(1) Let $G$ be the graph with vertices $V = \{2, 3, 4, 5, 6, 8, 9\}$ and edge set $E$ consisting of all pairs $(u, v)$ such that the numbers $u$ and $v$ share a factor other than 1. (That is, $u$ and $v$ are \textit{not} relatively prime.) Please answer the following short questions:

(a) Is the edge relation $E$ reflexive? Answer: The best answer is \textit{yes}: $u$ and $u$ share the factor $u$ which is greater than 1. (Or to use the alternative stipulation, $u$ and $u$ do not count as relatively prime even when $u$ is prime.)

(b) Is $E$ symmetric? Answer: \textit{Yes}—the predicate doesn’t require that $u$ be less than $v$, for instance, so it works to the same thing when you switch $u$ and $v$.

(c) Is $E$ transitive? If you say \textit{no}, please give a counterexample involving three different vertices. Answer: No—2 and 6 share a factor, and 6 and 3 share a factor, but 2 and 3 don’t—they are relatively prime.

(d) Draw a picture of the graph $G$ \textit{without} including any self-loops. Is it an undirected graph? Answer: \textit{Yes} the graph is undirected since $E$ is symmetric.

(e) Is $G$ connected? If not, is there a completely isolated vertex? Answer: No—the vertex 5 is isolated. The rest of the graph is however connected.

(f) Does $G$—again without the self-loops—have a triangle? Answer: \textit{Yes}—in fact it has a 4-clique formed by the nodes 2, 4, 6, 8 (what do we appreciate?).

(g) Does the \textit{complement} of $G$ have a triangle—which is the same as $G$ having an \textit{independent set} of size at least 3? Can you find such a set without using vertex 5? Answer: Yes, but only ones like 2, 3, 5 that involve vertex 5. The reason is that all the other numbers are divisible by 2 and/or 3, so there is no three-way relative primeness without 5 (or some other prime factor). (2+2+5+4+3+3+5 = 24 pts.)

(2) Take $\Sigma = \{0, 1\}$. Write regular expressions that express the following logical conditions on strings $x \in \Sigma^*$:

(a) $x$ begins with a 0.

(b) $x$ ends with a 0.

(c) $x$ begins with a 0 and ends with a 0.

(d) $x$ begins and ends with the same symbol (0 or 1).
Then answer: which of these conditions is most analogous to the language of the text’s automatic-door example? Note that the text’s language has 4 inputs which could be labeled 0, 1, 2, 3 with 3 meaning “Both,” say, and the text’s desired final state is that the door ends up being closed. (3 + 3 + 3 + 3 + 3 = 15 pts.)

Answers: (a) $0(0 \cup 1)^*$, (b) $(0 \cup 1)^*0$, (c) $0(0 \cup 1)^*0 \cup 0$—the extra $\cup 0$ is needed because the string “0” does indeed begin with a ‘0’ and end with a ‘0’. (d) $0(0 \cup 1)^*0 \cup 0 \cup 1(0 \cup 1)^*1 \cup 1$. The last question could be argued several ways but the clear best analogy is to (b), since the final input is what matters most to the door and you really want to end with the ‘NEITHER’ input which is analogous to ‘0’ so the door can safely close.

(3) Text, exercise 1.5, parts (d) and (g) only. That is:

(d) Design a DFA $M$ that accepts all and only those strings that don’t consist of zero-or-more $a$’s followed by zero-or-more $b$’s.

(g) Design a DFA $M$ that accepts all and only the strings that don’t have exactly two $a$’s in them.

In both cases the alphabet is $\Sigma = \{a, b\}$. The text suggests that you first design a DFA $M'$ for the complements of these languages—that is, changing “don’t” to “do” in the stated conditions—then make $M$ by swapping accepting and rejecting states. But you can also treat this as a case of rewriting the definitions to “Acc-Centuate the Positive, E-Liminate the Negative…” (incidentally I was wrong about it being written for Disney; it was by Johnny Mercer and Buffalo’s own Harold Arlen). Doing this for (d) is like the Tue. 2/2 lecture example but a little different; for (g) think of categories “zero,” “one,” “two,” and “three-or-more.” (12 + 12 = 24 pts., for 63 total on the problem set)

Answers in words: For (d), let the start state mean “no $b$’s yet” with a self-loop on $a$ and an arc on $b$ going to a second state with the meaning “Saw at least one $b$, but thankfully no more $a$’s after it.” This second state has a self-loop on $b$ but goes to “dead” on an $a$, and the dead state self-loops on $a$ and $b$. The start and second states are both accepting.

For (g) we allocate four states counting the number of $a$’s seen thus far: “zero,” “one,” “two,” and “three-or-more.” Since $b$ is a don’t-care char here, each state has a self-loop on $b$, whereas each state goes to the next on $a$ since the count ups by 1, except that the last state self-loops on $a$ too. The last state would be a dead state if the “don’t” were a “do,” but as things stand it’s a “nirvana” state, to go with the start and “one” states as the other accepting states.