Curves

How to represent curved surfaces?

How to represent curves?

True representations

Nonparametric:

Explicit functions

\[ y = f(x) \]

Implicit functions

\[ x^2 + y^2 - r^2 = 0 \]

Parametric:

\[ P(u) = (x(u), y(u)) \]

Approximate representations

Short straight line segments

polyline
What representations should be used in graphics?

Say use DDA for circles

Problems:

Don't just represent end points  
thus complex display structures

Also hard to transform  
don't just transform end points  
eg - circle to ellipse

Clipping  
algorithm only work for points and straight lines

Not simple analytical expression  
must consider which quadrant of circle being plotted  
(different expressions for different octants)

Better if use parametric representations

Allow multivalued functions

Must allow for easy interpolation

unknown curve

want smooth interpolation
How to draw arbitrary curves?

i.e. CAD - design car body, (without polygons)

old technique - artist draw car and sculpted model
draftsman copied curves onto a technical drawing
(used spline - stiff metal strip - bend around curve
stays bent - move and copy to paper)

with CAD

how input curve?

1) digitize points on drawing and interpolate

2) design body using graphics system
draw it using a bit pad

but if want to scale and rotate, then better
to have representation that is not just
a set of points

have designer specify curves in parametric form
B-splines - polynomial of degree n
Designer specifies and manipulates
the control points
the system draws in the curve
B-Splines

piecewise polynomial functions

countrol points: v1, v2, v3, v4

set of data points on the curve: x1, x2, x3, x4

want to interpolate the set of data points, x_i, i = 1 to n

to get x(s), the interpolated points, (s is the parameter)

assume x(i) = x_i

(s takes on integer values at data points)

\[ x(s) = \sum_{i=0}^{n+1} v_i B_i(s) \]

\[ v_i \] are the coefficients representing the curve

the vertices of the guiding polygon

\[ B_i \] are the basis functions

the blending functions

Linear

Quadratic
B-Splines are variation diminishing
they vary less than their guiding polygon

\[ B_i \]

1 2 3 4 5

Cubic

\[ B_i \]

B-Splines are variation diminishing
they vary less than their guiding polygon

\[ B_i \]

Cubic give the most continuity:
position
slope
curvature

\[ B_i \]

If spans of uniform size, then get uniform shape basis functions
Assume uniform spans

n data points

(n + 2) $v_i$'s

2 extra give the boundary conditions - what happens at the ends

if closed curve, then $v_0 = v_n$ and $v_{n+1} = v_1$

For cubic B-spline

only four nonzero basis function at each point on the curve

so for any point $x(s)$,
find span
use four terms in summation
How Represent the Basis Functions?

Four Basis Functions / Span

\[ x(s) = C_{i-1,3}(s) v_{i-1} + C_{i,2}(s) v_i + C_{i+1,1}(s) v_{i+1} + C_{i+2}(s) v_{i+2} \]

Note that \( C_{i-1,3} = C_{i,3} \) etc

So only define over 0 to 1

\[ C_{i,j}(s) = C_j(s-i) \quad i = 0, 1, \ldots, n+1 \quad j = 0, 1, 2, 3 \]

\[ x(s) = C_3(s-i) v_{i-1} + C_2(s-i) v_i + C_1(s-i) v_{i+1} + C_0(s-i) v_{i+2} \]

What are the C's?

define over \( t = 0 \) to 1

For cubic B-spline

\[ C_0(t) = \frac{t^3}{6} \]

\[ C_1(t) = \frac{-3t^3 + 3t^2 + 3t + 1}{6} \]

\[ C_2(t) = \frac{3t^3 - 6t^2 + 4}{6} \]

\[ C_3(t) = \frac{-t^3 + 3t^2 - 3t +1}{6} \]
In matrix form

\[
\begin{bmatrix}
C_3 & C_2 & C_1 & C_0
\end{bmatrix} = \frac{1}{6} \begin{bmatrix}
t^3 & t^2 & t & 1
\end{bmatrix} \begin{bmatrix}
-1 & 3 & -3 & 1 \\
3 & -6 & 3 & 0 \\
-3 & 0 & 3 & 0 \\
1 & 4 & 1 & 0
\end{bmatrix}
\]

If calculate $x(t)$ over $t = 0, 1$

\[
\begin{bmatrix}
C_3 & C_2 & C_1 & C_0
\end{bmatrix} = \begin{bmatrix}
vi-1 \\
v_i \\
v_{i+1} \\
v_{i+2}
\end{bmatrix}
\]

So, given the basis functions

segment them to get the $C_j$

have $v_i$

find $x_i(t)$
How to find just the data points?

just the $x_i$'s

t = 0

What is $[ C_3 \ C_2 \ C_1 \ C_0 ]$ ?

$$x_i = \frac{1}{6} \begin{bmatrix} 1 & 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} v_{i-1} \\ v_i \\ v_{i+1} \\ v_{i+2} \end{bmatrix}$$

For closed curve with $V_0 = V_n$ and $V_{n+1} = V_1$

$$\begin{bmatrix} x_0 \\ x_1 \\ \cdot \cdot \cdot \\ x_{n-1} \\ x_n \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ \cdot \cdot \cdot \\ 1 & 4 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ \cdot \cdot \cdot \\ v_{n-1} \\ v_n \end{bmatrix}$$

Now easy to see how to find $x_i$ from $v_i$
and $v_i$ from $x_i$
How do you get a sharp corner in a B-spline?

If add additional identical data points?

B-splines have local control

move one control point and how much of the curve changes

with linear basis functions?

with quadratic?

with cubic?
Bezier Splines

another common spline

global control

number of blending functions is a function of the number of data points

\[ B_i \]
\[ s \]
3 data points

\[ B_i \]
\[ s \]
4 data points