A Simple Linear Time Algorithm for Proper Box Rectangular Drawing of Plane Graphs

Xin He
Department of Computer Science and Engineering
State University of New York at Buffalo
Buffalo, NY 14260, USA
Email: xinhe@cse.buffalo.edu

Abstract

In this paper we introduce a new drawing style of a plane graph G, called *proper box rectangular* (PBR) drawing. It is defined to be a drawing of G such that every vertex is drawn as a rectangle, called a box, each edge is drawn as either a horizontal or a vertical line segment, and each face is drawn as a rectangle. We establish necessary and sufficient conditions for G to have a PBR drawing. We also give a simple linear time algorithm for finding such drawings. The PBR drawing is closely related to the *box rectangular* (BR) drawing defined by Rahman, Nakano and Nishizeki [17]. Our method can be adapted to provide a new simpler algorithm for solving the BR drawing problem.

1 Introduction

The problem of "nicely" drawing a graph G has received increasing attention [5]. Typically, we want to draw the edges and the vertices of G on the plane so that certain aesthetic quality conditions and/or optimization measures are met. Such drawings are very useful in visualizing planar graphs and find applications in fields such as computer graphics, VLSI layout, algorithm animation and so on.

Among different drawing styles, an *orthogonal drawing* has attracted much attention due to its applications in circuit layouts, database diagrams, entity-relationship diagrams etc. [4, 9, 19, 20, 21]. In an orthogonal drawing of a plane graph G, each vertex is drawn as an integer grid point and each edge is drawn as a sequence of alternate horizontal and vertical line segments along grid lines as illustrated in Fig 1 (a). Every plane graph G with maximum degree ≤ 4 has an orthogonal drawing. However, a plane graph with maximum degree ≥ 5 has no orthogonal drawing.

A *box-orthogonal drawing* of a plane graph G is a drawing of G on an integer grid such that each vertex is drawn as a rectangle, called a *box*, and each edge is drawn as a sequence of alternate horizontal and vertical line segments along grid lines as illustrated in Fig 1 (b). Some of the boxes may be degenerated rectangles (i.e. points). A box-orthogonal drawing is a natural generalization of an orthogonal drawing. Every plane graph has a box-orthogonal drawing even if its maximum degree is ≥ 5. The box-orthogonal drawing is studied in [3, 6, 15].

An orthogonal drawing of a plane graph G is called a *rectangular drawing* if each edge of G is drawn as a straight line seqment without bends and the contour of each face of G is

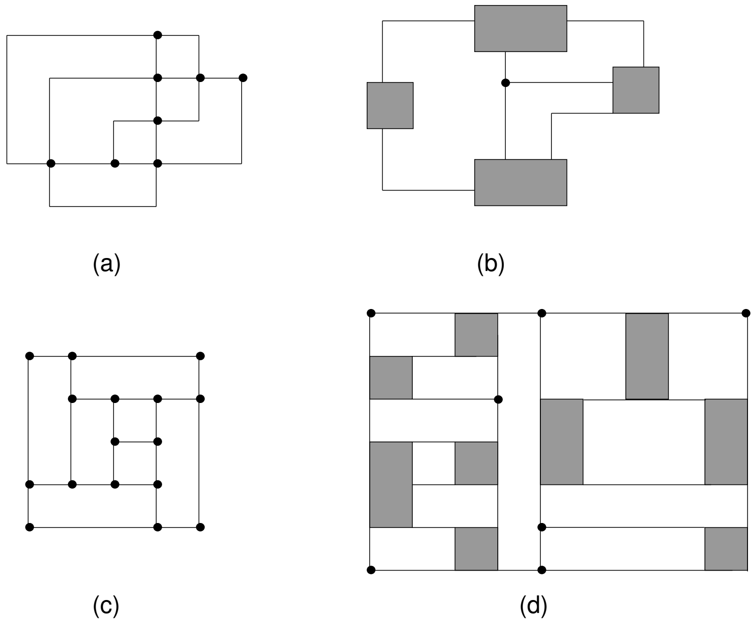

Figure 1: (a) An orthogonal drawing, (b) a box-orthogonal drawing, (c) a rectangular drawing, and (d) a box-rectangular drawing.

drawn as a rectangle as illustrated in Fig 1 (c). Since rectangular drawings have applications in VLSI floorplanning, it has been extensively studied [11, 13, 14, 22]. Not every plane graph has a rectangular drawing. Necessary and sufficient conditions for G to have such a drawing were given in [23]. Several linear time algorithms for constructing such drawings are known [2, 8, 10, 16].

As it can be easily seen, a box-orthogonal drawing is a generalization of an orthogonal drawing which in turn is a generalization of a rectangular drawing. Thus, an orthogonal drawing is an intermediate of a box-orthogonal drawing and a rectangular drawing. Another intermediate of the latter two drawings, *box-rectangular drawing* (BR drawing for short), was introduced in [17]: It is a drawing of G on an integer grid such that each vertex is drawn as a rectangle, called a *box* (possibly degenerated) and the contour of each face is drawn as a rectangle, as illustrated in Fig 1 (d). (A degenerated box will be called a *point*. A non-degenerated box will be called a *real box*).

As mentioned above, rectangular drawings have applications in VLSI floorplanning problem. In this problem the input is a plane graph H as illustrated in Fig 2 (a). Each vertex v of H represents a functional entity (called a *module*) of a chip, and an edge between two vertices u and v represents the interconnection of the two corresponding modules. The output of the problem is a partition of a rectangular chip area into smaller rectangles as illustrated in Fig 2 (d): each vertex of H corresponds to a small rectangle, and if two vertices u and v are adjacent in H then their corresponding rectangles share a common boundary. Thus a floorplan provides a physical layout of the modules of a circuit.

The conventional floorplanning algorithm uses rectangular drawing and works as follows: First obtain a graph H' by triangulating all interior faces of H (see Fig 2 (b) where dotted lines indicate new edges added into H). Then obtain a dual-like graph G of H' (see Fig 2

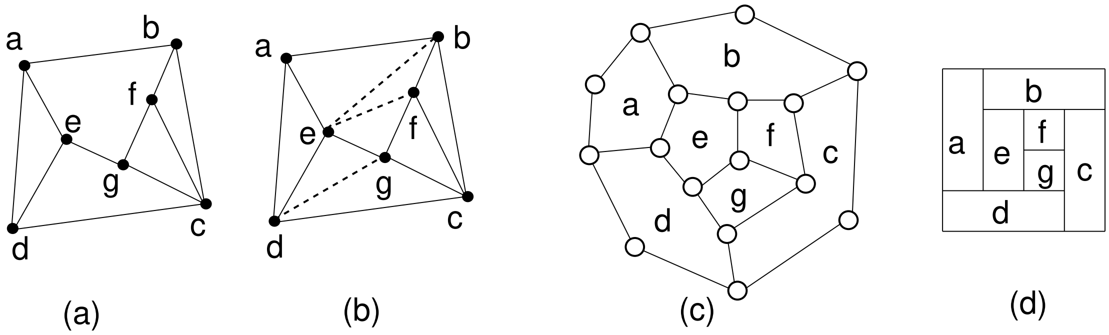

Figure 2: Floopplanning by a rectangular drawing.

(c) where the four degree-2 vertices correspond to the four corners of the drawing). Finally, find a rectangular drawing of G and this gives a floorplan of H as illustrated in Fig 2 (d). In this approach, two rectangles are always adjacent if their corresponding modules have interconnections. However, two rectangles may be adjacent even if the modules corresponding to them have no interconnections. For example, the modules e and f have no interconnection in Fig 2 (a), but their corresponding rectangles are adjacent in the floorplan in Fig 2 (d). Such unwanted adjacencies are not desirable in some applications. For example, in floorplanning of MultiChip Module (MCM), two modules generating excessive heat should not be adjacent, or two modules operating on high frequency should not be adjacent to avoid interference [17, 18]. Another example is in architectural floorplanning [7]: in a chemical plant, a processing unit that deals with poisonous chemical should not be adjacent to a cafeteria.

The unwanted adjacency can be avoided if we use the BR drawing instead of the rectangular drawing: First, without triangulating the interior faces of H, find a dual-like graph G of H as illustrated in Fig 3 (b). Then by finding a BR drawing of G, obtain a floorplan of H as illustrated in 3 (c). In this figure, the rectangles e and f are not adjacent although there is a "dead space" (drawn by a shade rectangular box) corresponding to a vertex of G.

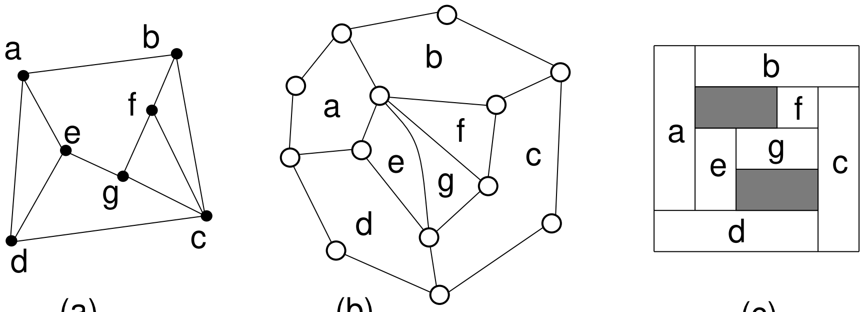

Figure 3: Floorplanning by a BR drawing.

For other applications of BR drawings, see [17]. A linear time algorithm for finding such a drawing is presented in [17]. Necessary and sufficient conditions for the existence of a BR drawing of a plane graph are also given in [17]. These conditions are rather complicated.

The BR drawing defined in [17] allows vertices be drawn as points. This is not desirable for applications such as floorplanning. A BR drawing is called a *proper box rectangular drawing*

(PBR drawing for short) if every vertex of G is drawn as a real box. In this paper, we establish necessary and sufficient conditions for a graph G to have a PBR drawing. We also present a simple linear time algorithm for constructing PBR drawings. Although BR drawing and PBR drawing are similar, our approach for solving this problem is totally different from that in [17]. Our necessary and sufficient conditions are logically simple and clean. Our algorithm is conceptually simpler. With slight modification, our method can be applied to the BR drawing problem also. This provides a simpler linear time algorithm and another set of necessary and sufficient conditions for a graph to have BR drawing. Moreover, our method can be easily adapted to solve other similar drawing problems.

The rest of the paper is organized as follows. The definitions and preliminary results are given in section 2. In section 3, we consider the PBR drawing problem for the case where the four corner vertices of the drawing are designated. In section 4, we consider the PBR drawing problem for the case where the four corner vertices are not specified. In section 5, we describe how to apply our method to the BR drawing and other drawing problems.

2 Preliminaries

In this section, we give definitions and present preliminary results.

Throughout the paper, $G = (V, E)$ denotes a connected plane graph with no selfloops, but may have multiple edges. Let $n = |V|$ and $m = |E|$. The *degree* of a vertex $v \in V$, denoted by $deg(v)$, is the number of edges incident to v. For a subset $E_1 \subseteq E$, $G - E_1$ denotes the graph $(V, E - E_1)$. For a subset $V_1 \subseteq V$, $G - V_1$ denotes the graph obtained from G by deleting the vertices in V_1 and their incident edges. A vertex v is called a *cut vertex* if $G - \{v\}$ is disconnected. G is *biconnected* if it has no cut vertices.

The embedding of G divides the plane into a number of regions. The unbounded region is called the *exterior face*. Other regions are called *interior faces*. The vertices and the edges on the exterior face are called *exterior vertices* and *exterior edges*, respectively. The *degree* of a face F of G is the number of edges on its boundary.

The dual graph $G^* = (V^*, E^*)$ of G is defined as follows: For each face F of G, G^* has a vertex v_F. For each edge e in G, G^* has an edge $e^* = (v_{F_1}, v_{F_2})$ where F_1 and F_2 are the two faces of G with e on their common boundaries. e^* is called the *dual edge* of e. The mapping $e \Leftrightarrow e^*$ is a one-to-one correspondence between E and E^*. For a subset $E_1 \subseteq E$, E_1^* denotes the set of the dual edges in G^* corresponding to the edges in E_1.

A *cycle* C of G is a sequence of distinct vertices $u_1, u_2, \ldots, u_k$ such that $(u_i, u_{i+1}) \in E$ for $1 \leq i < k$ and $(u_k, u_1) \in E$. We also use C to denote the set of the edges in it. If C contains k vertices, it is a k-cycle. A *triangle* (*quadrangle*, respectively) is a 3-cycle (4-cycle, respectively). A cycle C divides the plane into its interior region and exterior region. If there is at least one vertex in its interior region, C is called a *separating cycle* of G.

Consider a cut vertex v of G. Suppose v is an interior vertex. Let F be the interior face of G such that v appears on its boundary more than once. Let H be a connected component of $G - \{v\}$ surrounded by F (see Fig 4 (a)). It is impossible to draw G on the plane so that the face F is a rectangle since there is no room to draw the vertices in H. Thus G has no PBR drawing. Suppose v is an exterior vertex. Let H_1 and H_2 be two connected components of $G - \{v\}$ (see Fig 4 (b)). To obtain a PBR drawing of G, we can find the PBR drawings of H_1 and H_2 separately, then merge the rectangle for v in the drawing of H_1 and the rectangle for v in the drawing of H_2. This gives a PBR drawing of G (see Fig 4 (c)). Thus, without loss of generality, we will always assume G is biconnected from now on.

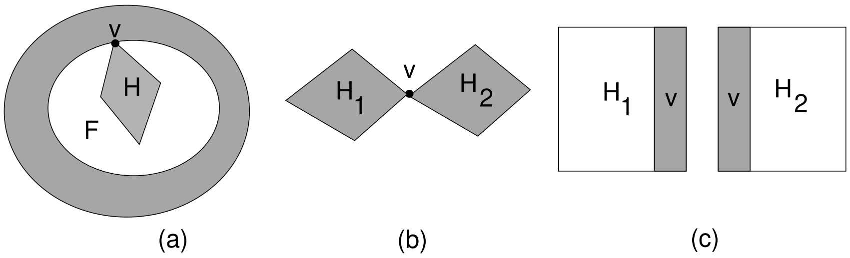

Figure 4: (a) An interior cut vertex, (b) an exterior cut vertex, (c) PBR drawings of H_1 and H_2.

Our PBR drawing algorithm is based on the concept of the *rectangular dual* defined as follows. Let R be a rectangle. A *rectangular subdivision system* of R is a partition of R into a set $\Phi = \{R_1, R_2, \ldots, R_n\}$ of non-intersecting smaller rectangles such that no four rectangles in Φ meet at the same point. A *rectangular dual* of a graph $G = (V, E)$ is a rectangular subdivision system Φ and a one-to-one correspondence $f : V \to \Phi$ such that two vertices u and v are adjacent in G if and only if their corresponding rectangles $f(u)$ and $f(v)$ share a common boundary. Fig 5 (c) illustrates a rectangular dual of the plane graph in Fig 5 (b).

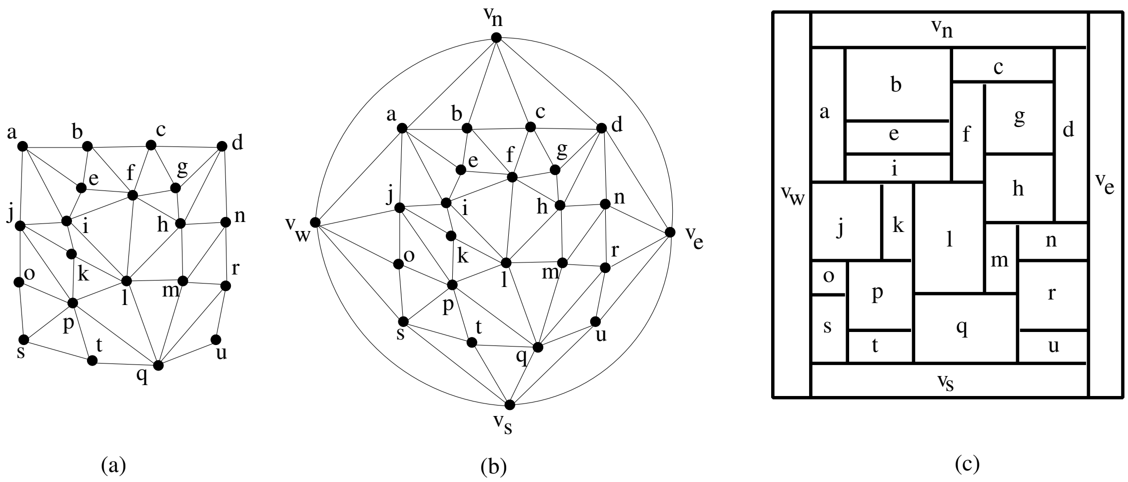

Figure 5: Rectangular dual of plane graphs.

Consider a plane graph $H = (V, E)$. Let v_0, v_1, v_2, v_3 be four vertices on the exterior face of H in counterclockwise order. Let P_i $(i = 0, 1, 2, 3)$ be the four paths on the exterior face of H consisting of the vertices between (and including) v_i and v_{i+1} (where the addition is mod 4). We seek a rectangular dual R_H of H such that the four vertices v_0, v_1, v_2, v_3 correspond to the four corner rectangles of R_H and the vertices on P_0 $(P_1, P_2, P_3$, respectively) correspond to the rectangles located along the north (west, south, east, respectively) boundary of R_H. Necessary and sufficient conditions for testing if H has a rectangular dual were discussed in [2, 11, 12, 13]. These conditions, however, can be easily reduced to the following simpler form.

In order to simplify the problem, we modify H as follows: Add four new vertices v_n, v_w, v_s, v_e and connect v_n (v_w, v_s, v_e, respectively) to every vertex on P_0 (P_1, P_2, P_3, respectively). Then add four new edges $(v_n, v_w), (v_w, v_s), (v_s, v_e), (v_e, v_n)$. Let G be the resulting graph. (Fig 5 (a) shows a graph H and Fig 5 (b) shows the corresponding graph G. The vertices d, a, s, u correspond to the vertices v_0, v_1, v_2, v_3 respectively). It is easy to see that H has a rectangular dual R_H with v_0, v_1, v_2, v_3 corresponding to the four corner rectangles of R_H if and only if G has a rectangular dual R.

If G has a rectangular dual R, then every face of G, except the exterior face, must be a triangle (since no four rectangles of R meet at the same point). Moreover, since at least four rectangles are needed to fully enclose some non-empty area on the plane, any separating cycle of G must have length at least 4. The following theorem states that these two conditions are also sufficient for G to have a rectangular dual.

Theorem 1 [12] *A plane graph $G = (V, E)$ has a rectangular dual R with four rectangles on the boundary of R if and only if the following two conditions hold: (1) Every interior face of G is a triangle and the exterior face of G is a quadrangle; (2) G has no separating triangles.*

In [12], it is assumed that the input graph has no multiple edges. In this paper, the graphs may have multiple edges. Note that two multiple edges form a 2-cycle. If G has a separating 2-cycle, then G cannot have a rectangular dual. (To see this, suppose that G has a separating 2-cycle C consisting of two vertices u and v. Let H be the subgraph of G consisting of the vertices in the interior of C. Suppose that G has a rectangular dual R. The common boundary of the rectangles in R corresponding to u and v is a line segment L. The vertices in H must be drawn in the "area" in the line segment L. This is clearly impossible). Thus, to meet our need, we restate Theorem 1 in the following equivalent form:

Theorem 2 *Let $G = (V, E)$ be a plane graph which may have multiple edges. G has a rectangular dual R with four rectangles on the boundary of R if and only if the following three conditions hold: (1) Every interior face of G is a triangle and the exterior face of G is a quadrangle; (2) G has no separating triangles; (3) G has no separating 2-cycles.*

A graph satisfying the three conditions of Theorem 2 is called a *proper triangular planar* (PTP for short) graph. The following theorem was given in [8, 10]:

Theorem 3 *Given a plane graph G, in linear time, we can check the conditions in Theorem 2, and construct a rectangular dual of G if these conditions are satisfied.*

3 Proper Box Rectangular Drawing with Designated Corner Vertices

In this section we establish necessary and sufficient conditions for the existence of a PBR drawing of a plane graph G when the four corner vertices are designated. We also give a simple linear time algorithm for finding such a drawing if it exists.

Let G be a biconnected plane graph. Let v_0, v_1, v_2, v_3 be the four vertices on the exterior face (in counterclockwise order) of G designated as corner vertices. These designated vertices may be repeated. For example, if $v_0 = v_1$, then in the PBR drawing of G, a single box occupying two corners represents the vertex $v_0 = v_1$. The example presented in this section has four distinct corner vertices. However, our method and theorems also apply to the case with repeated designated corner vertices.

We use P_i ($i = 0, 1, 2, 3$) to denote the four paths on the exterior face of G consisting of the vertices between (and including) v_i and v_{i+1} (where the addition is mod 4).

Definition 1 Let G be a biconnected plane graph with designated corner vertices v_0, v_1, v_2, v_3. The *extended graph* G_x is obtained from G as follows:

1. Add four new vertices v_n, v_w, v_s, v_e and connect v_n (v_w, v_s, v_e, respectively) to every vertex on P_0 (P_1, P_2, P_3, respectively). Add four new edges $(v_n, v_w), (v_w, v_s), (v_s, v_e), (v_e, v_n)$. For each interior face F of G, add a new vertex v_F in F and connect it to the vertices on the boundary of F.

2. For each edge e in the original graph G, delete e and add a new edge between v_{F_1} and v_{F_2} where F_1 and F_2 are the two faces of G with e on their boundary.

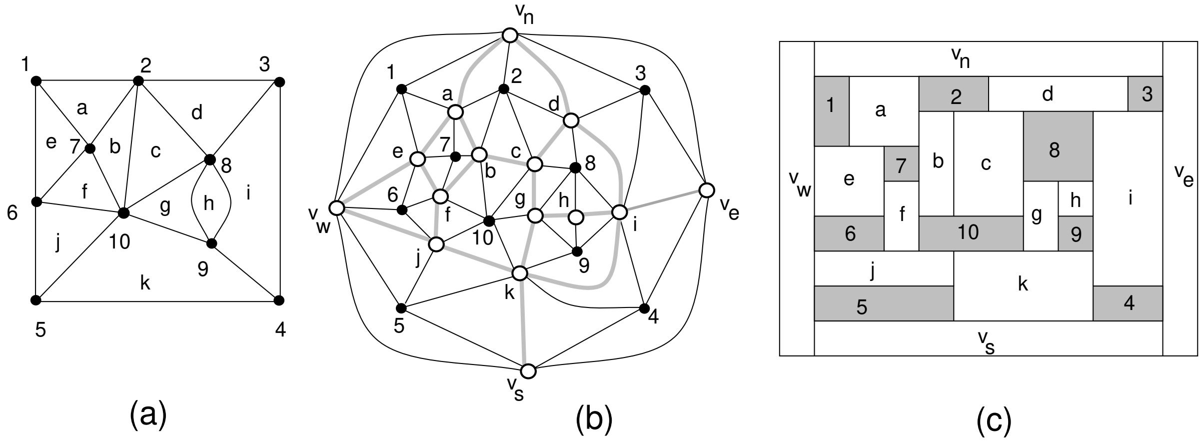

Figure 6: (a) a plane graph G, (b) the extended graph G_x (the added vertices are drawn as empty circles, the dual edges are drawn as light curves), (c) a rectangular dual of G_x which is also a PBR drawing of G after removing the four rectangles corresponding to v_n, v_w, v_s, v_e.

Fig 6 (b) shows the extended graph G_x of the graph G in Fig 6 (a). The vertices of G will be called *original vertices*. The vertices added in Step 1 are called *added vertices* of G_x and denoted by V_a. The edges added in Step 1 are called *added edges* of G_x and denoted by E_a. The edges introduced in Step 2 are called *dual edges* of G_x and denoted by E_d. Thus $G_x = (V \cup V_a, E_a \cup E_d)$. Note that the edge set E_d is almost identical to the edge set E^* of the dual graph $G^* = (V^*, E^*)$. The only difference is that in G^*, the dual edges corresponding to the exterior edges of G are incident to a single vertex v_o^* corresponding to the exterior face of G. In contrast, these dual edges are partitioned and incident to four added vertices v_n, v_w, v_s, v_e in G_x.

Note: After step 1, every interior face of the resulting graph $G_a = (V \cup V_a, E \cup E_a)$ is a triangle, and the exterior face of G_a is a quadrangle. Consider any edge $e = (u, v)$ in the original graph G. Let F_1 and F_2 be the two faces in G with e on their boundaries. In G_a, e is a diagonal of the quadrangle $Q = \{u, v, v_{F_1}, v_{F_2}\}$. In Step 2, $e = (u, v)$ is replaced by another diagonal (v_{F_1}, v_{F_2}) of Q. Thus, all internal faces of G_x are still triangles.

Lemma 1 *A biconnected plane graph G has a* PBR drawing if and only if its extended graph G_x has a rectangular dual.

Proof: If: Let R_x be a rectangular dual of G_x. Each vertex of G is drawn as a rectangle in R_x. For each face F of G, a new vertex v_F is added into G_x, and is drawn as a rectangle in R_x. This rectangle represents F in R_x. It is easy to see that, after removing the four rectangles in R_x corresponding to the vertices v_n, v_w, v_s, v_e, we get a PBR drawing of G (see Fig 6 (c)).

Only If: Let R be a PBR drawing of G. It is easy to see that, after adding four rectangles for the vertices v_n, v_w, v_s, v_w, we get a rectangular dual of G_x (see Fig 6 (c)). □

The extended graph G_x can be easily constructed in linear time from the embedding data structure of G. Thus by Theorems 2 and 3, we have:

Theorem 4 *Let G be a biconnected plane graph with four designated vertices v_0, v_1, v_2, v_3. In linear time, we can test whether G has a* PBR drawing with v_0, v_1, v_2, v_3 as the four corner vertices, and construct one if it exists.

Next we present necessary and sufficient conditions for the existence of a PBR drawing solely in terms of G. An *edge-cut* of $G = (V, E)$ is a *minimal* subset $X \subseteq E$ such that the graph $G - X = (V, E - X)$ is disconnected. If $|X| = k$, we say X is a k-edge-cut. A *2-mixed-cut* of G is a pair $\{v, e\}$ where $v \in V$ and $e \in E$ such that $G - \{e\} - \{v\}$ is disconnected.

Lemma 2 *Let $G = (V, E)$ be a biconnected plane graph and X a 2-edge-cut or a 3-edge-cut of G. Then X contains either 0 or 2 exterior edges of G.*

Proof: Consider the dual graph G^* of G. Let v_o^* be the vertex in G^* corresponding to the exterior face of G. For $k = 2$ or 3, a subset $X \subseteq E$ is a k-edge-cut of G if and only if its corresponding set X^* of dual edges is a k-cycle in G^*. If X contains an exterior edge of G, then v_o^* is on the cycle X^*. To complete the cycle, X^* must contain exactly two dual edges corresponding to two exterior edges of G. So X must contain exactly two exterior edges. □

Let $X = \{e_1, e_2\}$ be a 2-edge-cut (or $X = \{e_1, e_2, e_3\}$ a 3-edge-cut, respectively). By Lemma 2, X can be classified as follows:

- X is an *interior edge-cut* if all of its edges are interior edges;
- X is a *cross exterior edge-cut* if $e_1 \in P_i$, $e_2 \in P_j$ and $|i - j| = 2$.
- X is a *corner exterior edge-cut* if $e_1 \in P_i$, $e_2 \in P_j$ and $|i - j| = 1$ or 3.
- X is a *1-side exterior edge-cut* if $e_1 \in P_i$, $e_2 \in P_j$ and $i = j$.

Theorem 5 *A biconnected plane graph G has a* PBR drawing with v_0, v_1, v_2, v_3 as the four corner vertices if and only if the following hold:

1. All 2-edge-cuts of G are cross exterior edge-cuts; and
2. All 3-edge-cuts of G are cross exterior or corner exterior edge-cuts; and

3. G has no 2-mixed-cut.

Proof: If: If we can show the extended graph G_x is a PTP graph, then G has a PBR drawing by Theorem 2 and Lemma 1. As noted before, all interior faces of G_x are triangles and the exterior face of G_x is a quadrangle.

Since G is biconnected, G has no 1-edge-cuts. Thus G_x has no selfloops. Next we show G_x has no separating 2-cycles. Towards a contradiction, suppose G_x has a separating 2-cycle C. From the construction of G_x, an added edge (v, v_F) in G_x between an original vertex v and an added vertex v_F can be in C only if v appears on the boundary of the face F (of G) more than once. Then v is a cut vertex of G and this contradicts the assumption that G is biconnected. Thus C must consist of two dual edges e_1^* and e_2^*. Let e_1 and e_2 be the two edges in G corresponding to e_1^* and e_2^*, respectively. If both e_1 and e_2 are interior edges of G, then $\{e_1, e_2\}$ is an interior 2-edge-cut of G. This contradicts our assumption. Suppose e_1 is an exterior edge of G. Then C contains one of exterior vertices, say v_n, of G_x. Hence $e_1 \in P_0$. From the construction of G_x, e_2 must be in P_0 also. Thus $\{e_1, e_2\}$ is a 1-side exterior 2-edge-cut of G. This contradicts our assumption.

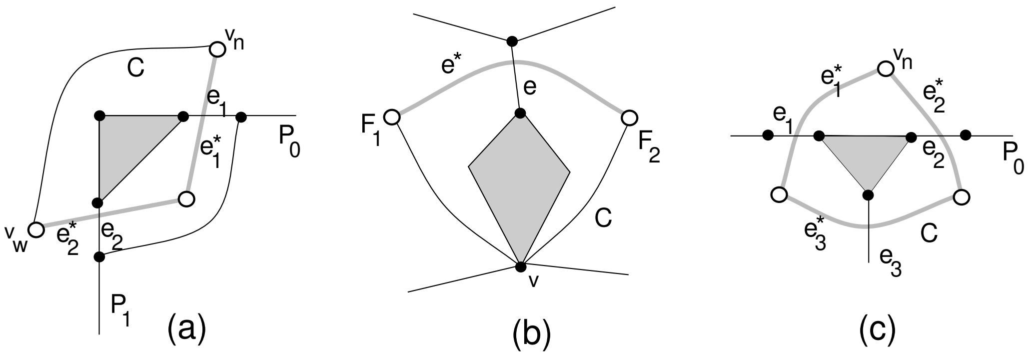

Figure 7: The illustrations of the proof of the "if" part of Theorem 5.

Next we show that G_x has no separating triangles. Towards a contradiction, suppose G_x has a separating triangle C.

Case 1: C contains an exterior edge of G_x, say (v_n, v_w). From the construction of G_x, the other two edges of C must be two dual edges e_1^* and e_2^* such that the edge e_1 corresponding to e_1^* is on P_0 and the edge e_2 corresponding to e_2^* is on P_1. Then $\{e_1, e_2\}$ is a corner exterior 2-edge-cut of G (see Fig 7 (a)). This contradicts our assumption.

Case 2: C contains no exterior edge of G_x, but contains at least one added edge in G_x. By the construction of G_x, this can happen only when C consists of an original vertex v of G, two added vertices v_{F_1} and v_{F_2} in G_x. Thus the three edges of C are: (v, v_{F_1}), (v, v_{F_2}) and the dual edge $e^* = (v_{F_1}, v_{F_2})$. Let e be the edge of G corresponding to e^*. Then $\{v, e\}$ is a 2-mixed-cut of G (see Fig 7 (b)). This contradicts our assumption.

Case 3: C contains no added edges. Then C consists of three dual edges e_1^*, e_2^*, e_3^*. Let e_1, e_2, e_3 be the three edges in G corresponding to e_1^*, e_2^*, e_3^*, respectively. If e_1, e_2, e_3 are all interior edges, they form an interior 3-edge-cut of G. This contradicts our assumption. Suppose e_1 is an exterior edge, say on P_0. Then C contains v_n. From the construction of G_x, another edge of C must be in P_0. Then $\{e_1, e_2, e_3\}$ form a 1-side exterior 3-edge-cut (see Fig 7 (c)), a contradiction again.

Since all cases lead to contradictions, G_x has no separating triangles. So G_x is a PTP graph.

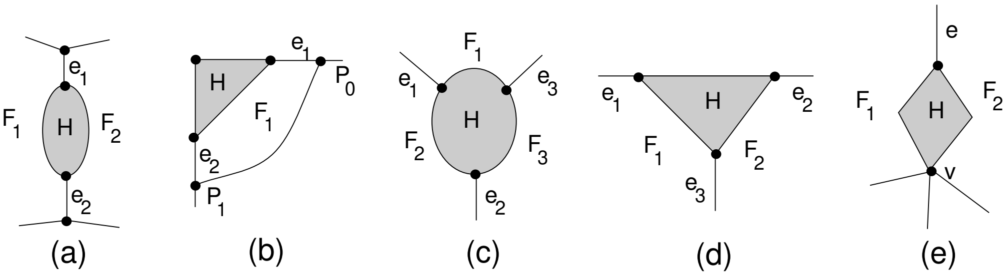

Figure 8: The illustrations of the proof of the "only if" part of Theorem 5.

Only If: We show that if any of the three conditions of the theorem fails, then G has no PBR drawing. Suppose that G has a 2-edge-cut X that is not a cross exterior edge-cut. There are three possible cases:

- X consists of two interior edges e_1, e_2. Let F_1, F_2 be the two faces of G with e_1 and e_2 on their boundary. Let H be the connected component of $G - \{e_1, e_2\}$ that is between F_1 and F_2. It is impossible to draw the faces F_1, F_2 and the vertices in H as rectangles (see Fig 8 (a)).

- X consists of two exterior edges e_1, e_2 located on the paths P_i and P_j where $|i - j| = 1$ or 3. Let F_1 be the interior face of G with e_1 and e_2 on its boundary. Let H be the connected component of $G - \{e_1, e_2\}$ that is between F_1 and the exterior face of G. It is impossible to draw the face F_1 and the vertices in H as rectangles (see Fig 8 (b)).

- X consists of two exterior edges e_1, e_2 located on the same path P_i. Let F_1 be the interior face of G with e_1 and e_2 on its boundary. Let H be the connected component of $G - \{e_1, e_2\}$ that is between F_1 and the exterior face of G. It is impossible to draw the face F_1 and the vertices in H as rectangles.

Suppose that G has a 3-edge-cut X that is neither a cross exterior nor a corner exterior edge-cut. There are two possible cases:

- X consists of three interior edges e_1, e_2, e_3. Let F_1, F_2 and F_3 be the three faces of G separated by e_1, e_2 and e_3. Let H be the connected component of $G - \{e_1, e_2, e_3\}$ that is surrounded by F_1, F_2, F_3. It is impossible to draw the faces F_1, F_2, F_3 and the vertices in H as rectangles (see Fig 8 (c)).

- X consists of two exterior edges e_1, e_2 located on the same path P_i. Let F_1 and F_2 be the two interior faces of G separated by e_1, e_2 and e_3. Let H be the connected component of $G - \{e_1, e_2, e_3\}$ that is surrounded by F_1, F_2 and the exterior face of G. It is impossible to draw the faces F_1, F_2 and the vertices in H as rectangles (see Fig 8 (d)).

Suppose that G has a 2-mixed-cut $\{v, e\}$. Let F_1 and F_2 be the two faces of G with e on their boundary. Note that v is also on the boundaries of F_1 and F_2. (If e is an exterior edge of G, then one of F_1 and F_2 is the exterior face of G). Let H be the connected component of

$G - \{e\} - \{v\}$ that is between F_1 and F_2. It is impossible to draw the faces F_1, F_2 and the vertices in H as rectangles (see Fig 8 (e)). $\square$

Note: The two edges incident to a degree-2 vertex v form a 2-edge-cut which cannot be a cross exterior 2-edge-cut, unless v is a repeated designated corner vertex. The three edges incident to a non-corner degree-3 vertex form either an interior or a 1-side exterior 3-edge-cut. Thus if G has a PBR drawing, then any degree-2 vertex of G must be a repeated designated corner vertex, and any degree-3 vertex of G must be a corner vertex.

4 Proper Box Rectangular Drawing with No Designated Corner Vertices

In this section, we consider the case where no vertex of G is designated as a corner vertices. Let $G = (V, E)$ be a biconnected plane graph. Let C_o be the cycle that form the exterior face of G. Let $u_1, u_2, \ldots, u_t$ be the exterior vertices of G in counterclockwise order. Let $e_1, e_2, \ldots, e_t$ be the exterior edges of G where $e_i = (u_i, u_{i+1})$ $(1 \le i \le t-1)$ and $e_t = (u_t, u_1)$.

We want to chose four vertices v_0, v_1, v_2, v_3 on C_o so that G has a PBR drawing with v_0, v_1, v_2, v_3 as corner vertices. To make this possible, we must chose the corner vertices in a way such that the conditions in Theorem 5 are satisfied. Note that the notions of 2-mixed-cuts and internal edge-cuts are defined solely in terms of G, and are independent from the choice of v_0, v_1, v_2, v_3. On the other hand, the notions of cross, corner and 1-side exterior edge-cuts depend on both G and the choice of v_0, v_1, v_2, v_3.

Let G^* be the dual graph of G. Let v_o^* be the vertex in G^* corresponding to the exterior face of G. Let e_i^* $(1 \le i \le t)$ be the dual edge in G^* corresponds to e_i. The *exterior face enlarged dual graph* of G, denoted by $\bar{G}^*$, is obtained from G^* as follows: (a) Remove the vertex v_o^* from G^*; (b) add a cycle C_o^* consisting of t new vertices v_i^* $(1 \le i \le t)$; (c) make the edge e_i^* $(1 \le i \le t)$ incident to v_i^*. (See Fig 9 (a)). Note that if we shrink the cycle C_o^* into a single vertex (and delete selfloops), then $\bar{G}^*$ becomes G^*. The method described in this section can be presented using the dual graph G^* only. However, it is easier to illustrate the ideas by using $\bar{G}^*$. Note that each vertex v_i^* $(1 \le i \le t)$ on C_o^* corresponds to the exterior edge e_i of G, and each exterior edge (v_{i-1}^*, v_i^*) on C_o^* corresponds to the exterior vertex u_i on C_o. An *arc* of $\bar{G}^*$ is a continuous section of C_o^*. Let A be an arc of $\bar{G}^*$. Later in this section by saying "chose a corner vertex on A", we mean "chose a vertex in C_o that corresponds to an edge in A as a corner vertex".

A *bridge 2-path* (3-path, respectively) of $\bar{G}^*$ is a path P in $\bar{G}^*$ consisting of two (three, respectively) edges such that its two end vertices are on C_o^* and its internal vertex (two internal vertices, respectively) are interior vertices of $\bar{G}^*$. Let $\hat{G}^*$ be the subgraph of $\bar{G}^*$ consisting of C_o^* and the edges in all bridge 2- and 3-paths (see Fig 9 (b)). Note that a subset $X \subseteq E$ forms an exterior 2-edge-cut (3-edge-cut, respectively) if and only if the corresponding set X^* of dual edges forms a bridge 2-path (3-path, respectively) of $\hat{G}^*$.

Let P be a bridge 2-path with end vertices u, v consisting of two dual edges e_i^*, e_j^*. Then $X = \{e_i, e_j\}$ is a 2-edge-cut of G. The vertices u, v divides C_o^* into two arcs A_1 and A_2. In order to make X a cross exterior edge-cut of G, we must chose two corner vertices on A_1 and two corner vertices on A_2.

Let P be a bridge 3-path with end vertices u, v consisting of three dual edges e_i^*, e_j^*, e^*.

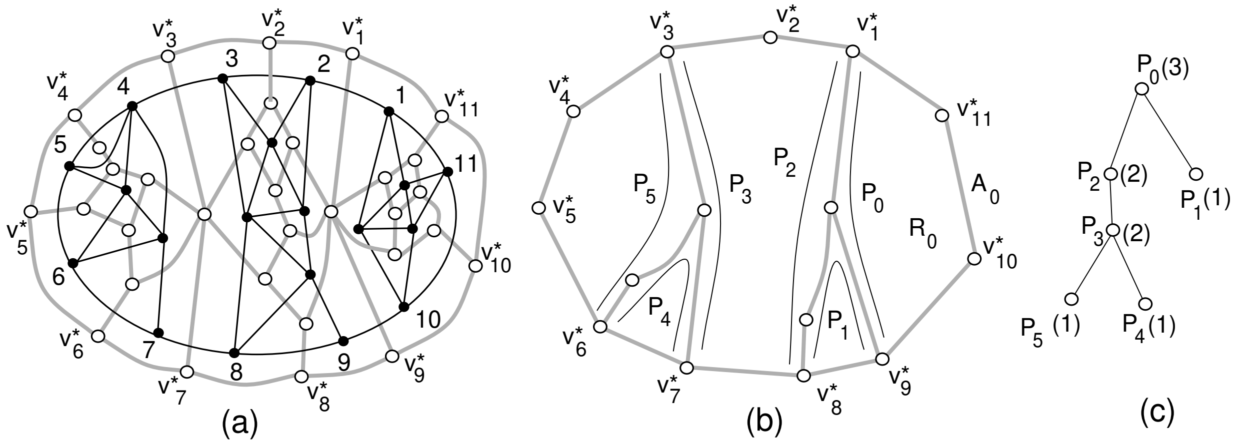

Figure 9: (a) A graph G (drawn as solid lines and dark circles) and the corresponding graph $\bar{G}^*$ (drawn as empty circles and light lines); (b) the graph $\hat{G}^*$; (c) the tree T (the demand value for each bridge 2- and 3-path is given in parenthesis).

Then $X = \{e_i, e_j, e\}$ is an exterior 3-edge-cut of G. The vertices u, v divides C_o^* into two arcs A_1 and A_2. In order to make X a corner exterior edge-cut of G, we must chose at least one corner vertex on A_1 and at least one corner vertex on A_2.

As we have seen, each exterior 2- or 3-edge-cut X of G corresponds to a bridge 2-path or 3-path of $\hat{G}^*$. To ensure that X satisfies the conditions in Theorem 5, it demands either 1 or 2 corner vertices on certain arcs. For an arc A of C_o^*, let demand(A) be the number of corner vertices we must chose on A in order to satisfy these demands. In particular, demand(C_o^*) is the total number of corner vertices demanded by all bridge 2- and 3-paths. Since we can chose only four corner vertices, if demand(C_o^*) ≥ 5, then G has no PBR drawing. In the following, we describe how to calculate demand(C_o^*).

Consider a face R_0 of $\hat{G}^*$. Let A_0 be the portion of the boundary of R_0 that is on C_o^*. Let P_0 be the portion of the the boundary of R_0 that is in the interior of C_o^*. Chose such a face R_0 such that P_0 is a bridge 2- or 3-path, and call R_0 the *root face*, A_0 the *root arc*, and P_0 the *root path*. (See Fig 9 (b)).

Let $\bar{A}_0 = C_o^* - A_0$. The two end vertices of each bridge 2- or 3-path P of $\hat{G}^*$ divides C_o^* into two arcs. One of them is completely contained in $\bar{A}_0$ and is denoted by arc(P). In particular, arc(P_0) $= \bar{A}_0$. We will calculate demand(arc(P)) for each bridge 2- and 3-path P.

Let P_1 and P_2 be two bridge 2- or 3-paths. If arc(P_1) overlaps with arc(P_2), but neither of them contains another, we call P_1 and P_2 *redundant*. If a bridge 2- or 3-path is not redundant, it is called *primary*. Given two redundant bridge 2- or 3-paths P_1 and P_2, let $A_1 = \text{arc}(P_1) - \text{arc}(P_2)$, $A_2 = \text{arc}(P_2) - \text{arc}(P_1)$ and $A_3 = \text{arc}(P_1) \cap \text{arc}(P_2)$. See Fig 10 for an example. In this figure, $P_1 = \{e_1, e_2, e_3\}$ and $P_2 = \{e_4, e_5, e_6\}$. Let $P_3 = \{e_1, e_2, e_4\}$, $P_4 = \{e_3, e_4\}$ and $P_5 = \{e_3, e_5, e_6\}$. Note that $A_1 = \text{arc}(P_3)$, $A_2 = \text{arc}(P_5)$, $A_3 = \text{arc}(P_4)$, and P_3, P_4, P_5 are all bridge 2- or 3-paths. Since the number of corner vertices demanded by P_1 on arc(P_1) is 1, and demand(A_1) and demand(A_3) are at least 1 each, the demand of corner vertices on arc(P_1) by P_1 is automatically satisfied by demand(A_1) and demand(A_3). Similarly, the demand of corner vertices on arc(P_2) by P_2 is automatically satisfied by demand(A_2) and demand(A_3). For other cases (P_1 is a 2-path and P_2 is a 3-path, and both P_1 and

P_2 are 2-paths), it can be shown that demand(arc(P_1)) and demand(arc(P_2)) are automatically satisfied by demand(A_1), demand(A_2) and demand(A_3). Thus, we only need to calculate demand(arc(P)) for all primary bridge 2- and 3-paths.

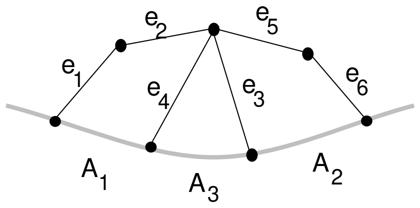

Figure 10: Redundent bridge 2- and 3-paths.

Let P_1, P_2 be two primary bridge 2- or 3-paths. If $\text{arc}(P_1) \subseteq \text{arc}(P_2)$, we say P_1 is a *descendent* of P_2. If P_1 is a descendent of P_2, and there is no other primary bridge 2- or 3-path P_3 ($\neq P_1, P_2$) such that $\text{arc}(P_1) \subseteq \text{arc}(P_3) \subseteq \text{arc}(P_2)$, then P_1 is called a *child* of P_2. This parent-child relation defines a rooted-tree structure T with the root path P_0 as its root (see Fig 9 (c)). We will calculate demand(arc(P_0)) in a bottom-up fashion on T as follows.

For a leaf node P of T, if P is a 2-path, let $\text{demand}(\text{arc}(P)) = 2$. If P is a 3-path, let $\text{demand}(\text{arc}(P)) = 1$.

Consider an internal node P of T. Let $P_1, P_2, \ldots, P_k$ be the children of P in T. If P is a 2-path, then let $\text{demand}(\text{arc}(P)) = \max\{2, \sum_{i=1}^{k} \text{demand}(\text{arc}(P_i))\}$. If P is a 3-path, then let $\text{demand}(\text{arc}(P)) = \sum_{i=1}^{k} \text{demand}(\text{arc}(P_i))$.

We can calculate demand(A_0) as follows. Let P_k be the highest descendent of P_0 that has at least 2 children in T. Let Q be the path of T between P_k and P_0 including P_0 and P_k. (If P_0 has at least two children in T, then $P_k = P_0$ and Q contains a single node. If every node of T has either 1 or 0 child, then T is a single path which is also Q).

- If there is a node on Q that is a bridge 2-path, let $\text{demand}(A_0) = 2$.
- If all nodes on Q are bridge 3-paths, let $\text{demand}(A_0) = 1$.

Finally, let $\text{demand}(C_o^*) = \text{demand}(A_0) + \text{demand}(\text{arc}(P_0))$.

Theorem 6 *A biconnected plane graph G has a* PBR drawing with proper choice of corner vertices if and only if the following conditions hold:

1. G has no internal 2-edge-cuts nor internal 3-edge-cuts.
2. G has no 2-mixed-cuts.
3. $\text{demand}(C_o^*) \leq 4$.

Proof: Only if: Suppose that G has either an internal 2-edge-cut, or a 3-edge-cut, or a 2-mixed-cut. Regardless of how we chose the four corner vertices v_0, v_1, v_2, v_3 on the exterior face of G, the conditions in Theorem 5 cannot be satisfied. Thus G cannot have a PBR drawing for any choice of corner vertices.

Suppose that $\text{demand}(C_o^*) > 4$. Then regardless of how we chose the four corner vertices, at least one exterior 2-edge-cut will be a corner or a 1-side exterior edge-cut; or at least one exterior 3-edge-cut will be a 1-side edge-cut. By Theorem 5, G cannot have a PBR drawing.

If: Let $l = \text{demand}(C_o^*) \leq 4$. We can chose l corner vertices on C_o according to demands. If $l < 4$, we can arbitrarily chose the remaining corner vertices. With such choice of the four corner vertices, all conditions in Theorem 5 are satisfied. Thus G has a PBR drawing with the chosen corner vertices. □

In Fig 9 (c), the demand for each P_i is given in parenthesis. In this example, since $\text{demand}(A_0) = 2$, we have $\text{demand}(C_o^*) = 5$ and hence G has no PBR drawing. This is because we must chose at least one corner vertex on each of $\text{arc}(P_1), \text{arc}(P4), \text{arc}(P_5)$, and at least two corner vertices on the arc A_0. (Namely, the vertex 9 and 7 must be corner vertices, one of the vertices in $\{4, 5, 6\}$ must be a corner vertex; and two of the vertices in $\{10, 11, 1\}$ must be corner vertices).

The dual graph G^* can be obtained in linear time from the embedding data structure of G. The graph $\bar{G}^*$ is just a slight modification of G^*. $\hat{G}^*$ is a subgraph of $\bar{G}^*$. Both can be constructed in linear time from G^*. The tree T and the calculation of $\text{demand}(\text{arc}(P_0))$ on T can also be done in linear time. The other two conditions in Theorem 6 can be tested in linear time by known algorithms. Thus we have:

Theorem 7 *Given a biconnected plane graph G, in linear time, we can test whether G has a* PBR drawing with proper choice of corner vertices; and construct such a drawing if one exists.

5 Box Rectangular Drawing and Other Applications

In this section, we describe how to apply our method to solve the BR drawing problem as defined in [17]. We only consider the case when the four corner vertices v_0, v_1, v_2, v_3 are designated. We also outline application of our method in other similar drawing problems.

Let G be a biconnected plane graph. If G has a PBR drawing then, as noted at the end of section 3, every degree-2 vertex of G must be a repeated corner vertex and every degree-3 vertex of G must be a corner vertex. In contrast, in a BR drawing, a vertex of G can be drawn as a point. Thus, even if G has other degree-2 and degree-3 vertices, it may still have a BR drawing.

Let G_x be the extended graph of G. A separating 2-cycle C of G_x is *simple* if it has exactly one vertex v in its interior. Clearly, v must be adjacent to the two vertices of C and have degree 2. v is called a *special degree-2* vertex.

A separating triangle C of G_x is *simple* if it has exactly one vertex v in its interior. Clearly, v must be adjacent to the three vertices of C and have degree 3. v is called a *special degree-3* vertex. It is easy to see that:

- Each non-corner degree-2 vertex v of G is a special degree-2 vertex in G_x.
- Each corner degree-2 vertex v of G is a special degree-3 vertex in G_x.
- Each non-corner degree-3 vertex v of G is a special degree-3 vertex in G_x.

Let G'_x be the graph obtained from G_x as follows:

- For each simple separating 2-cycle C in G_x, delete the special degree-2 vertex in C, and delete one edge of C.

- For each simple separating triangle C in G_x, delete the special degree-3 vertex in C.

Theorem 8 *Let G be a biconnected plane graph. Then G has a* BR drawing if and only if G'_x has a rectangular dual.

Proof: Let R_x be a rectangular dual of G'_x. Then R_x gives a BR drawing of G which can be seen as follows. Consider a degree-3 vertex v in G that is deleted from G_x. Let C be the simple separating triangle in G_x containing v in its interior. Let $v_{F_1}, v_{F_2}, v_{F_3}$ be the three vertices of C. In R_x, $v_{F_1}, v_{F_2}, v_{F_3}$ are drawn as three rectangles sharing a common point. This point represents v in R_x.

Consider a degree-2 vertex v of G that is deleted from G_x. Let C be the simple separating 2-cycle in G_x containing v in its interior. Let v_{F_1}, v_{F_2} be the two vertices of C. In R_x, V_{F_1}, v_{F_2} are drawn as two rectangles with a line segment L as their common boundary. We can draw a point at the middle of L, and this point represents v in R_x. After these modifications, and deleting the four rectangles corresponding to the four vertices v_n, v_w, v_s, v_n, we get a BR drawing of G.

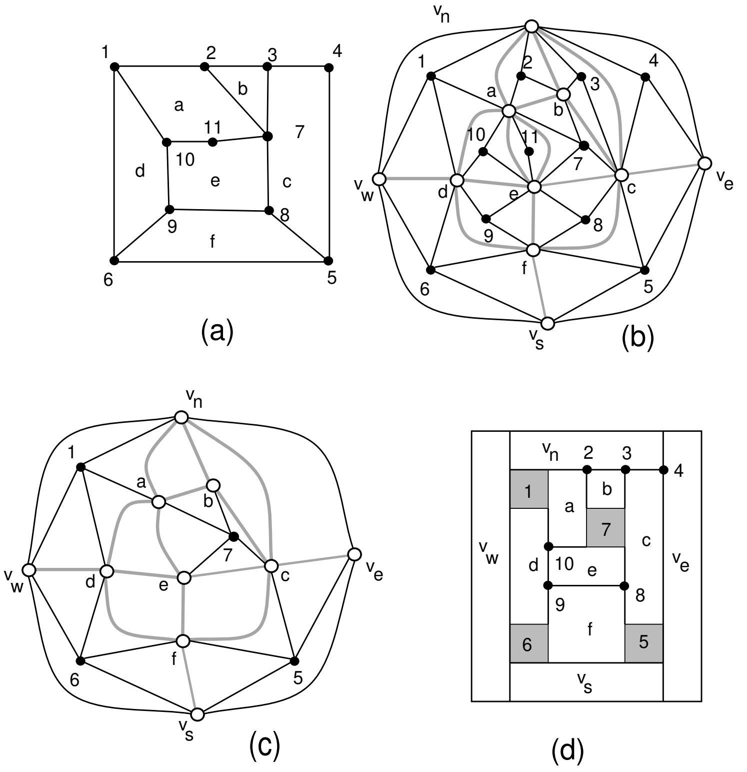

Figure 11: (a) A plane graph G, (b) G_x, (c) G'_x, (d) a rectangular dual of G'_x.

Figure 11 shows an example. Fig 11 (a) shows a plane graph G. Fig 11 (b) shows its extended graph G_x. The vertex 11 is a special degree-2 vertex. The vertices 2, 3, 4, 8, 9, 10 are special degree-3 vertices. After deleting these special vertices, we get the graph G'_x as shown in Fig 11 (c). Fig 11 (d) illustrates a rectangular dual of G'_x. After removing the rectangles corresponding to the vertices v_n, v_w, v_s, v_e, and adding a point (representing the vertex 11) at the middle of the common boundary of the faces a and e, we obtain a BR drawing of G. Note that the vertices 2, 3, 4, 8, 9, 10, 11 are all drawn as points.

Conversely, suppose that G has a BR drawing R. Clearly, each degree-2 vertex that is not a repeated corner vertex and each non-corner degree-3 vertex of G must be drawn as a point in R. As mentioned above, these vertices are special degree-2 and degree-3 vertices in G_x. All other vertices and all faces of G are drawn as rectangles in R. Thus, after adding four rectangles surrounding R, we get a rectangular dual of G'_x. $\Box$

Based on Theorem 8, the necessary and sufficient conditions for G to have a BR drawing (with no designated corner vertices) can be developed. We omit the details here.

Our method can also be used to solve other similar drawing problems. For example, suppose that we allow the faces of G be represented as degenerated rectangles. Clearly, only degree-3 faces can be drawn as points. Our method can be easily adapted to solve this drawing problem: In the definition of the extended graph G_x, instead of adding a new vertex v_F for *every* face F of G, we do this only for the faces with degree ≥ 4. No other changes are needed. The corresponding necessary and sufficient conditions for having such a drawing can be established accordingly.

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