(1) For each of the following stated relationships between complexity classes, say whether it is known to be true or not. In all cases where it is “known,” you can prove it by applying theorems relating time and space complexity classes.

Answers:

(a) $\text{DSPACE}[(\log n)^2] \subseteq \text{P}$. Not known: the theorem $\text{DSPACE}[s(n)] \subseteq \text{NSPACE}[s(n)] \subseteq \text{DTIME}[2^{O(s(n))}]$ only gives $\text{DSPACE}[(\log n)^2] \subseteq \text{DTIME}[2^{O((\log n)^2)}]$. The function $2^{O((\log n)^2)}$ equals $n^{\log_2 n}$, which is not bounded above by any polynomial in $n$. (Most researchers believe that $\text{DSPACE}[(\log n)^2]$ contains languages that are not in P.)

(b) $\text{PSPACE} \subseteq \text{EXP}$. True: Let $A \in \text{PSPACE}$. Then there exists $k > 0$ such that $A \in \text{DSPACE}[n^k]$.

By Theorem 5.12, $A \in \text{DTIME}[2^{n^k}]$, which is contained in EXP. (It was also AOK to cite Theorem 5.12 as saying that $\text{NSPACE}[n^{O(1)}] \subseteq \text{DTIME}[2^{n^{O(1)}}] = \text{EXP}$ directly.)

(c) $\text{NP} \subseteq \text{E}$. Not known: We know $\text{NTIME}[n^2] \subseteq \text{NSPACE}[n^2] \subseteq \text{DTIME}[2^{O(n^2)}]$ by Theorem 5.12, but no more than that—and $\text{DTIME}[2^{O(n^2)}]$ is not contained in E.

(d) $\text{NP} \subseteq \text{DSPACE}[n^2]$. Not known: the theorem $\text{NTIME}[t(n)] \subseteq \text{DSPACE}[t(n)]$ only gives us $\text{NP} \not\subseteq \text{PSPACE}$.

(e) $\text{NTIME}[O(n)] \subseteq \text{E}$. True: $\text{NTIME}[O(n)] \subseteq \text{NLBA} \subseteq \text{DTIME}[2^{O(n)}] = \text{E}$.

(2) Prove that for any $k \geq 0$ and $\epsilon > 0$, $\text{DTIME}[n^k]$ is properly contained in $\text{DTIME}[n^{k+\epsilon}]$. Show the numerical estimates needed for the conditions of the deterministic time hierarchy theorem to apply. (12 pts.)

Answer: We want to apply the Time Hierarchy Theorem with $t_1(n) = n^k$ and $t_2(n) = n^{k+\epsilon}$. To meet the condition $t_1(n) \log t_1(n) = o(t_2(n))$ we need the following to go to 0 as $n \to \infty$:

$$\frac{n^k \log(n^k)}{n^{k+\epsilon}} = \frac{k \log n}{n^\epsilon}.$$

Using L’Hopital’s Rule, we take the derivative of numerat and denominator to get

$$\frac{k/n}{\epsilon n^{\epsilon-1}} = \frac{k}{\epsilon} \frac{n^{1-\epsilon}}{n} = C \frac{1}{n^\epsilon},$$

where $C = k/\epsilon$. This clearly goes to 0 as $n \to \infty$, so the condition is met and the conclusion follows.

(3) Show, with reference to 1(d) above, that $\text{NP}$ cannot be equal to $\text{DSPACE}[n^2]$. Use the fact that QBF is complete for $\text{PSPACE}$ under $\leq^p_m$, the space hierarchy theorem, and the closure of $\text{NP}$ under $\leq^p_m$. (12 pts., for 54 on the set)

Answer: $\text{NP}$ is closed downward under $\leq^p_m$. If we show that $\text{DSPACE}[n^2]$ isn’t closed downward under $\leq^p_m$, then it ipso-fact must be different from NP. Well, QBF belongs to $\text{DSPACE}[n^2]$ (actually to $\text{DSPACE}[O(n)]$ per Arora-Barak’s footnote covered later in lecture) and is PSPACE-complete under $\leq^p_m$. So the closure of $\text{DSPACE}[n^2]$ under $\leq^p_m$ contains (in fact, equals) PSPACE. Finally, by the Space Hierarchy Theorem, PSPACE properly contains $\text{DSPACE}[n^2]$.  