operation locally on the B_i matrices and store it in S^s where S^s_i which denotes the prefix product of i^{th} B matrix from the serial step.

Parallelizing the Algorithm

- The next step is parallel prefix computation.
- After computing the parallel prefix of each matrix in S^s, the solutions can be calculated using:

$$Y_{i+1} = B_i Y_i = B_i B_{i-1} Y_{i-1} = \cdots = B_i B_{i-1} B_{i-2} \cdots B_1 Y_1$$
$$= S_i Y_1 \quad \text{where } S_i = B_i B_{i-1} B_{i-2} \cdots B_1.$$
$$= \begin{bmatrix} x_{i+1} \\ x_i \\ 1 \end{bmatrix}$$

- Here, Y1 is calculated as follows: $Y_1 = [x_1\ x_0\ 1]^T$

  where x_0 is a zero matrix from boundary conditions and

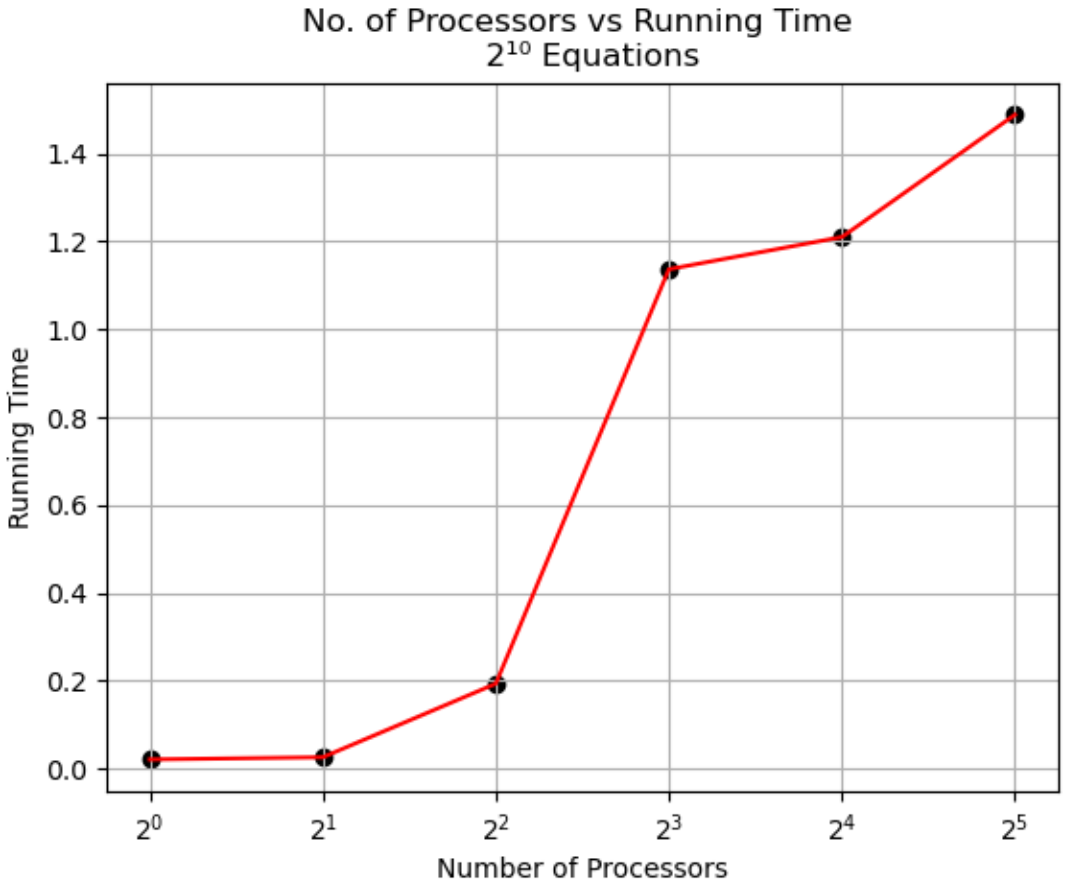
  x_1 is $x_1 = -[S_N^{11}]^{-1} S_N^{13}$

Running Times

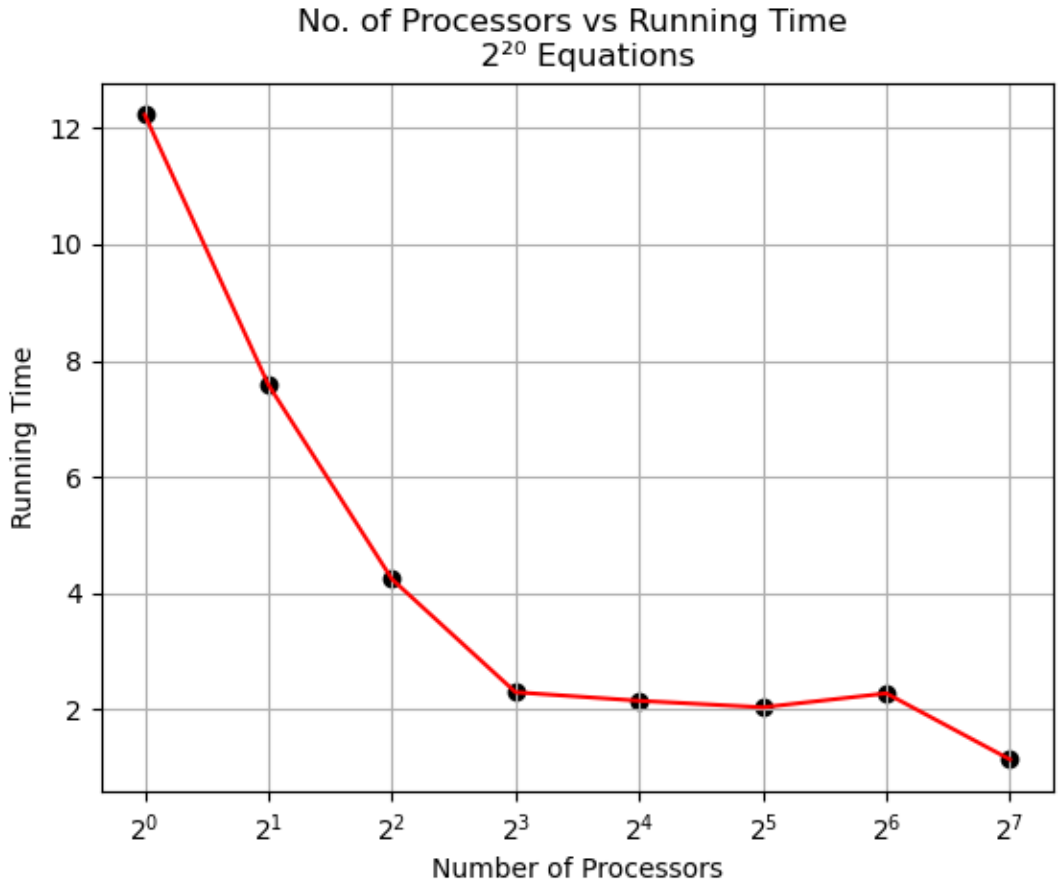
Processors > Equations v	1	2	4	8	16	32	64	128
1,024	0.021	0.0261	0.193	1.136	1.209	1.488	-	-
1,048,576	12.228	7.587	4.243	2.299	2.153	2.040	2.280	1.144

(in seconds)

Graphs


No. of Processors vs Running Time
2^{10} Equations
Running Time
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
2^0
2^1
2^2
2^3
2^4
2^5
Number of Processors

Graphs


No. of Processors vs Running Time
2^{20} Equations
Running Time
12
10
8
6
4
2
2^0
2^1
2^2
2^3
2^4
2^5
2^6
2^7
Number of Processors

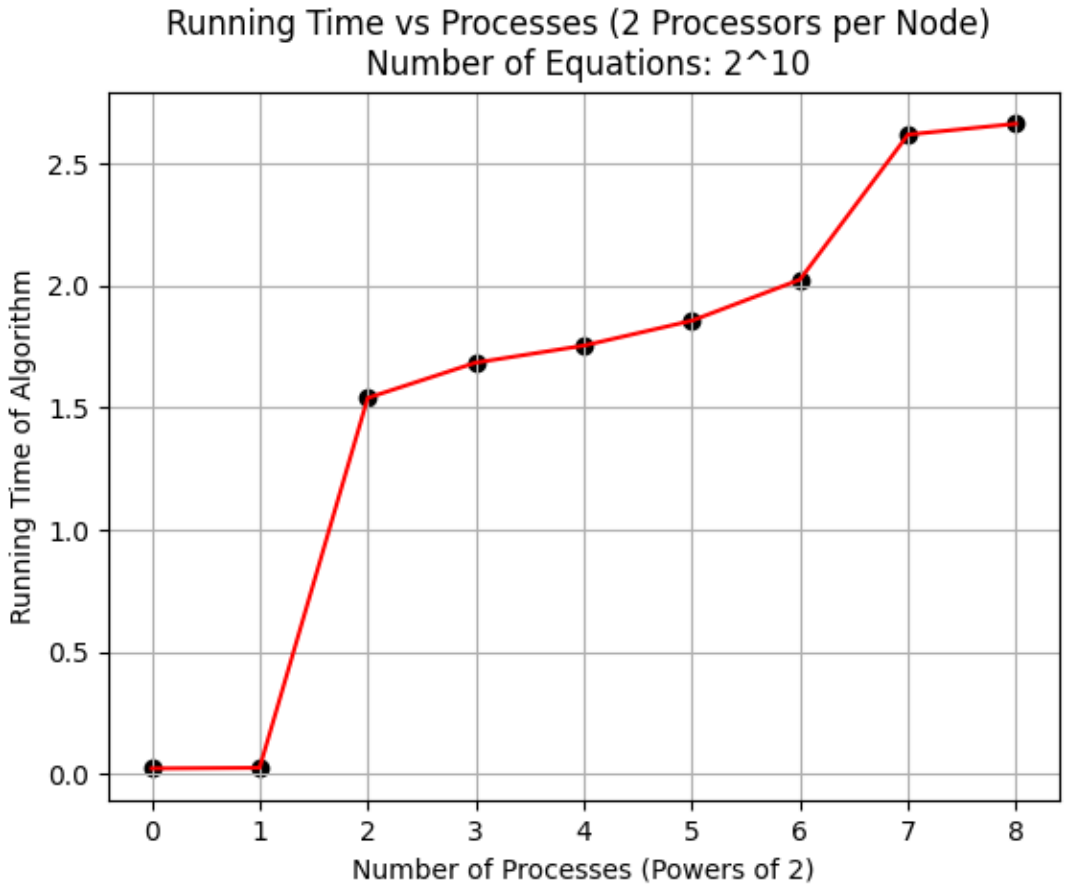
Running Times

(2 tasks per node)

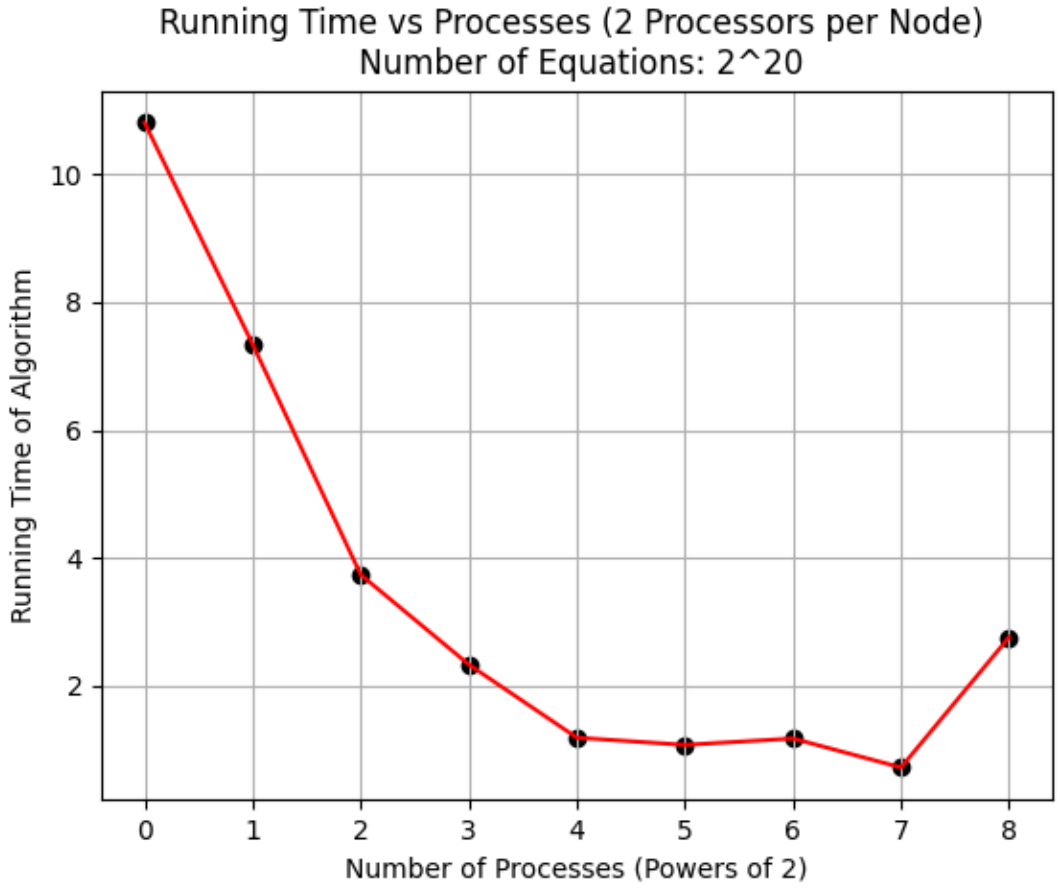
Processes > Equations v	1	2	4	8	16	32	64	128	256
1,024	0.0209	0.0236	1.5407	1.6852	1.7550	1.8570	2.0251	2.6191	2.6633
1,048,576	10.8038	7.3399	3.7387	2.3350	1.1916	1.0790	1.1787	0.7259	3.7616
33,554,432	-	-	118.1773	65.6128	41.2065	35.7604	26.8380	24.7228	20.2991

(in seconds)

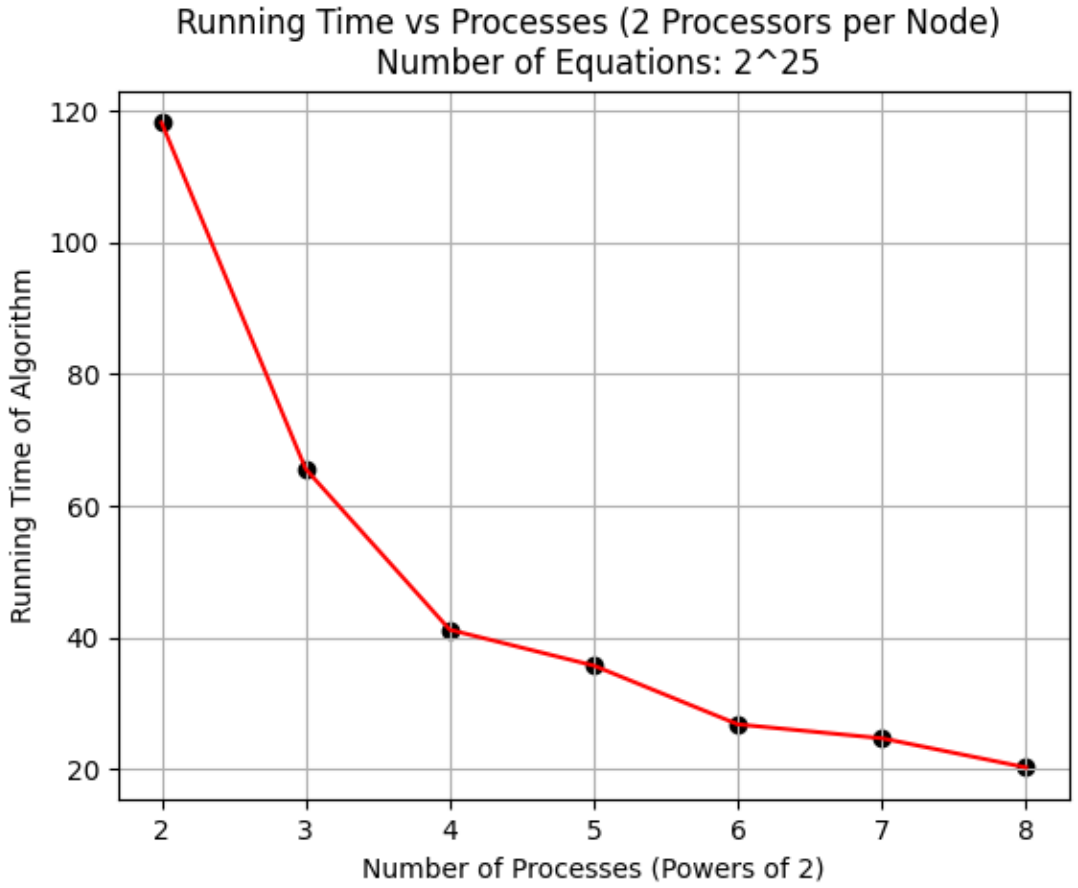
Running Times



Running Times



Running Times



References

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- Revisiting Parallel Cyclic Reduction and Parallel Prefix-based Algorithms for Block Tridiagonal Systems of Equations (Sudip K. Seal, Kalyan S. Perumalla, Steven P. Hirshman)
- Teaching Parallel Computing through Parallel Prefix (Srinivas Aluru)
- https://ubccr.freshdesk.com/support/solutions/articles/13000026245-tutorials-and-training-documents
- https://mathworld.wolfram.com/TridiagonalMatrix.html

THANK YOU!