Lecture Thu Mar 26.

\[ E = \{ x \in \{a, b\}^* : \#a(x) = \#b(x) \} \]

1. Prove \( L(g) \subseteq E \). (By SI.)

2. Is \( E \subseteq L(g) \), so \( L(g) = E \)

3. If so, can we remove some rules?

\( P_s = \) "Every \( x \) that I derive has \( \#a(x) = \#b(x) \)."

\( P_x = \) "Every \( y \) I derive has \( \#a(y) = \#b(y) + 1 \)."

\( P_y = \) "Every \( z \) I derive has \( \#b(z) = \#a(z) + 1 \)."

Every soundness of each rule for these properties in any order:

### A \( \rightarrow \) BAA:

Suppose \( A \Rightarrow^* y \) utpf. Then \( y = u \cdot v \cdot w \) where \( B \Rightarrow^* u, A \Rightarrow^* v, \) and \( A \Rightarrow^* w \). By IH \( P_B \) and \( P_A \) (twice) on RHS, \( \#b(u) = \#a(u) + 1, \#a(v) = \#b(v) + 1, \) and \( \#a(w) = \#b(w) + 1 \).

Hence for \( \Psi = uvw \) we have \( \#a(\Psi) = \#a(u) + \#a(v) + \#a(w) \)

Here is where \( \Phi \)-es are being used:

\[ = \#b(u) - 1 + \#b(v) + 1 + \#b(w) + 1 \]

\[ = \#b(u) + \#b(v) + \#b(w) + 1 = \#b(\Psi) + 1. \]

### B \( \rightarrow \) ABB:

\( \) Symmetrical.

### A \( \rightarrow \) aS:

Suppose \( A \Rightarrow^* y \) utpf. Then \( y = au \) where \( S \Rightarrow u \). By IH \( P_S \) on RHS, \( \#a(u) = \#b(u) \). Hence \( \#a(y) = 1 + \#a(u) = 1 + \#b(u) \).

This upholds \( P_A \) on LHS:

### B \( \rightarrow \) S:\:

\( \) Symmetrical.

### S \( \rightarrow \) SS:

Suppose \( S \Rightarrow^* x \) utpf. Then \( x = uv \) where \( S \Rightarrow u \) and \( S \Rightarrow v \). By IH \( P_S \) on RHS (twice), \( \#a(u) = \#b(u) \) and \( \#a(v) = \#b(v) \). Hence \( \#a(uv) = \#b(uv) \).

### S \( \rightarrow \) AB:

Suppose \( S \Rightarrow^* x \) utpf. Then \( x = uv \) where \( A \Rightarrow u \) and \( B \Rightarrow v \). By IH \( P_B \) on RHS.
and \( L(H) \) on RHS, \( \#a(u) = \#b(u) + 1 \) and \( \#a(v) = \#b(v) - 1 \).

Hence \( \#a(x) = \#a(u) + \#b(v) = \#b(u) + 1 + \#b(v) - 1 = \#b(u) + \#b(v) = \#b(x) \). \( \therefore \) Rs on LHS. \( S \Rightarrow BA \): Similar. All rules check out. \( \therefore L(G) \subseteq L(G) \).

\( S \Rightarrow \varepsilon | a B | a B A | B a A | A B A \) Is \( R \subseteq L(G) \). We need to prove:
\( \forall x \in \{0, 1\}^* : x \in E \Rightarrow x \in L(G) \).

Two needed steps: 0. Rewrite in terms of lengths \( n \) of \( x \) and \( m, p, k \) of substrings. Define two other languages: 2. Write similar objectives for the other variables.

\( L_A = \{ x \in \Sigma^* : \#a(x) = \#b(x) + 1 \} \) \( L_B = \{ x \in \Sigma^* : \#b(x) = \#a(x) + 1 \} \)

(1) \( \forall n \geq 0 \) \( P(n) \) where \( P(n) \) is For each string \( x \) of length \( n \):

- If \( \#a(x) = \#b(x) \) then \( S \Rightarrow ^* x \) and we have strengthened the induction to include mutual comprehensiveness for all the variables.

(2) If \( \#a(x) = \#b(x) + 1 \) then \( A \Rightarrow ^* x \) and

(3) If \( \#a(x) = \#b(x) - 1 \) then \( B \Rightarrow ^* x \).

Basis \( \{ n = 0 \} \) The only \( x \) to consider is \( x = \varepsilon \). \#a(\varepsilon) = \#b(\varepsilon) \), so we need to handle the if-then clause for \( S \). \( S \Rightarrow \varepsilon \) is a rule \( \varepsilon \), so \( S \Rightarrow \varepsilon \). \( \checkmark \)

Extra Basis \( \{ n = 1 \} \). The only \( x \) of length \( 1 \) are \( x = a \) and \( x = b \). Then \( x = a \) needs handling the if-then clause for \( L_A \). We have \( A \Rightarrow \alpha S \Rightarrow a \). So \( A \Rightarrow \varepsilon a \). Similarly, for \( x = b \) we need \( B \Rightarrow b S \Rightarrow b \). \( \checkmark \)

Let any \( x \in \Sigma^* \) be given. We need to allow for possibility that \( x \) "behaves" as any of the three if-then clauses, and possibly break into subcases of that.
Let any $x \in \Sigma^*$ be given (we have $n \geq 2$ with \textit{extra base})

\begin{itemize}
  \item \textbf{cases: (i)} \textit{x begins with a, ie. $x = au$ where $u \in \Sigma^{n-1}$ -- exhaustive of $x$}.
  \item \textbf{or (ii)} \textit{x begins with b, ie $x = bu$ where $|u| = n - 1$}.
\end{itemize}

\begin{itemize}
  \item \textbf{for Ls:} IF $\#a(x) = \#b(x)$ and $x = au$, then $S \rightarrow \varepsilon \quad \Rightarrow \quad AB \rightarrow BA \rightarrow \varepsilon$. \\
  \text{And} $|u| = n - 1$ and $u \in \Sigma^L$ by IH $\rho(n-1)$ for $L_b$, we get $B \Rightarrow ^* u$. \\
  \text{Thus we can build the derivation} $S \rightarrow AB \rightarrow aSB \rightarrow aB \rightarrow au = x$. \\
  \text{Hence} $S \Rightarrow ^* x$, which satisfies the \textit{then} part that was activated. \\
\end{itemize}

\begin{itemize}
  \item \textbf{for Lb:} IF $\#a(x) = \#b(x) + 1$ then $x = au$ where $\#a(u) = \#b(u)$. \\
  \text{Thus} $u \in \Sigma^L$ and $|u| = n - 1 < n$, so we may apply IH $\rho(n-1)$ for $S$ to conclude $S \Rightarrow ^* u$. This gives $A \Rightarrow aS \Rightarrow au = x$, so $A \Rightarrow ^* x$ as needed.
\end{itemize}

\begin{itemize}
  \item \textbf{for Ls:} IF $\#a(x) = \#b(x) - 1$ and $x = au$, then $\#a(u) = \#b(u) - 2$.
  \text{Keypoint observation:} $u$ can be broken as $u = uv$ such that $\#a(v) = \#b(v) - 1$ and $\#a(u) = \#b(u) - 1$.
  \text{on} $v \in \Sigma^L$ and $w \in \Sigma^L$ and $|v| < n - 1 < n$, $w$ within $\Sigma^L$.
  \text{Let} $B \Rightarrow v$ and $B \Rightarrow w$ by IH $\rho(|v|), \rho(|w|)$. This enables us to build the derivation $B \Rightarrow AB \Rightarrow aSB \Rightarrow aB \Rightarrow ^* avB \Rightarrow ^* awv = au = x$. \textbf{\Rightarrow} $B \Rightarrow ^* x$. \\
\end{itemize}

\begin{itemize}
  \item \textbf{case (ii):} This is similar for $L_b$ using rules $B \Rightarrow bS$, $S \Rightarrow \varepsilon$, and $S \Rightarrow bA$.
  \text{X begins with b, and for Ls, where $\#a(x) = \#b(x)$}. $S \Rightarrow bA \Rightarrow bSA \Rightarrow ^* ahx^x$. \\
  \text{For Ls we have $x = bu$ where $\#a(x) = \#b(x) + 1$. Break $u = uv$ as before.} \\
  \text{A} \Rightarrow bAA \Rightarrow bAA \Rightarrow bVA \Rightarrow ^* bwv = bu = x. \text{For Lb, $x = bu$ where $\#a(u) = \#b(u)$. Get $B \Rightarrow bS \Rightarrow bu = x$. \textbf{\Rightarrow} }$.
\end{itemize}