Define \( L^* = \left\{ x \in \{1, 0\}^* : x \text{\ is survivable with unlimited }$ saving $\right\} \).

Same argument with \( \left\lceil \frac{m}{n} \right\rceil \) shows that the language set 

\[ S = \{ \# \}^* \cup \{ 0 \}^* \] is PD for \( L^* \).

Take \( \Sigma = \{1, 2\} \) and \( L = \left\{ x \in \Sigma^* : x \text{\ can be \ closed \ into \ a \ balanced \ parentheses \ string} \right\} \).

Example: \( x = ( ( ) )'' \) \( \in L \) but \( x' = ( ( ) ( ) ')' \) \( \notin L \) because the marked right paren is unbalanced. 

\( L \) is the same as \( L^* \) with \( ( = \# ) = 0, \) no 0. Infinite PD set: \( S = ( \).
Recall $D = \{ u \in \{0, 1\}^* \mid \exists u, v \in \{0, 1\}^* : u \leq v \leq u \}$.

Some critical cases:

1. $u \leq v \leq u$
   - If $u = 000$, then nothing forces $v = 000$ too. $v = 0^*$ is fine.

Tip: Anytime you have a compound explosion like $u \cdot v$ on the LHS, just rewrite in form $D = \Sigma W = W$ such that $W = u \cdot v$ such that $W = u \cdot v$.

Hence the $\Sigma$ is important. Let $S$ be defined by:

$$S = \{ 0^m \mid m \geq 0 \} = 0^*.$$  

Or take $S = \{ 0^m 1^m 1^m \mid m \geq 0 \} = 0^+ 1$.

Still clearly infinite and "tighter."

$S$ is PD: let any $x, y \in S$ and $x \neq y$ be given. Then there are $m, n \in \mathbb{N}$ (we could say "why $m < n$", but we won't need it) such that $m \geq 1$ and $x = 0^m 1^m$, $y = 0^n 1^n$.

Take $z = 0^m 1^m$. Then $xz = 0^m 10^m \in D$ but $yz = 0^n 10^m \not\in D$ since $n \neq m$.

So $D$ is non-regular by MNT. \(\Box\)

Can do $PAL = \{palindromes\}$ similarly $S = 0^+ 1$ etc.

We could take $S = 0^*$ anyway: let any $x = 0^m$, $y = 0^m$ be given with $m \neq n$. Take $z = 10^m$.

Then $xz = 0^m 10^m \in PAL$ but $yz = 0^n 10^m \not\in PAL$ since $n \neq m$.

Since $C$ is infinite, $PAL$ is not regular. \(\Box\)
One more example that starts regular but “blows up”:

$L_K = \{ x \in \{0, 1\}^* : \text{the } k\text{th bit from the right is a } 1 \} = \{ (01)^* 1 (01)^* \}_{\lceil \log_2 K \rceil}$

Strictly according to the regexp induction:

$$R_K = (01)^* 1 (01)^* (01)^* (01)^* \cdots (01)^*$$

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$$R_K = (01)^* 1 (01)^* (01)^* (01)^* \cdots (01)^*$$

VFA $N_K = \begin{array}{c}
0 \rightarrow 1 \rightarrow 0 \rightarrow 0 \\
\vdots
\end{array}$

$$K+1 = \Theta(K) \text{ states}$$

$$2K+1 = \Theta(K) \text{ transitions}$$

$$K-1 \text{ progressions using } K \text{ states}$$

$$\text{ at each } 0 \text{ and } 1 \text{ go to dead state.}$$

How many states does a DFA $M_K$ such that $L(M_K) = L_K$ need?

Define $S = \{0, 1\}^K$. Then $S$ is finite but $|S| = 2^K$.

Claim: $S$ is PD for $L_K$, which $\Rightarrow M_K$ needs (at least) $2^K$ states!

Let any $x, y \in S$, $x \neq y$ be given. Then $x$ and $y$ differ arbitrarily

strings in at least one place $j \leq K$. $x$

Take $z = 0^{j-1}$, For counterexample, $i \geq 1$, $y$

say “$x$" refers to the string that has the $1$ in place $j$, “$y$" to the string with $0$ in place $j$.

Then $xz = x 0^{j-1}$ now has that $1$ in position $K$ from the end, so $xz \in L_K$.

Because $yz = y 0^{j-1}$ has a $0$ in place $K$ from the end, so $yz \notin L_K$. $\Rightarrow S$ is PD.

By MPT, $M_K$ needs (at least) $2^K$ states. $K = 2.90$ makes $|M_K| > 10^{1000}$
Brief Intro to Context Free Grammars

Which can express many nonregular languages.
Examples in Advance.

1. $G = S \rightarrow \text{0S1 | } \varepsilon$. Can produce

   $S \Rightarrow \varepsilon$ directly.
   $S \Rightarrow \text{0S1} \Rightarrow \text{01}$
   $S \Rightarrow \text{0S1} \Rightarrow \text{00S11} \Rightarrow \text{0011}$

   $G$ generates the language $\{0^n 1^n : n \geq 0\}$.

2. $G = S \rightarrow \varepsilon | \text{SS | (S)}$. Generates all balanced parentheses.

   $G' = S \rightarrow \varepsilon | (S)S | (S)$

3. $G = S \rightarrow \varepsilon | (S)S | (S)$ generates all closable strings.

4. $G = S \rightarrow \text{0S0 | 1S1 | 011 | } \varepsilon$. Generates PAL.

   Can we generate the doubleword language $D$?