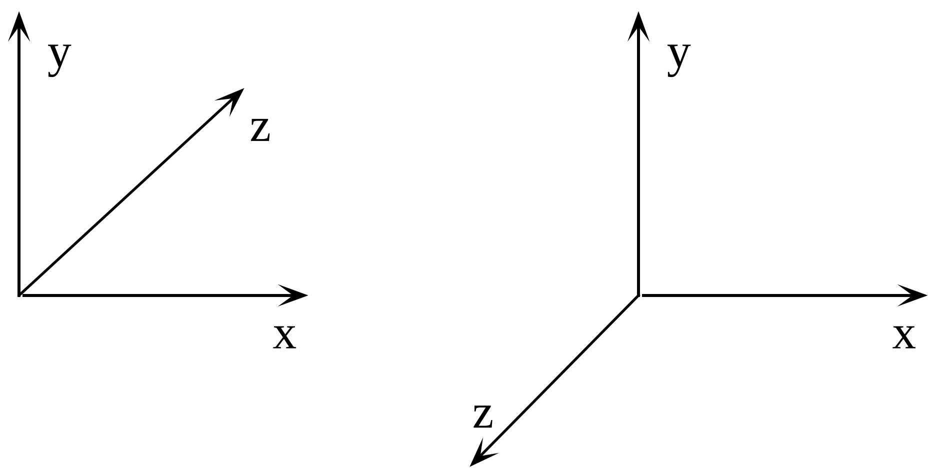
3-D Transformations

3-D space

Two conventions for coordinate systems

Left-Hand vs Right-Hand

(Thumb is the x axis, index is the y axis)



Which is which?

Most graphics systems use left hand coordinates

Homogeneous Coordinates

3-D point (x,y,z)

Homogeneous (x,y,z,1)

Translation

$$(x',y',z',1) = (t_x, t_y, t_z, 0) + (x,y,z,1)$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Scaling

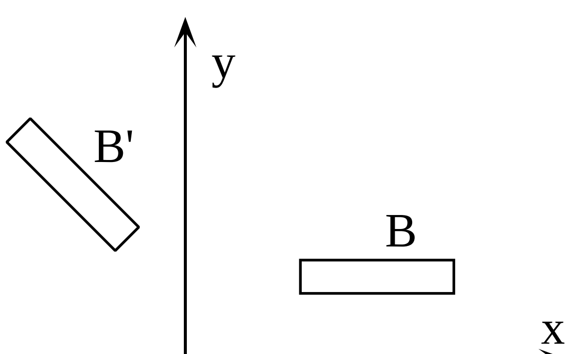
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Rotation

Not as simple to extend to 3D

3 possible axes to rotate about

All rotations counter clockwise - when looking down the axis (see figure)



2D is rotation about what axis?

2D

Notes:

When rotating about a given axis, the coordinates of that axis are always unchanged

So in the rotation matrices, the row and column of the rotation axis are as the identity matrix

Rotation Matrices differ for right and left hand systems (Use right-hand here (assume in modelling coordinate system))

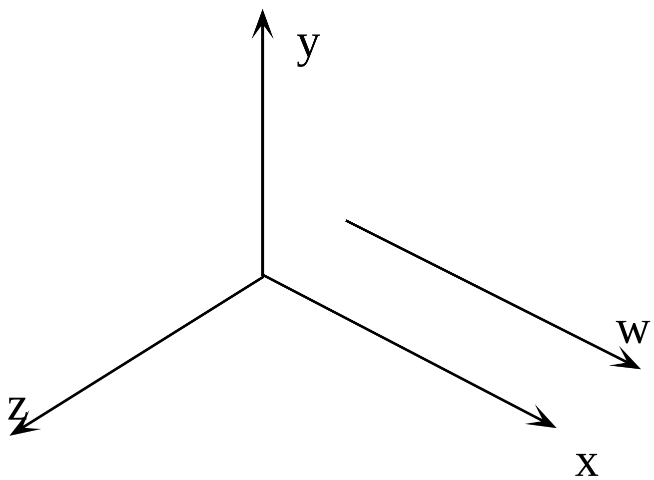
So what is the rotation about the z axis?

$$R_z(\P) = \begin{bmatrix} \cos \P & -\sin \P & 0 & 0 \\ \sin \P & \cos \P & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_x(\P) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \P & -\sin \P & 0 \\ 0 & \sin \P & \cos \P & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_y(\P) = \begin{bmatrix} \cos \P & 0 & \sin \P & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \P & 0 & \cos \P & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

How would one rotate an object about an axis that is parallel to an axis of the coordinate system?



origin of w = (1,1,0)

w is parallel to x

Does this completely specify x?

Translate w to x
Rotate by ¶ about x
Translate back
(see figure)

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -t_x \\ 0 & 1 & 0 & -t_y \\ 0 & 0 & 1 & -t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \P & -\sin \P & 0 \\ 0 & \sin \P & \cos \P & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

For example above, what are values for translations?

How to rotate about an arbitrary axis

Translate and rotate arbitrary axis onto a coordinate axis

Perform rotation

Back transform

How transform arbitrary axis onto a coordinate axis?

Translate origin of arbitrary axis to origin of coordinate axes (see figure)

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

If use P_1 - get rotation one way about the line P_1P_2

If use P_2 - get rotation other way

$P_1 = (x_1, y_1, z_1)$

$P_2 = (x_2, y_2, z_2)$

What values of t_x, t_y, and t_z?

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Now rotate P_2 onto the z axis

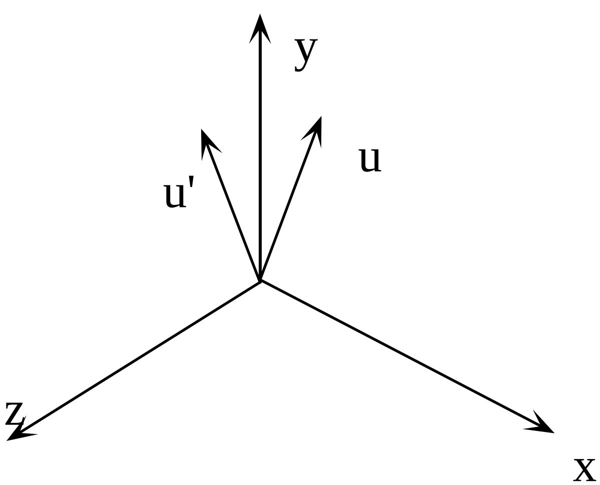
(we will rotate about the z axis - could choose any axis)

But can't rotate about an arbitrary axis!

Solution: break into two rotations about coordinate axes

Rotate about x axis until P_2 lies in xz plane
(see figure)

Rotate about y axis until P_2 lies on z axis
(see figure)

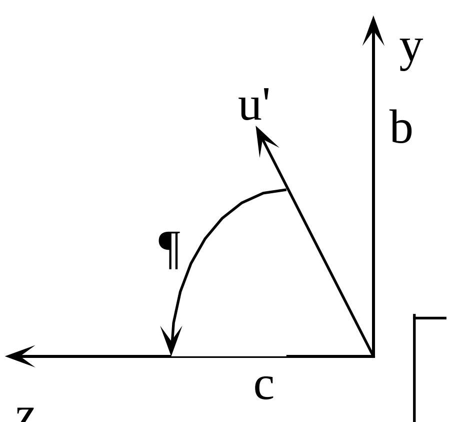


Look at unit vector, u, along P_1P_2

Look at projection, u', of u onto yz plane

u' = (0, b, c)

Need to rotate u such that u' lies on z axis



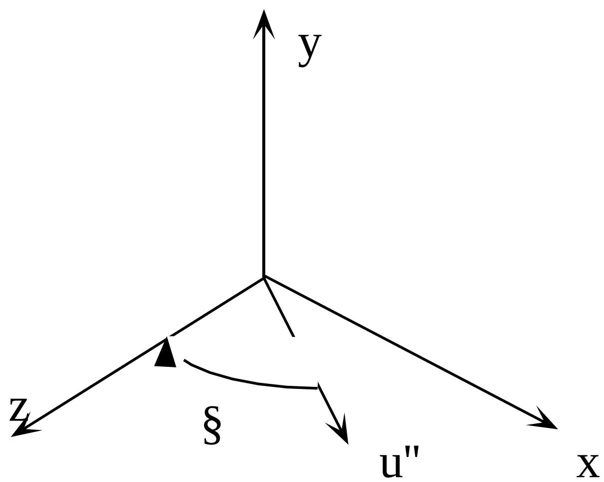
Let d = Ã $b^2 + c^2$

sin ¶ = b/d

cos ¶ = c/d

$$R_x(\P) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \P & -\sin \P & 0 \\ 0 & \sin \P & \cos \P & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c/d & -b/d & 0 \\ 0 & b/d & c/d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now need to rotate about y until P_2 lies on z axis



u" is the result of last rotation

u" = (a, 0, d)

sin § = -a

cos § = d

$$R_y(§) = \begin{bmatrix} \cos § & 0 & \sin § & 0 \\ 0 & 1 & 0 & 0 \\ -\sin § & 0 & \cos § & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} d & 0 & -a & 0 \\ 0 & 1 & 0 & 0 \\ a & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now can rotate by desired amount about z axis
(see figure)

$$R_z(½) = \begin{bmatrix} \cos ½ & -\sin ½ & 0 & 0 \\ \sin ½ & \cos ½ & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation transformation up to this point =

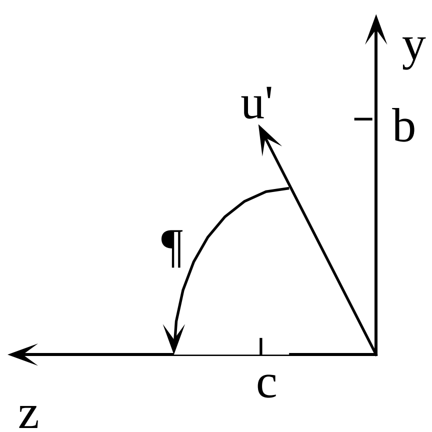
$$\begin{bmatrix} \cos ½ & -\sin ½ & 0 & 0 \\ \sin ½ & \cos ½ & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d & 0 & -a & 0 \\ 0 & 1 & 0 & 0 \\ a & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c/d & -b/d & 0 \\ 0 & b/d & c/d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \cos ½ & -\sin ½ & 0 & 0 \\ \sin ½ & \cos ½ & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d & -ab/d & -ac/d & 0 \\ 0 & c/d & -b/d & 0 \\ a & b & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

And don't forget that initial translation:

$$\begin{bmatrix} \cos ½ & -\sin ½ & 0 & 0 \\ \sin ½ & \cos ½ & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d & -ab/d & -ac/d & 0 \\ 0 & c/d & -b/d & 0 \\ a & b & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -x_1 \\ 0 & 1 & 0 & -y_1 \\ 0 & 0 & 1 & -z_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

What are a, b, c, and d in terms of P_1 and P_2?



How transform P_1P_2 to unit vector?

Already moved P_1 to origin

$P'_2 = (x_2-x_1, y_2-y_1, z_2-z_1)$

$P'_2 = (x',y',z')$

Divide by its length

to give it unit length

D =length = $Ã(x'^2+y'^2+z'^2)$

$b = y'/D$, $c = z'/D$, $d = Ã(b^2+c^2)$

$a = x'/D$

To complete rotation must now back transform with two rotations and the translation, all with minus values (see figure)

What about scaling an object relative to a fixed point? (see figure)

Translate fixed point to the origin

Scale

Back translate

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -t_x \\ 0 & 1 & 0 & -t_y \\ 0 & 0 & 1 & -t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & -t_x \\ 0 & s_y & 0 & -t_y \\ 0 & 0 & s_z & -t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

What are t_x, t_y and t_z here?

Properties of 3-D Affine Transformations

Lines are transformed into lines

Parallel lines are transformed into parallel lines

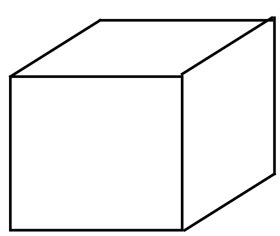
Proportional distances are preserved

Volumes are scaled by the determinant of M,
where M is the transformation matrix

$P' = M P$

$\text{Vol}(P') = |M| \, \text{Vol}(P)$

Note:
It's not true that the 2D projections of these 3D transformations have these properties!



This is the projection of a unit cube

Is the length preserved?