2-D Transformations

Translation

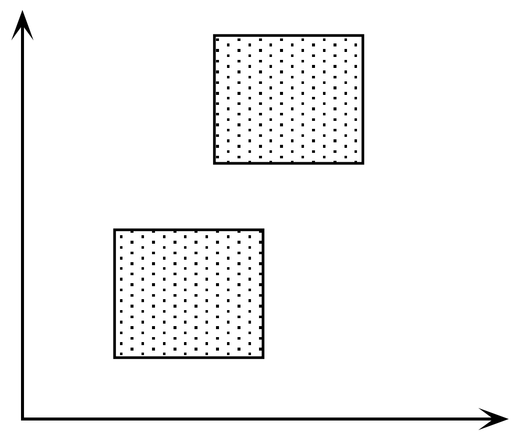
Moves an object by a given amount

$x' = x + t_x$

$y' = y + t_y$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

Graphically indicate the values of t_x and t_y for this example



Another Example:

Line L from (4,2) to (9,6)

$t_x = 1$

$t_y = 5$

Line L' = (,) to (,)

Scaling

Enlarge / reduce an object by a given amount

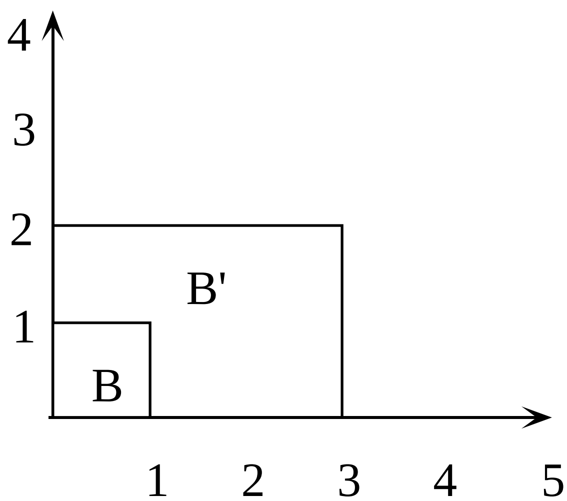
relative to the origin

x and y scale factors may be different

$x' = s_x x$

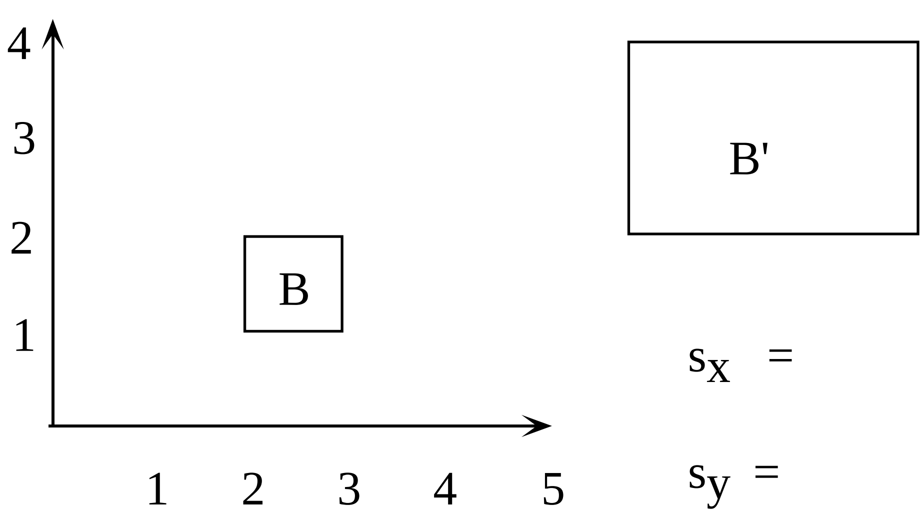
$y' = s_y y$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



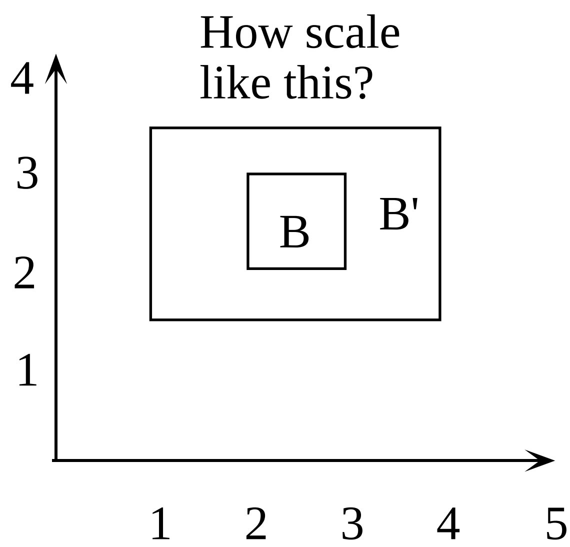
s_x =

s_y =



s_x =

s_y =



Translate B to be centered on origin

t_x = , t_y =

Scale

s_x = , s_y =

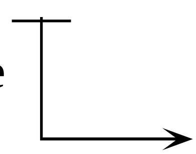
Translate back

t_x = , t_y =

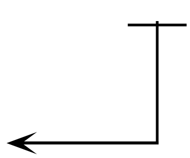
Using Scaling to Reflect

By setting scale factors to +/- 1, can reflect

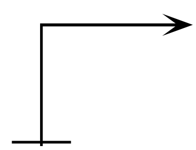
Original shape



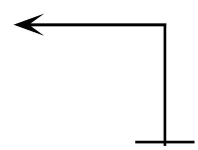
$s_x = -1$
$s_y = 1$



$s_x = 1$
$s_y = -1$

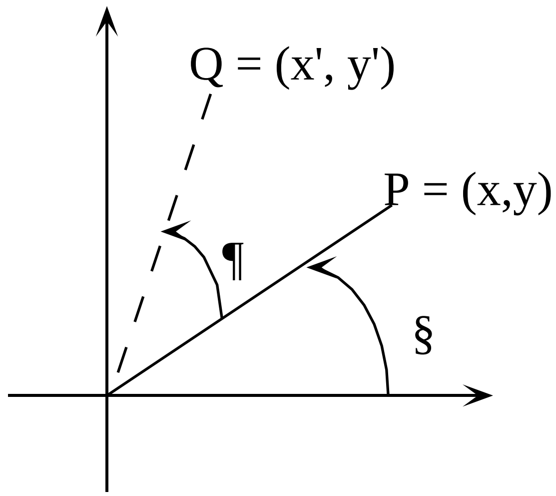


$s_x = -1$
$s_y = -1$



Is this the same as rotation by pi?

Rotation



Length of lines is R

$$x' = x \cos \P - y \sin \P$$
$$y' = x \sin \P + y \cos \P$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \P & -\sin \P \\ \sin \P & \cos \P \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x = P_x = R \cos \S$$
$$y = P_y = R \sin \S$$

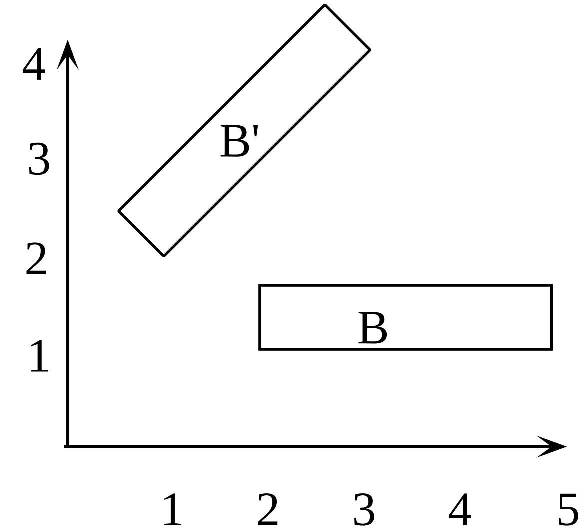
$$x' = Q_x = R \cos (\S+\P)$$
$$y' = Q_y = R \sin (\S+\P)$$

But $\cos (\S+\P) = \cos \S \cos \P - \sin \S \sin \P$
$\sin (\S+\P) = \sin \S \cos \P + \cos \S \sin \P$

So $x' = R \cos \S \cos \P - R \sin \S \sin \P$
$= x \cos \P - y \sin \P$

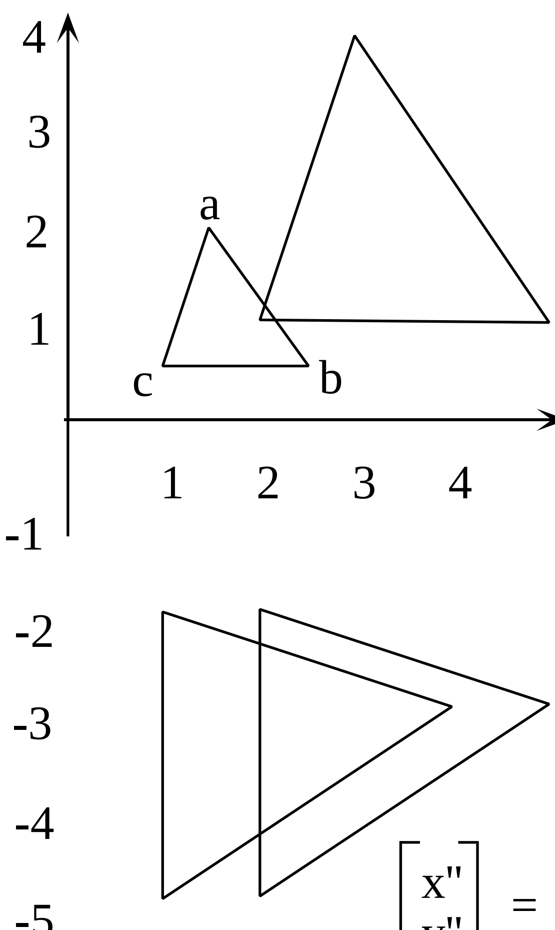
$y' = R \sin \S \cos \P + R \cos \S \sin \P$
$= x \sin \P + y \cos \P$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos ¶ & \sin ¶ \\ -\sin ¶ & \cos ¶ \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



Rotation is relative to the origin

What is ¶?



Triangle abc
Scaled by $s_x = s_y = 2$
Rotated by ¶ = -pi/2
Translate by $t_x = 1$, $t_y = 0$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} \cos ¶ & -\sin ¶ \\ \sin ¶ & \cos ¶ \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\begin{bmatrix} x''' \\ y''' \end{bmatrix} = \begin{bmatrix} x'' \\ y'' \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} \cos ¶ & -\sin ¶ \\ \sin ¶ & \cos ¶ \end{bmatrix} \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} \cos \P & -\sin \P \\ \sin \P & \cos \P \end{bmatrix} \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Could multiple the matrixes before applying transform to points
Why is this more efficient?

$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} s_x\cos \P & -s_y\sin \P \\ s_x\sin \P & s_y\cos \P \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Would like to do this for translation too

$$\begin{bmatrix} x''' \\ y''' \end{bmatrix} = \begin{bmatrix} x'' \\ y'' \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

$$\begin{bmatrix} x''' \\ y''' \end{bmatrix} = \begin{bmatrix} s_x\cos \P & -s_y\sin \P \\ s_x\sin \P & s_y\cos \P \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

Have a problem

Solution: Homogeneous Coordinates
- Notice that scaling and rotation expressed as matrix multiplications
- Not so for translation
- Want to have all be matrix multiplication
  - (since matrix multiplication is associative: (A B) C = A (B C))
- Use homogeneous coordinates
  - (x,y) becomes (x,y,1)

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = M \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = M \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Translation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Scaling

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotation

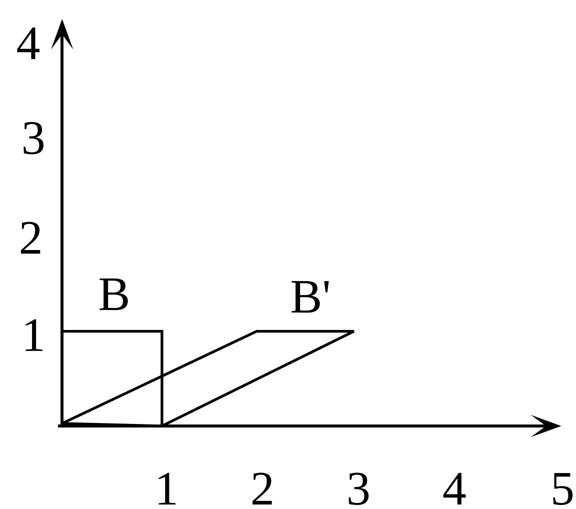
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \text{cos¶} & \text{-sin¶} & 0 \\ \text{sin¶} & \text{cos¶} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Shear

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & g & 0 \\ h & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$x' = x + g\,y$
$y' = y + h\,x$

$g = 2$
$h = 0$



Rigid Body Transformations

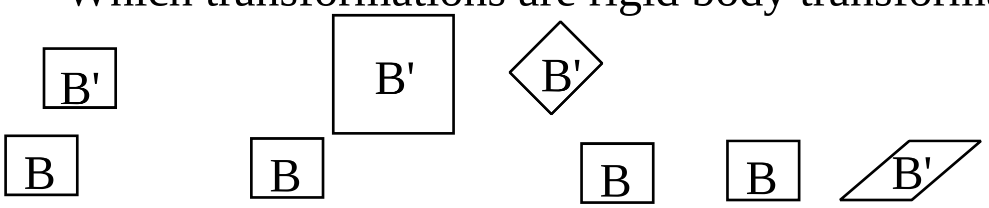
Preserve lines, angles and lengths

A line transforms to a line

Two lines forming angle ¶ transform to two lines with angle ¶

A line of length A transforms to a line of length A

Which transformations are rigid body transformations?



Affine Transformations

Preserve lines and parallelism of lines, but not lengths and angles

A line transforms to a line

Two parallel lines transform to two parallel lines

Which transformations are affine?

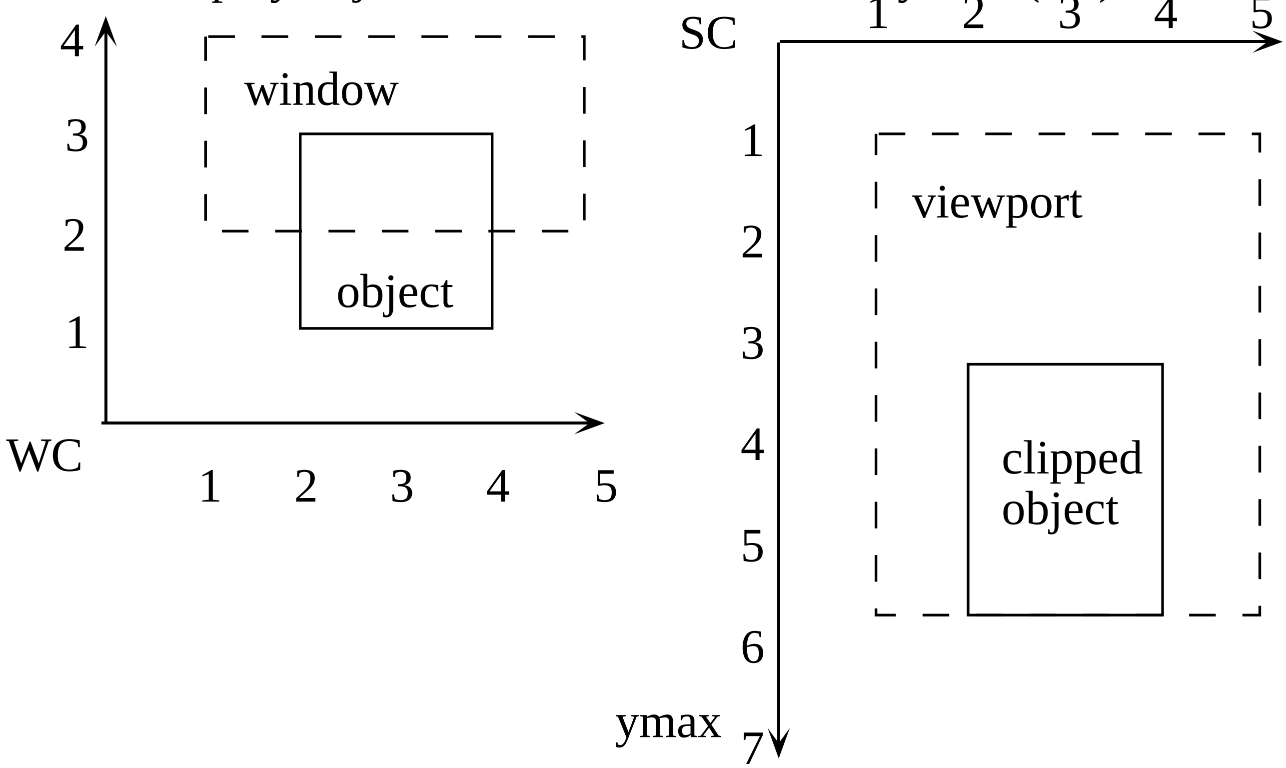
Which type of transformations would you use to display an object in motion?

Does it depend on the type of object?

The Window to Viewport Transformation

Define objects in World Coordinate System (WC)

Display objects in Screen Coordinate System (SC)



Define window (world coordinate window - not X window) in WC

Define viewport in screen coordinates (or normalized device coordinates)

Transform from WC to SC such that window maps to viewport

Lets first just look at transformation due to change in origin

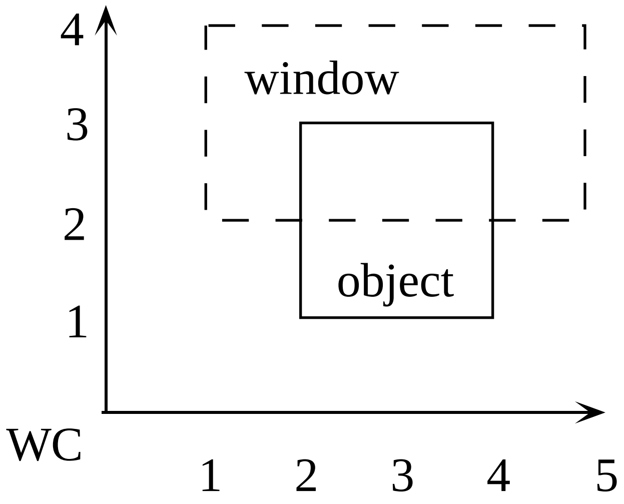
Want to map (0,0) in WC to (0,7) in SC

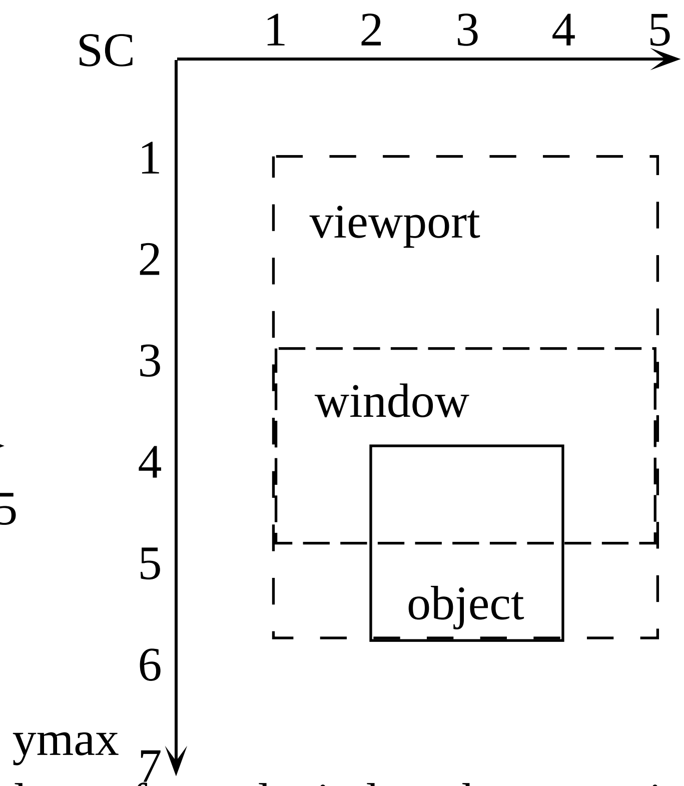
(2,2) (2,5)

For p in WC and p' in SC

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & ymax \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

After that transformation





Now need to make viewport and transformed window the same size

$s_x = 1$

$s_y = 5/2$

$xl = 1,\ x'l = 1 \qquad xr = 5,\ x'r = 5$

$yb = 5,\ y'b = 12.5 \qquad yt = 3,\ y't = 7.5$

Now need to translate to correct position

$t_x = 0$

$t_y = -6.5$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & ymax \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x'' \\ y'' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x''' \\ y''' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x'' \\ y'' \\ 1 \end{bmatrix}$$

Compose the matrices

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & ymax \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x'' \\ y'' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x''' \\ y''' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x'' \\ y'' \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x''' \\ y''' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & ymax \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x''' \\ y''' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & t_x \\ 0 & s_y & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & ymax \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x''' \\ y''' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & t_x \\ 0 & -s_y & t_y - s_y\, ymax \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

So this is the transformation to go from WC to SC

Called the Viewing Transformation

Can compose Viewing Transformation with Modeling Transformations

Say store model of an object in Object Coordinates (OC)

Create an instance of the object and scale, rotate and translate it (Modeling transformation)

P' = M P

Then Clip object with window

P'' = C(P')

Then apply viewing transformation

P''' = V P''

What is the problem in terms of efficiency?

So when to clip?

MT	C	VT	Can't compose matrices
MT	VT	C	Must transform all points, not just viewable points
C	MT	VT	Must compute backtransformations on the clip box (but just once)

Tradeoff, with choice determined by application and how doing clipping

Matrix multiplication is associative

Which combinations of transforms are commutative?

Eg. $T_1 T_2 = T_2 T_1$

True for two rotations? (How tell?)

for two scalings?

for two shears?

$$\begin{bmatrix} 1 & g_1 & 0 \\ h_1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & g_2 & 0 \\ h_2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} =? \begin{bmatrix} 1 & g_2 & 0 \\ h_2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & g_1 & 0 \\ h_1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} g_1h_2+1 & g_2+g_1 & 0 \\ h_1+h_2 & g_2h_1+1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} g_2h_1+1 & g_2+g_1 & 0 \\ h_2+h_1 & g_1h_2+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Symmetric scaling with rotation is commutative

Translation and rotation?