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22. Infinite Regress

Chairman of a meeting of the Society of Logicians: "Before we put the motion: 'That the motion be now put,' should we not first put the motion: 'That the motion: "That the motion be now put" be now put'?"

From an old issue of *Punch*

THE INFINITE REGRESS, along which thought is compelled to march backward in a never ending chain of identical steps, has always aroused mixed emotions. Witness the varied reactions of critics to the central symbol of Broadway's most talked-about 1964 play, Edward Albee's *Tiny Alice*. The principal stage setting—the library of an enormous castle owned by Alice, the world's richest woman—is dominated by a scale model of the castle. Inside it lives Tiny Alice. When lights go on and off in the large castle, corresponding lights go on and off in the small one. A fire erupts simultaneously in castle and model. Within the model is a smaller model in which a tinier Alice perhaps lives, and so on down, like a set of nested Chinese boxes. ("Hell to clean," comments the butler, whose name is Butler.) Is the castle itself, into which the play's audience peers, a model in a still larger model, and that in turn . . .? A similar infinite nesting is the basis of E. Nesbit's short story, "The Town in the Library in the Town in the Library" (in her *Nine Unlikely Tales*); perhaps this was the source of Albee's idea.

For many of the play's spectators the endless regress of castles stirs up feelings of anxiety and despair: Existence is a mysterious, impenetrable, ultimately meaningless labyrinth; the regress is an endless corridor that leads nowhere. For theological students, who are said to be flocking to the play, the regress deepens an awareness of what Rudolf Otto, the German theologian, called the *mysterium tremendum*: the ultimate mystery, which one must approach with awe, fascination, humility and a sense of "creaturehood." For the mathematician and the logician the regress has lost most of its terrors; indeed, as we shall soon see, it is a powerful, practical tool even in recreational mathematics. First, however, let us glance at some of the roles it has played in Western thought and letters.

Aristotle, taking a cue from Plato's *Parmenides*, used the regress in his famous "third man" criticism of Plato's doctrine of ideas. If all men are alike because they have something in common with Man, the ideal and eternal archetype, how (asked Aristotle) can we explain the fact that one man and Man are alike without assuming another archetype? And will not the same reasoning demand a third, fourth, and fifth archetype, and so on into the regress of more and more ideal worlds?

A similar aversion to the infinite regress underlies Aristotle's argument, elaborated by hundreds of later philosophers, that the cosmos must have a first cause. William Paley, an eighteenth-century English theologian, put it this way: "A chain composed of an infinite number of links can no more support itself than a chain composed of a finite number of links." A finite chain does indeed require support, mathematicians were quick to point out, but in an infinite chain *every* link hangs securely on the one above. The question of what supports the entire series no more arises than the question of what kind of number precedes the infinite regress of negative integers.

Agrippa, an ancient Greek skeptic, argued that nothing can be proved, even in mathematics, because every proof must be proved valid and its proof must in turn be proved, and so on. The argument is repeated by Lewis Carroll in his paper "What the Tortoise Said to Achilles" (*Mind*, April, 1895). After finishing their famous race, which involved an infinite regress of smaller and smaller distances, the Tortoise traps his fellow athlete in a more disturbing regress. He refuses to accept a simple deduction involving a triangle until Achilles has written down an infinite series of hypothetical assumptions, each necessary to make the preceding argument valid.

F. H. Bradley, the English idealist, argued (not very convincingly) that our mind cannot grasp *any* type of logical relation. We cannot say, for example, that castle A is smaller than castle B and leave it at that, because "smaller than" is a relation to which both castles are related. Call these new relations c and d. Now we have to relate c and d to the two castles and to "smaller than." This demands four more relations, they in turn call for eight more, and so on, until the shaken reader collapses into the arms of Bradley's Absolute.

In recent philosophy the two most revolutionary uses of the regress have been made by the mathematicians Alfred Tarski and Kurt Gödel. Tarski avoids certain troublesome paradoxes in semantics by defining truth in terms of an endless regress of "metalanguages," each capable of discussing the truth and falsity of statements on the next lower level but not on its own level. As Bertrand Russell once explained it: "The man who says 'I am telling a lie of order n' *is* telling a lie, but a lie of order $n + 1$." In a closely related argument Gödel was able to show that there is no single, all-inclusive mathematics but only an infinite regress of richer and richer systems.

The endless hierarchy of gods implied by so many mythologies and by the child's inevitable question "Who made God?" has appealed to many thinkers. William James closed his *Varieties of Religious Experience* by suggesting that existence includes a collection of many gods, of different degrees of inclusiveness, "with no absolute unity realized in it at all. Thus would a sort of polytheism return upon us. . . ." The notion turns up in unlikely places. Benjamin Franklin, in a quaint little work called *Articles of Belief and Acts of Religion*, wrote: "For I believe that man is not the most perfect being but one, but rather that there are many degrees of beings superior to him." Our prayers, said Franklin, should be directed only to the god of our solar system, the deity closest to us. Many writers have viewed life as a board game in which we are the pieces moved by higher intelligences who in turn are the pieces in a vaster game. The prophet in Lord Dunsany's story "The South Wind" observes the gods striding through the stars, but as he worships them he sees the outstretched hand of a player "enormous over Their heads."

Graphic artists have long enjoyed the infinite regress. Who can look at the striking cover of the April, 1965, issue of *Scientific American* (showing the magazine cover reflected in the pupil of an eye) without recalling, from his childhood, a cereal box or magazine cover on which a similar trick was played? The cover of the November, 1964, *Punch* showed a magician pulling a rabbit out of a hat. The rabbit in turn is pulling a smaller rabbit out of a smaller hat, and this endless series of rabbits and hats moves up and off the edge of the page. It is not a bad picture of contemporary particle physics. The latest theory proposes a smaller, yet undetected, group of particles called "quarks" to explain the structure of known particles. Is the cosmos itself a particle in some unthinkably vast variety of matter? Are the laws of physics an endless regress of hat tricks?

The play within the play, the puppet show within the puppet show, the story within the story have amused countless writers. Luigi Pirandello's *Six Characters in Search of an Author* is perhaps the best-known stage example. The protagonist in Miguel de Unamuno's novel *Mist*, anticipating his death later in the plot, visits Unamuno to protest and troubles the author with the thought that he too is only the figment of a higher imagination. Philip Quarles, in Aldous Huxley's *Point Counter Point*, is writing a novel suspiciously like *Point Counter Point*. Edouard, in André Gide's *The Counterfeiters*, is writing *The Counterfeiters*. Norman Mailer's story "The Notebook" tells of an argument between the writer and his girl friend. As they argue he jots in his notebook an idea for a story that has just come to him. It is, of course, a story about a writer who is arguing with his girl friend when he gets an idea. . . .

J. E. Littlewood, in *A Mathematician's Apology*, recalls the following entry, which won a newspaper prize in Britain for the best piece on the topic: "What would you most like to read on opening the morning paper?"

OUR SECOND COMPETITION

The First Prize in the second of this year's competitions goes to Mr. Arthur Robinson, whose witty entry was easily the best of those we received. His choice of what he would like to read on opening his paper was headed "Our Second Competition" and was as follows: "The First Prize in the second of this year's competitions goes to Mr. Arthur Robinson, whose witty entry was easily the best of those we received. His choice of what he would like to read on opening his paper was headed 'Our Second Competition,' but owing to paper restrictions we cannot print all of it."

One way to escape the torturing implications of the endless regress is by the topological trick of joining the two ends to make a circle, not necessarily vicious, like the circle of weary soldiers who rest themselves in a bog by each sitting on the lap of the man behind. Albert Einstein did exactly this when he tried to abolish the endless regress of distance by bending three-dimensional space around to form the hypersurface of a four-dimensional sphere. One can do the same thing with time. There are Eastern religions that view history as an endless recurrence of the same events. In the purest sense one does not even think of cycles following one another, because there is no outside time by which the cycles can be counted; the *same* cycle, the *same* time go around and around. In a similar vein, there is a sketch by the Dutch artist Maurits C. Escher of two hands, each holding a pencil and sketching the other [*see Figure 157*]. In *Through the Looking Glass* Alice dreams of the Red King, but the King is himself asleep and, as Tweedledee points out, Alice is only a "sort of thing" in *his* dream. *Finnegans Wake* ends in the middle of a sentence that carries the reader back for its completion to the broken sentence that opens the book.

Since Fitz-James O'Brien wrote his pioneer yarn "The Diamond Lens" in 1858 almost countless writers have played with the theme of an infinite regress of worlds on smaller and smaller particles. In Henry Hasse's story "He Who Shrank" a man on a cosmic level much larger than ours is the victim of a scientific experiment that has caused him to shrink. After diminishing through hundreds of subuniverses he lingers just long enough in Cleveland to tell his story before he vanishes again, wondering how long this will go on, hoping that the levels are joined at their ends so that he can get back to his original cosmos.

Even the infinite hierarchy of gods has been bent into a closed curve by Dunsany in his wonderful tale "The Sorrow of Search." One night as the prophet Shaun is observing by starlight the four mountain gods of old—Asgool, Trodath, Skun, and

Escher Foundation—Collection Haags Gemeentemuseum, The Hague

157. Maurits C. Escher's "Drawing Hands"

Rhoog—he sees the shadowy forms of three larger gods farther up the slope. He leads his disciples up the mountain only to observe, years later, two larger gods seated at the summit, from which they point and mock at the gods below. Shaun takes his followers still higher. Then one night he perceives across the plain an enormous, solitary god looking angrily toward the mountain. Down the mountain and across the plain goes Shaun. While he is carving on rock the story of how his search has ended at last with the discovery of the ultimate god, he sees in the far distance the dim forms of

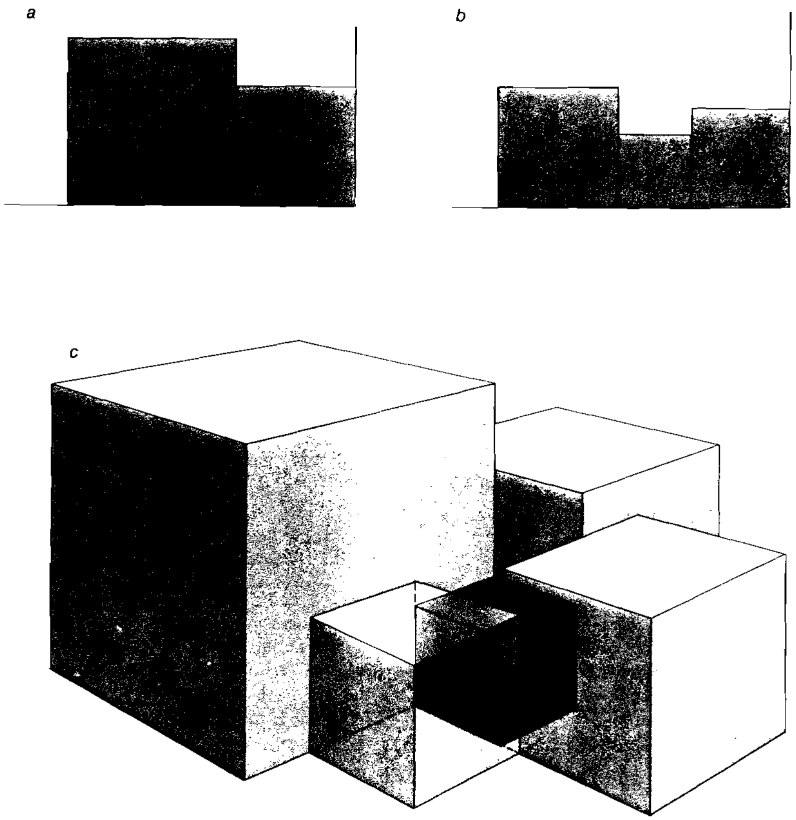
four higher deities. As the reader can guess, they are Asgool, Trodath, Skun, and Rhoog.

No branch of mathematics is immune to the infinite regress. Numbers on both sides of zero gallop off to infinity. In modular arithmetics they go around and around. Every infinite series is an infinite regress. The regress underlies the technique of mathematical induction. Georg Cantor's transfinite numbers form an endless hierarchy of richer infinities. A beautiful modern example of how the regress enters into a mathematical proof is related to the difficult problem of dividing a square into other squares no two of which are alike (see Chapter 17 of my *Second Scientific American Book of Mathematical Puzzles and Diversions*; New York: Simon and Schuster, 1965). The question arises: Is it possible similarly to cut a cube into a finite number of smaller cubes no two of which are alike? Were it not for the deductive power of the regress, mathematicians might still be searching in vain for ways to do this. The proof of impossibility follows.

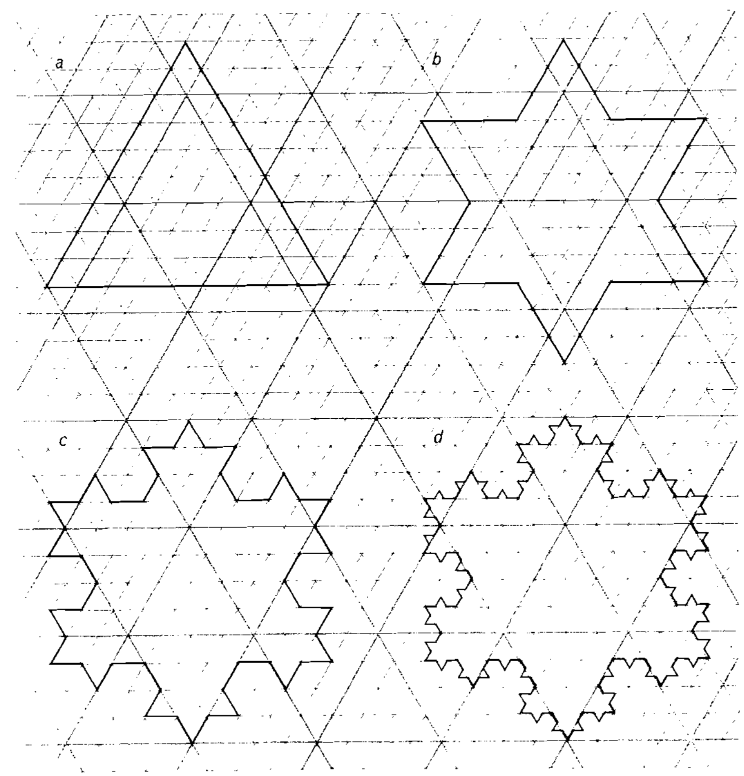
Assume that it is possible to "cube the cube." The bottom face of such a dissected cube, as it rests on a table, will necessarily be a "squared square." Consider the smallest square in this pattern. It cannot be a corner square, because a larger square on one side keeps any larger square from bordering the other side [*see "a" in Figure 158*]. Similarly, the smallest square cannot be elsewhere on the border, between corners, because larger squares on two sides prevent a third larger square from touching the third side [*b*]. The smallest square must therefore be somewhere in the pattern's interior. This in turn requires that the smallest cube touching the table must be surrounded by cubes larger than itself. This is possible [*c*], but it means that four walls must rise above all four sides of the small cube—preventing a larger cube from resting on top of it. Therefore on this smallest cube there must rest a set of smaller cubes, the bottoms of which will form another pattern of squares.

The same argument is now repeated. In the new pattern of squares the smallest square must be somewhere in the interior. On this smallest square must rest the smallest cube, and the little cubes on top of it will form another pattern of squares. Clearly the argument leads to an endless regress of smaller cubes, like the endless hierarchy of fleas in Dean Swift's jingle. This contradicts the original assumption that the problem is solvable.

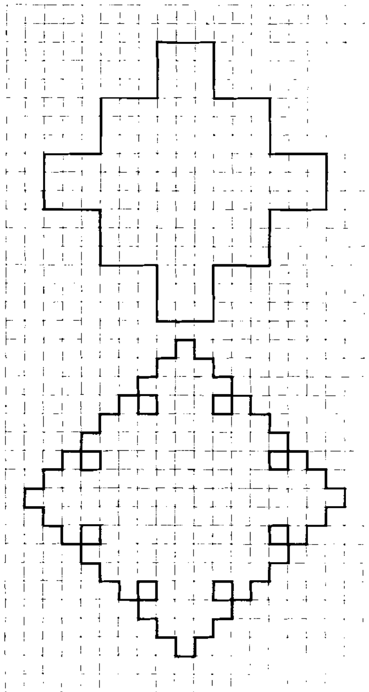
Geometric constructions such as this one, involving an infinite regress of smaller figures, sometimes lead to startling results. Can a closed curve of infinite length enclose a finite area of, say, one square inch? Such pathological curves are infinite in number. Start with an equilateral triangle [*see "a" in Figure 159*] and on the central third of each side erect a smaller equilateral triangle. Erase the base lines and you have a six-pointed star [*b*]. Repeating the construction on each of the star's 12 sides produces a 48-sided polygon [*c*]. The third step is shown in *d*. The limit of this infinite construction, called the snowflake curve, bounds an area 8/5 that of the original triangle. It is easy to show that successive



158. Proof that the cube cannot be "cubed"



159. The snowflake curve



160. The cross-stitch curve

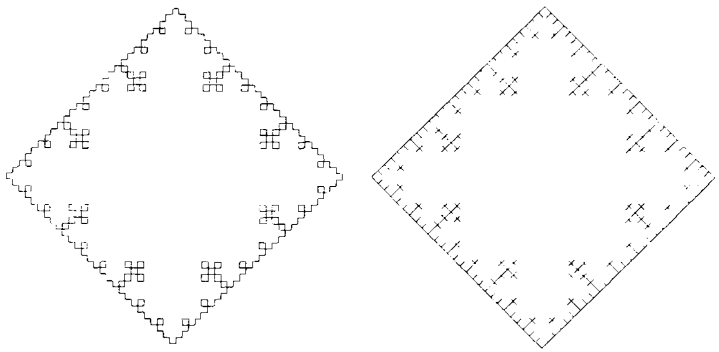
additions of length form an infinite series that diverges; in short, the length of the snowflake's perimeter is infinite. (In 1956 W. Grey Walter, the British physiologist, published a science-fiction novel, *The Curve of the Snowflake,* in which a solid analogue of this crazy curve provides the basis for a timetravel machine.)

Here are two easy puzzles about the less well known square version of the snowflake, a curve that has been called the cross-stitch. On the middle third of each side of a unit square erect four smaller squares as shown at the top of Figure 160. The second step is shown at the bottom. (The squares will never overlap, but corners will touch.) If this procedure continues to infinity, how long is the final perimeter? How large an area does it enclose?

Answers

The cross-stitch curve has, like its analogue the snowflake, an infinite length. It bounds an area twice that of the original square. The drawing at the left in Figure 161 shows its appearance after the third construction. After many more steps it resembles (when viewed at a distance) the drawing at the right. Although the stitches seem to run diagonally, actually every line segment in the figure is vertical or horizontal. Similar constructions of pathological curves can be based on any regular polyhedron, but beyond the square the figure is muddied by overlapping, so that certain conventions must be adopted in defining what is meant by the enclosed area.

Samuel P. King, Jr., of Honolulu, supplied a good analysis of curves of this type, including a variant of the cross-stitch discovered by his father. Instead of erecting four squares outwardly each time, they are



161. Solution to cross-stitch curve problem

erected inwardly from the sides of each square. The limit curve has an infinite length, but encloses zero area.

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