Curves

- How to represent curved surfaces?
- How to represent curves?
    - True representations
        - Nonparametric:
            - Explicit functions
                - $y = f(x)$
            - Implicit functions
                - $x^2 + y^2 - r^2 = 0$
        - Parametric:
            - $P(u) = (x(u), y(u))$
    - Approximate representations
        - Short straight line segments
            - polyline

What representations should be used in graphics?

Say use DDA for circles

Problems:

Don't just represent end points
thus complex display structures

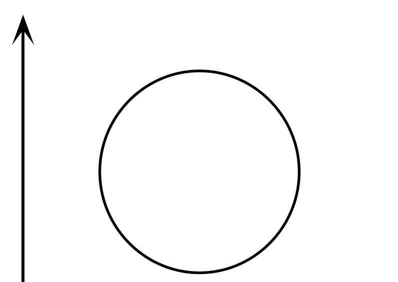
Also hard to transform
don't just transform end points
eg - circle to ellipse

Clipping
algorithms only work for points and straight lines

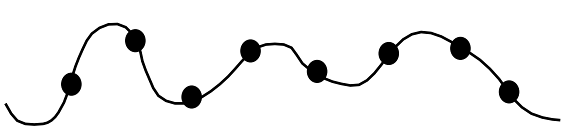
Not simple analytical expression
must consider which quadrant of circle being plotted
(different expressions for different octants)

Better if use parametric representations

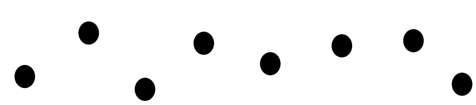
Allow multivalued functions



Must allow for easy interpolation



unknown curve



samples

want smooth interpolation

How to draw arbitrary curves?

i.e. CAD - design car body, (without polygons)

old technique - artist draw car and sculpted model
draftsman copied curves onto a technical drawing
(used spline - stiff metal strip - bend around curve
stays bent - move and copy to paper)

with CAD

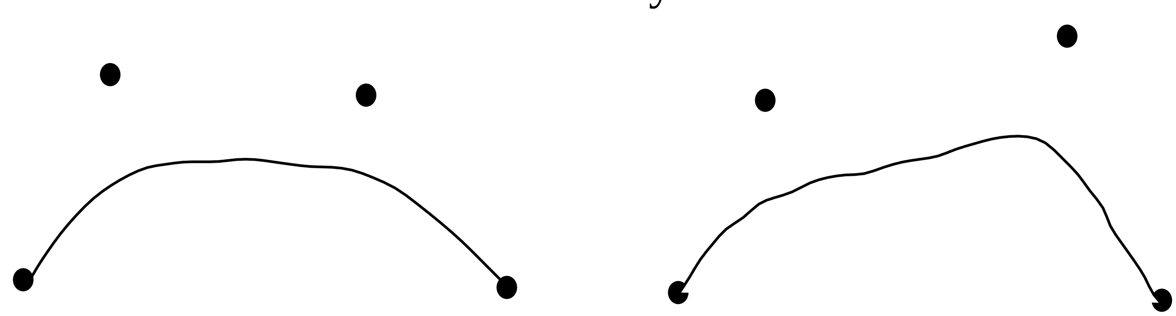
how input curve?

1) digitize points on drawing and interpolate

2) design body using graphics system
draw it using a bit pad

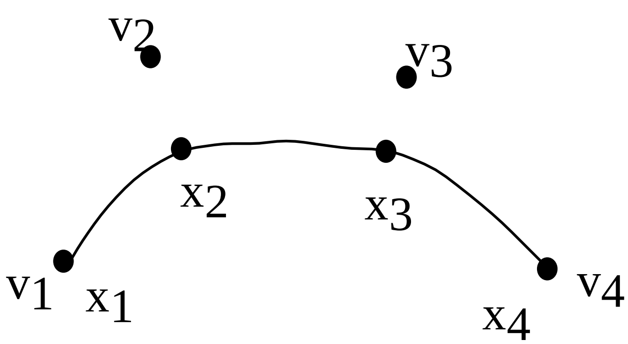
but if want to scale and rotate, then better
to have representation that is not just
a set of points

have designer specify curves in parametric form
B-splines - polynomial of degree n
Designer specifies and manipulates
the control points
the system draws in the curve



B-Splines

piecewise polynomial functions



control points: v_1, v_2, v_3, v_4

set of data points on the curve:
x_1, x_2, x_3, x_4

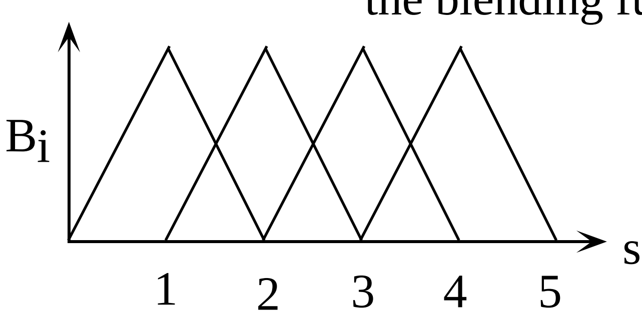
want to interpolate the set of data points, x_i, $i = 1$ to n
to get $x(s)$, the interpolated points, (s is the parameter)

assume $x(i) = x_i$
(s takes on integer vales at data points)

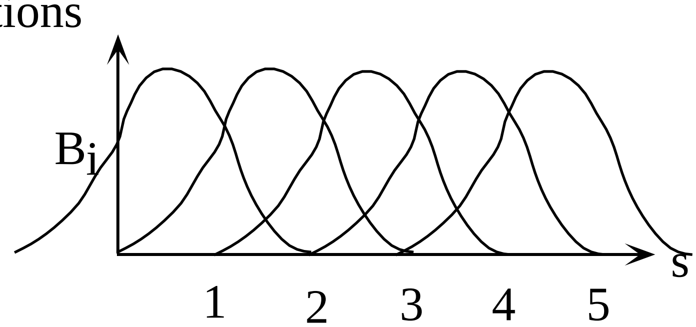
$$x(s) = \sum_{i=0}^{n+1} v_i B_i(s)$$

v_i are the coefficients representing the curve
the vertices of the guiding polygon

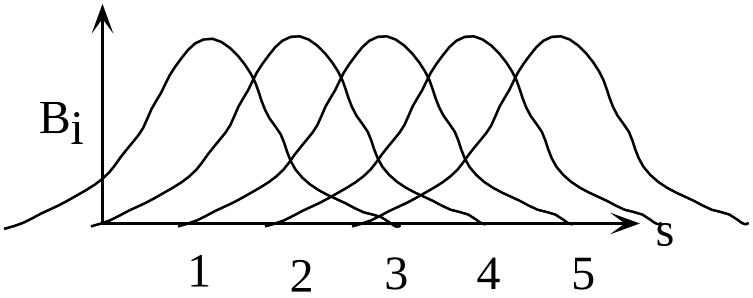
B_i are the basis functions
the blending functions



Linear

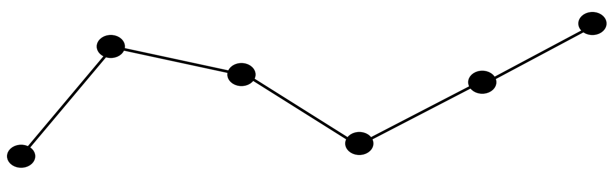


Quadratic

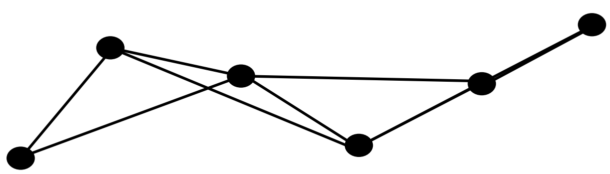


Cubic

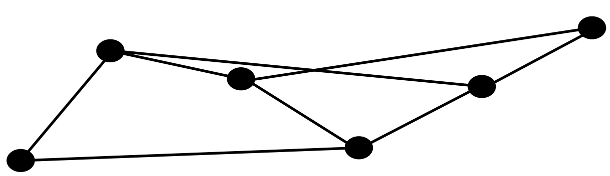
B-Splines are variation diminishing
they vary less than their guiding polygon



linear: n = 1



quadratic: n = 2



cubic: n = 3

n is the degree of the interpolating polynomial

Cubic give the most continuity:
- position
- slope
- curvature

If spans of uniform size, then get uniform shape basis functions

Assume uniform spans

n data points

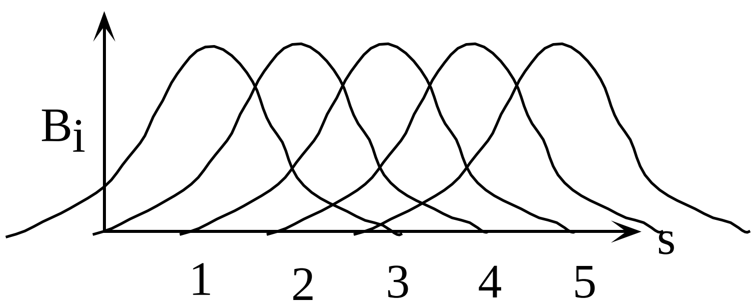
(n + 2) v_i's

2 extra give the boundary conditions - what happens at the ends

if closed curve, then $v_0 = v_n$ and $v_{n+1} = v_1$

For cubic B-spline

only four nonzero basis function at each point on the curve

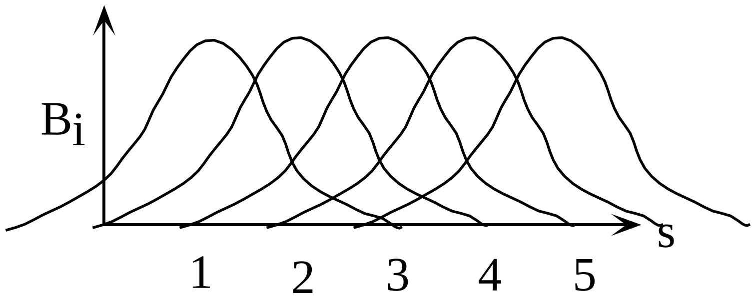


on (i, i+1)

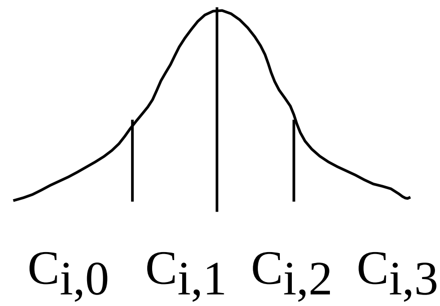
$B_{i-1}(s)$ $B_i(s)$ $B_{i+1}(s)$ $B_{i+2}(s)$

so for any point x(s),
find span
use four terms in summation

How Represent the Basis Functions?



Four Basis Functions / Span



Four parts / function

$x(s) = C_{i-1,3}(s)\ v_{i-1} + C_{i,2}(s)\ v_i + C_{i+1,1}(s)\ v_{i+1} + C_{i+2}(s)\ v_{i+2}$

Note that $C_{i-1,3} = C_{i,3}$ etc

So only define over 0 to 1

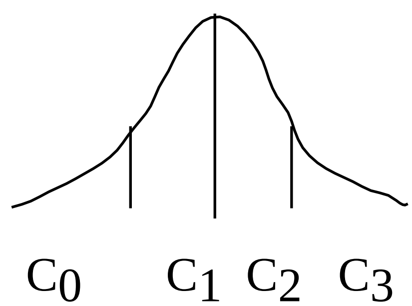
$C_{i,j}(s) = C_j(s-i) \qquad i = 0, 1, \ldots n+1 \qquad j = 0, 1, 2, 3$

$x(s) = C_3(s-i)\ v_{i-1} + C_2(s-i)\ v_i + C_1(s-i)\ v_{i+1} + C_0(s-i)\ v_{i+2}$

What are the C's?

define over t = 0 to 1

For cubic B-spline

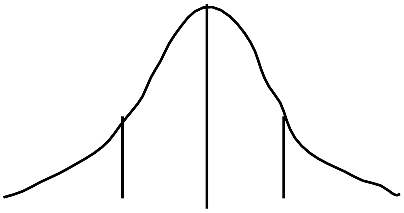


$C_0(t) = t^3 / 6$

$C_1(t) = (-3t^3 + 3t^2 + 3t + 1) / 6$

$C_2(t) = (3t^3 - 6t^2 + 4) / 6$

$C_3(t) = (-t^3 + 3t^2 - 3t + 1) / 6$



C_0 C_1 C_2 C_3

C_0 0 to 1/6

C_1 1/6 to 4/6

C_2 4/6 to 1/6

C_3 1/6 to 0

In matrix form

$$\begin{bmatrix} C_3 & C_2 & C_1 & C_0 \end{bmatrix} = 1/6 \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}$$

If calculate x(t) over t = 0, 1

$$x_i(t) = \begin{bmatrix} C_3 & C_2 & C_1 & C_0 \end{bmatrix} \begin{bmatrix} v_{i-1} \\ v_i \\ v_{i+1} \\ v_{i+2} \end{bmatrix}$$

So, given the basis functions

segment them to get the C_j

have v_i

find $x_i(t)$

How to find just the data points?

just the x_i's

$t = 0$

What is [C_3 C_2 C_1 C_0] ?

$$x_i = 1/6 \begin{bmatrix} 1 & 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} v_{i-1} \\ v_i \\ v_{i+1} \\ v_{i+2} \end{bmatrix}$$

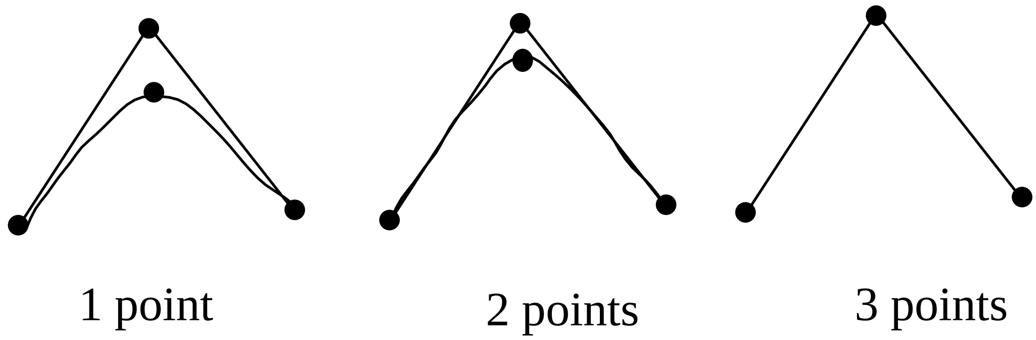
For closed curve with $V_0 = V_n$ and $V_{n+1} = V_1$

$$\begin{bmatrix} x_0 \\ x_1 \\ . \\ . \\ . \\ x_{n-1} \\ x_n \end{bmatrix} = 1/6 \begin{bmatrix} 4 & 1 & & & & 1 \\ 1 & 4 & 1 & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & 1 & 4 & 1 \\ 1 & & & & 1 & 4 \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ . \\ . \\ . \\ v_{n-1} \\ v_n \end{bmatrix}$$

Now easy to see how to find x_i from v_i
and v_i from x_i

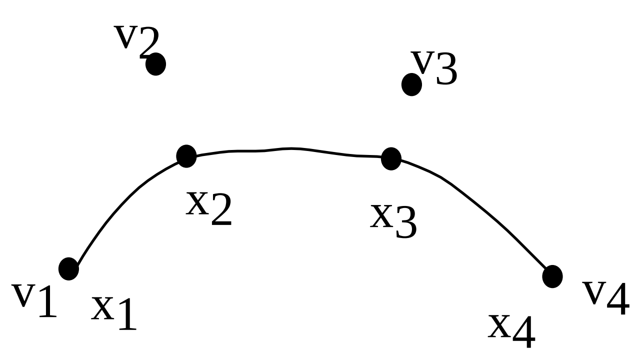
How do you get a sharp corner in a B-spline?

If add additional identical data points?



B-splines have local control

move one control point and how much of the curve changes



with linear basis functions?

with quadratic?

with cubic?

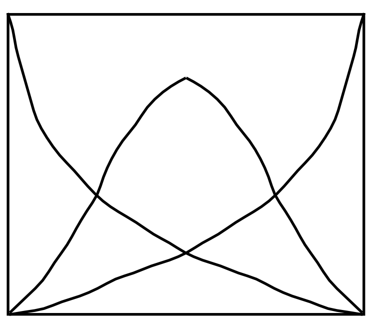
Bezier Splines

another common spline

global control

number of blending functions is a function of the number of data points

B_i



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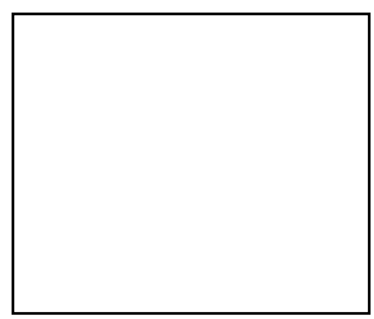
3 data points

B_i



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4 data points