Restricted Constrained Delaunay Triangulations

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Abstract

We introduce the *restricted constrained Delaunay triangulation* (restricted CDT). The restricted CDT generalizes the restricted Delaunay triangulation, allowing us to define a triangulation of a surface that includes a set of constraining segments. Under certain sampling conditions, the restricted CDT includes every constrained segment and suggests an algorithm that produces a triangulation of the surface that contains every constraining segment.

1. Introduction

Surface triangulations are used in computer graphics, simulations of thin plates and shells, and boundary element methods for solving partial differential equations. Given a surface $\Sigma \subset \mathbb{R}^3$ (without boundary) and a finite set of sample points $V \subset \Sigma$, the *restricted Delaunay triangulation* of V with respect to Σ is a rigorous way to define a Delaunay-like surface triangulation whose mathematical properties facilitate algorithms for generating meshes with guaranteed quality [1]. Here, we study a variant where we are also given a set S of line segments whose endpoints are in V. Our goal is to construct a triangulation $\mathscr{T}$ of Σ that contains every segment in S. See Figure 1.

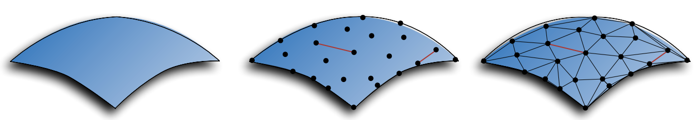

Figure 1: Given a set of points sampled from Σ and a set of segments, red, we wish to compute a triangulation of Σ that contains all of the red segments.

Let $\mathscr{T}$ be a simplicial complex. The *underlying space* of $\mathscr{T}$ is $|\mathscr{T}| = \cup_{\tau \in \mathscr{T}} \tau$, the union of all simplicies in $\mathscr{T}$. We say that $\mathscr{T}$ is a triangulation of Σ if $|\mathscr{T}|$ is homeomorphic to Σ. The goal of surface mesh generation is to compute a triangulation $\mathscr{T}$ of Σ that also approximates the geometry of Σ well.

The *medial axis* M of Σ is the closure of the set of all points in $\mathbb{R}^d$ that have at least two closest points on Σ. Intuitively, the medial axis of Σ is meant to capture the middle of the object. The *local feature size* function is $\operatorname{lfs} : \Sigma \to \mathbb{R}$, $p \mapsto d(p, M)$ where $d(p, M)$ denotes the distance from p to M. A finite point set V is called an ε-sample of Σ if for every point $p \in \Sigma$, $d(p, V) \leq \varepsilon \operatorname{lfs}(p)$. That is, there is some sample point $v \in V$ whose distance from p is no greater than $\varepsilon \operatorname{lfs}(p)$.

We assume that the reader is familiar with Delaunay triangulations and Voronoi diagrams, as well as their basic properties. Consider a Voronoi cell $\operatorname{Vor} v$ for some $v \in V$. We define the *Voronoi cell restricted to* Σ as $\operatorname{Vor}|_\Sigma v = \operatorname{Vor} v \cap \Sigma$. We can define every lower-dimensional Voronoi face similarly. The *restricted Voronoi diagram* $\operatorname{Vor}|_\Sigma V$ is the set of all restricted Voronoi cells and their faces. The *restricted Delaunay triangulation* $\operatorname{Del}|_\Sigma V$ is the simplicial complex dual to $\operatorname{Vor}|_\Sigma V$. A j-simplex σ is in $\operatorname{Del}|_\Sigma V$ if and only if $\cap_{v \in \sigma} \operatorname{Vor}|_\Sigma v \neq \emptyset$. In words, a simplex in $\operatorname{Del} V$ is in $\operatorname{Del}|_\Sigma V$ if and only if its dual Voronoi face intersects Σ [1].

Our main result is that, under certain sampling conditions on V and S, we can construct a triangulation $\mathscr{T}$ of Σ that contains the segments in S. To this end, we introduce the *restricted constrained Delaunay triangulation*, which is a generalization of the restricted Delaunay triangulation to enforce constraining edges.

2. Portals

Informally, a *portal* P is a subset of a topological space X that acts as a doorway between topological spaces. A portal has two "sides" along each of which we glue a new topological space, say Y and Y'. A path entering P from one side continues in Y, whereas a path entering from the other side continues in Y'. Let $X = \mathbb{R}^3$ and let S be a set of line segments. For each segment $s \in S$, the user specifies a plane h_s that includes s. Denote by $\mathbf{n}_s$ a unit vector normal to h_s.

Now consider the diametric ball B_s of s — the smallest circumscribing ball of s. The intersection $B_s \cap h_s$ is a disk which we call P_s. The disk P_s is the diametric ball of s with respect to the space h_s. The disk P_s will serve as our portal. The relative interior of P_s is the interior of P_s with respect to its affine hull h_s. By a slight abuse of notation we will denote the relative interior by $\operatorname{Int} P_s$.

We construct the space $X_s = X - \operatorname{Int} P_s$, $\mathbb{R}^3$ with the interior of P_s removed. The space X_s can be endowed with a metric as follows. Let $\gamma : [0,1] \to X_s$ be a continuous curve and define the length of γ as

$$L(\gamma) = \sup_{0=t_0 \leq t_1 \leq \ldots \leq t_{n-1} \leq t_n = 1} \sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i))$$

where the supremum is taken over all subdivisions of γ and d is the Euclidean metric on X. Then the *induced length metric* $\widehat{d}$ is given by

$$\widehat{d}(x,y) = \inf_{\gamma} L(\gamma). \tag{1}$$

It can be easily checked that the space $(X_s, \widehat{d})$ is a metric space. Notice that X_s is not complete as a metric space because $\operatorname{Int} P_s$ is missing. For any metric space Y, the completion of Y, denoted $\overline{Y}$, is a complete metric space that includes Y as a dense subset. Every metric space can be shown to have a completion by defining an equivalence relation over the set of all Cauchy sequences and adding a convergence point for each equivalence class of Cauchy sequences.

The metric $\widehat{d}$ on X_s distinguishes Cauchy sequences that approach P_s from different sides of h_s. Thus the completion $\overline{X}_s$ contains two distinct copies of $\operatorname{Int} P_s$, denoted by P_s^+, P_s^-, one for each side of h_s. Let $x \equiv y$ if x and y have the same coordinates. Let $\mathbb{R}^3_+, \mathbb{R}^3_-$ be two copies of $\mathbb{R}^3$ and define an equivalence relation $\sim$ as

$$x \sim y \iff \begin{cases} x = y & x, y \in \overline{X}_s \text{ or } x, y \in \mathbb{R}^3_+ \text{ or } x, y \in \mathbb{R}^3_- \\ x \equiv y & x \in \mathbb{R}^3_+ \text{ and } y \in P_s^+ \\ x \equiv y & x \in \mathbb{R}^3_- \text{ and } y \in P_s^-. \end{cases}$$

In words, we glue $\mathbb{R}^3_+$ to $\overline{X}_s$ along P_s^+ and glue $\mathbb{R}^3_-$ along P_s^-. With $\sim$ we construct the quotient space

$$\widetilde{X} = \overline{X}_s \sqcup \mathbb{R}^3_+ \sqcup \mathbb{R}^3_- / \sim .$$

We refer to $\overline{X}_s$ as the *principal branch* and refer to each of $\mathbb{R}^3_+, \mathbb{R}^3_-$ as *secondary branches*. Figure 2 illustrates this construction in $\mathbb{R}^2$.

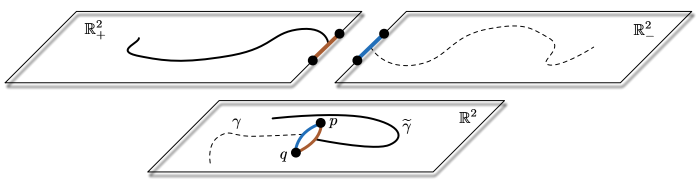

Figure 2: Completing the slitted plane creates a hole in $\mathbb{R}^2$ bounded by two curves marked in blue and orange. The equivalence relation $\sim$ identifies the blue path in the principal branch with the one in $\mathbb{R}^2_-$ and similarly for the orange path. A path in the principal branch that enters the portal on one side continues in the appropriate secondary branch.

The space $\widetilde{X}$ can then be endowed with the induced length metric $\widehat{d}$ as in Equation 1. When considering the length of a continuous curve γ the length of each segment in the subdivision is measured using the metric of the branch it is contained in. However since the metrics on each branch are identical this does not cause any difficulties. The following fact is immediate.

Lemma 1. *Let $(\widetilde{X}, \widehat{d})$ be the metric space defined above and let $\gamma : [0,1] \to \widetilde{X}$ be a shortest path between $x, y \in \widetilde{X}$. Then γ is a piecewise curve comprised of straight line segments.*

The construction works for any number of segments. We start by removing the portals P_s of all segments from X, $X_S = X - \cup_{s \in S} \operatorname{Int} P_s$, then take the completion $\overline{X}_S$. Then we construct the quotient space with $2m$ copies of $\mathbb{R}^3$ glued along the $2m$ portals bounding the m holes in the completion $\overline{X}_S$. The resulting space $\widetilde{X}$ can again be endowed with the induced length metric $\widehat{d}$. Lemma 1 still holds.

We also surgically modify Σ and embed an extended surface in $\widetilde{X}$. Consider a segment s with endpoints p, q and let $\gamma_s = h_s \cap \Sigma \cap B_s$. As h_s locally intersects Σ transversally, the intersection $h_s \cap \Sigma$ is a curve, possibly with multiple components. Thus γ_s is a curve along Σ contained in B_s from p to q.

We can then extrude the curve γ_s into each of the secondary branches connected to the portal P_s. For each point $x \in \gamma_s$ we extrude a ray in the direction of $\mathbf{n}_s$ into $\mathbb{R}^3_+$, and another in the direction of $-\mathbf{n}_s$ into $\mathbb{R}^3_-$. More precisely, we define the ruled surfaces

$$\Sigma_s^+ = \{\gamma_s(u) + v\mathbf{n}_s \in \mathbb{R}^3_+ : u \in [0,1], v \in [0,\infty)\}$$

and

$$\Sigma_s^- = \{\gamma_s(u) - v\mathbf{n}_s \in \mathbb{R}^3_- : u \in [0,1], v \in [0,\infty)\},$$

which are extruded into $\mathbb{R}^3_+$ and $\mathbb{R}^3_-$ respectively. Define an equivalence relation $\sim_\Sigma$ that identifies points along γ_s on all three surfaces. Our extended surface $\widetilde{\Sigma} = \Sigma \sqcup \bigsqcup_{s \in S} \Sigma_s^+ \sqcup \bigsqcup_{s \in S} \Sigma_s^- / \sim_\Sigma$. See Figure 3.

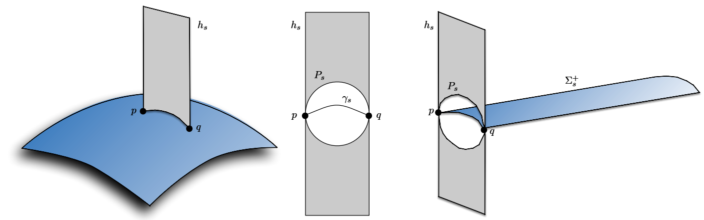

Figure 3: The plane h_s intersects Σ in a curve. We consider the portion of $h_s \cap \Sigma$ included in the diametric ball B_s. The curve γ_s is extruded into $\mathbb{R}^3_+$ in the direction $\mathbf{n}_s$ orthogonal to h_s. The surface Σ_s^+ thus defined is then glued to Σ along γ_s at the entrance to the portal P_s.

3. Restricted Constrained Delaunay Triangulations

Voronoi diagrams can be defined in an obvious way for any metric space. To define the restricted constrained Delaunay triangulation, we start by defining the *extended Voronoi diagram* in $\widetilde{X}$. For any $v \in V$, the *extended Voronoi cell* of v is defined as

$$\operatorname{Evor} v = \{x \in \widetilde{X} : \widehat{d}(x,v) \le \widehat{d}(x,u),\ \forall u \in V\}.$$

Then the extended Voronoi diagram $\operatorname{Evor} V$ is the set of all extended Voronoi cells and their faces.

Next we define the *restricted extended Voronoi cell*. Let $v \in V$ and consider the extended Voronoi cell $\operatorname{Evor} v$. Its restriction to $\widetilde{\Sigma}$ is

$$\operatorname{Evor}|_{\widetilde{\Sigma}} v = \operatorname{Evor} v \cap \widetilde{\Sigma}.$$

The *restricted extended Voronoi diagram* is the cell complex containing $\operatorname{Evor}|_{\widetilde{\Sigma}} v$ for all $v \in V$, along with all their faces. Finally we define the *restricted constrained Delaunay triangulation* (restricted CDT) $\operatorname{Del}|_{\widetilde{\Sigma}} V$ as the simplicial complex dual to the restricted extended Voronoi diagram. The restricted CDT $\operatorname{Del}|_{\widetilde{\Sigma}} V$ contains a Delaunay simplex if its dual Voronoi face intersects $\widetilde{\Sigma}$. Under a standard nondegeneracy assumption, no Voronoi vertex intersects $\widetilde{\Sigma}$ so $\operatorname{Del}|_{\widetilde{\Sigma}} V$ contains no Delaunay tetrahedra. The following results hold.

Lemma 2. *Let V be an ε-sample and let $s \in S$ be a segment with endpoints $p, q \in V$. If $d(p,q) \le \rho \operatorname{lfs}(p)$ for $\rho < 2 - \sqrt{2}$, then P_s is disjoint from the medial axis.*

Lemma 3. *Let $s \in S$ be a segment with endpoints $p, q \in V$. Then* $\operatorname{Evor}|_{\widetilde{\Sigma}} p \cap \operatorname{Evor}|_{\widetilde{\Sigma}} q \neq \emptyset$.

Lemma 3 is the reason for the seemingly complicated construction and gives merit to the name restricted constrained Delaunay triangulation. In the full paper we establish further properties of restricted CDTs and show how to use them for surface reconstruction.

4. References

[1] S.-W. Cheng, T. K. Dey, and J. R. Shewchuk. *Delaunay Mesh Generation*. CRC Press, Boca Raton, Florida, Dec. 2012.