$\mathbb{P}$-Species and the q-Mehler Formula

Hung Quang Ngo *

Abstract

In this paper, we present a bijective proof of the q-Mehler formula. The proof is in the same style as Foata's proof of the Mehler formula. Since Foata's proof was extended to show the Kibble-Slepian formula, a very general multilinear extension of the Mehler formula, we hope that the proof provided in this paper helps find some multilinear extension of the q-Mehler formula.

The basic idea to obtain this proof comes from generalizing a result by Gessel. The generalization leads to the notion of species on permutations and the q-generating series for these species. The bijective proof is then obtained by applying this new exponential formula to a certain type of species on permutations and a weight preserving bijection relating this species to the q-Mehler formula. Some by-products of the q-exponential formula shall also be derived.

* Computer Science and Engineering Department, 201 Bell Hall State University of New York at Buffalo, Amherst, NY 14260, USA. hungngo@cse.buffalo.edu

1 Introduction

In this paper, we present a bijective proof of the q-Mehler formula. The proof is in the same style as Foata's proof of the Mehler formula. Since Foata's proof was extended to show the Kibble-Slepian formula, a very general multilinear extension of Mehler formula, we hope that the proof provided in this paper helps find a multilinear extension of the q-Mehler formula.

The rest of the paper is organized as follows. Section 1.1 introduces the Hermite polynomials, the Mehler formula and its extensions. Section 1.2 presents a q-analogue of the Mehler formula. Section 1.3 discusses the only known version of the q-exponential formula, which is influential on the proof of the main result of the paper. Section 2 contains the Foata-style proof of the q-Mehler formula introduced earlier. Section 2.1 introduces a variation of the q-Hermite polynomials, whose corresponding q-Mehler formula is to be shown. Section 2.2 develops a new type of species on permutations, their generating series, and several nice consequences derived from this new species. Section 2.3 discusses a q-analogue of the bicolored n-involutionary graphs and their properties. This q-analogue also belongs to the new class of species introduced in Section 2.2, thus they satisfy certain identity. Section 2.4 finishes the Foata-style proof of the q-Mehler formula by bijectively showing relations between the q-analogue of the bicolored n-involutionary graphs and the new variation of the q-Hermite polynomials. Lastly, Section 3 concludes the paper and discusses related issues and future works arising from the paper.

1.1 The Mehler formula and its extensions

Throughout this paper, we use $\mathbb{R}[x]$ to denote the ring of polynomials in x on $\mathbb{R}$, $p_n(x)$ a polynomial in $\mathbb{R}[x]$ of degree n, and $\mathcal{L} : \mathbb{R}[x] \to \mathbb{C}$ a linear functional. We often think of $\mathcal{L}(p(x))$ as $\int_a^b p(x)d\alpha(x)$ for some non-decreasing function $\alpha(x)$ on the interval $[a, b]$.

A sequence $\{p_n(x)\}_{n=0}^{\infty}$ is called an *orthogonal polynomial sequence* with respect to $\mathcal{L}$ (or to the distribution $d\alpha(x)$) if for all $m, n \in \mathbb{N}$ we have

$$\mathcal{L}\big(p_m(x)p_n(x)\big) = h_n\delta_{mn}, \tag{1}$$

where $h_n \in \mathbb{C}$ and δ_{mn} is the Kronecker symbol. Relation (1) is often called the *orthogonality relation* of the sequence. In terms of the distribution $d\alpha(x)$, it reads

$$\int_a^b p_m(x)p_n(x)d\alpha(x) = h_n\delta_{mn}. \tag{2}$$

Obviously, not all linear functionals have an orthogonal polynomial sequence. Let $\mu_n :=$

$\mathcal{L}(x^n)$, called the *nth moment* of $\mathcal{L}$, then $\mathcal{L}$ has an orthogonal polynomial sequence iff none of the Hankel determinants for the sequence $\{\mu_n\}_{n=0}^{\infty}$ vanishes.

For convenience, we often normalize the polynomials so that all of them are monic. It is not difficult to show that any monic orthogonal polynomial sequence satisfies a *three term recurrence*:

$$p_{n+1}(x) = (x - c_n)p_n(x) - \lambda_n p_{n-1}(x), \tag{3}$$

where $p_0(x) = 1$ and $p_{-1}(x) = 0$.

There are 5 classes of the so-called *classical orthogonal polynomials* (see [1, 5]), including the Jacobi, the ultraspherical (or Gegenbauer), the Chebyshev, the Laguerre, and the Hermite polynomials.

Definition 1.1. The Hermite polynomials $H_n(x)$ are the orthogonal polynomials with respect to the normal distribution e^{-x^2}. They can be defined by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n e^{-x^2}}{dx^n}. \tag{4}$$

The orthogonality relation for the Hermite polynomials is

$$\int_{-\infty}^{\infty} H_m(x)H_n(x)e^{-x^2}dx = 2^n n! \sqrt{\pi}\delta_{mn}, \tag{5}$$

and the three term recurrence is

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x). \tag{6}$$

The proofs of these relations involve the use of a very powerful tool: the exponential generating function, as illustrated below.

The fact that

$$e^{-x^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} e^{2ixt} dt$$

can be used to repeatedly differentiate e^{-x^2} n times, yielding

$$\frac{d^n e^{-x^2}}{dx^n} = \frac{(2i)^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} t^n e^{2ixt} dt.$$

It is now not difficult to find the exponential generating function for the sequence $\{H_n(x)\}_{n=0}^{\infty}$:

$$\begin{aligned}
\sum_{n=0}^{\infty} H_n(x)\frac{r^n}{n!} &= \sum_{n=0}^{\infty} (-1)^n e^{x^2} \left(\frac{(2i)^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} t^n e^{2ixt} dt \right) \frac{r^n}{n!} \\
&= \frac{e^{x^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} e^{2ixt} \left(\sum_{n=0}^{\infty} \frac{(-2irt)^n}{n!} \right) dt \\
&= \frac{e^{x^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} e^{2it(x-r)} dt \\
&= e^{x^2} e^{-(x-r)^2}.
\end{aligned}$$

Hence,

$$\sum_{n=0}^{\infty} H_n(x)\frac{r^n}{n!} = e^{2xr-r^2}. \tag{7}$$

From (7), relations (5) and (6) can be readily verified.

Moreover, the right hand side of (7) "almost" looks like the exponential generating function for some set of weighted combinatorial objects. It thus makes sense to first transform $H_n(x)$ into an equivalent form, whose exponential generating function is also the exponential generating function for a simple set of weighted combinatorial objects. We then hopefully could derive nice relations about $H_n(x)$ from combinatorially studying these objects. This equivalent form, known as the *normalized* Hermite polynomials, is defined as follows.

$$\tilde{H}_n(x) := \frac{H_n(x/\sqrt{2})}{2^{n/2}}. \tag{8}$$

Replacing (8) into (7), and let $t = \sqrt{2}r$, we get the generating function for the normalized Hermite polynomials:

$$\sum_{n=0}^{\infty} \tilde{H}_n(x)\frac{t^n}{n!} = e^{xt-t^2/2}. \tag{9}$$

Throughout this paper, we shall use $[1,n]$ to denote the set of integers from 1 to n. The standard notation is $[n]$, but we do not use this to avoid confusion with $[n]_q$ which is also denoted by $[n]$. Let M_n be the set of all matchings (not necessarily perfect) on $[1,n]$. For each matching $\alpha \in M_n$, let $F(\alpha)$ denote the number of fixed points, and $|\alpha|$ the number of edges of α. Also define the weight function $w(\alpha)$ for each $\alpha \in M_n$ by

$$w(\alpha) := (-1)^{|\alpha|}x^{F(\alpha)}, \tag{10}$$

with x being an indeterminate. Basically, each fixed point of α is weighted by x and each edge of α is weighted by -1. Now, it is not difficult to see that

$$\tilde{H}_n(x) = \sum_{\alpha \in M_n} w(\alpha) \tag{11}$$

because the exponential generating function for both the left and the right hand side of (11) is $\exp(xt - t^2/2)$. Due to this combinatorial interpretation, the $\tilde{H}_n(x)$ are also called the *matching polynomials*. The matching interpretation allows us to show combinatorially many relations on the Hermite polynomials. Firstly, it allows us to easily write down a formula for the $\tilde{H}_n(x)$ polynomials:

$$\tilde{H}_n(x) = \sum_{0\le k\le n/2} \binom{n}{2k}\big((2k-1)(2k-3)\dots 1\big)(-1)^k x^{n-2k}. \tag{12}$$

Secondly, the well known Mehler formula:

$$\sum_{n=0}^{\infty} \tilde{H}_n(x)\tilde{H}_n(y)\frac{t^n}{n!} = \frac{1}{\sqrt{1-t^2}} \exp\left(\frac{2txy - t^2(x^2+y^2)}{2(1-t^2)}\right) \tag{13}$$

could now also be proved combinatorially (see Foata [6]). For a discussion of this proof and its relation to other combinatorial results on orthogonal polynomials, the reader is referred to Stanton [16].

The Mehler formula is often referred to as the bilinear extension of (9). Carlitz [3, 4] found several multilinear extensions. Kibble [13], and later independently Slepian [15] found an extension, known as the Kibble-Slepian formula, whose specializations include all other extensions. Louck [14] proposed another extension which was proved combinatorially to be equivalent to the Kibble-Slepian formula by Foata [7].

To describe the Kibble-Slepian formula, let us first introduce some notation. For each integer $n \geq 2$, define a symmetric $n \times n$ matrix R by

$$(R)_{ij} = \begin{cases} r_{ij} & \text{if } i \neq j, \\ 1 & \text{otherwise,} \end{cases}$$

where $\{r_{ij}\}_{i,j\geq 1}$ is an infinite sequence of indeterminates. Let $z = (z_1, \ldots, z_n)^T$ be a vector of n indeterminates. Let $\mathcal{N}$ be the set of all symmetric matrices $N = (\nu_{ij})$ $(1 \leq i, j \leq n)$ of order n such that $\nu_{ii} = 0$ for all $i \leq n$, and that ν_{ij} is a non-negative integer for all $i \neq j$. Also, for a fixed $N \in \mathcal{N}$, let the ith row sum of N be

$$s_i = \nu_{i1} + \nu_{i2} + \cdots + \nu_{in}.$$

The Kibble-Slepian formula reads

$$\sum_{N\in\mathcal{N}} \tilde{H}_{s_1}(z_1)\ldots\tilde{H}_{s_n}(z_n)\frac{\prod_{i<j} r_{ij}^{\nu_{ij}}}{\prod_{i<j} \nu_{ij}!} = \frac{1}{\sqrt{\det R}} \exp\left(\frac{1}{2}\left(z^T z - z^T R^{-1} z\right)\right). \tag{14}$$

Example 1.2. When $n = 2$, $R = \left(\begin{smallmatrix} 1 & t \\ t & 1 \end{smallmatrix}\right)$, $x = z_1$, and $y = z_2$, the Kibble-Slepian formula (14) reduces to the Mehler formula (13).

Example 1.3. When $n = 3$, let

$$R = \begin{pmatrix} 1 & r & s \\ r & 1 & t \\ s & t & 1 \end{pmatrix}, \text{ and } z = \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

then the Kibble-Slepian formula becomes

$$
\sum_{m,n,k=0}^{\infty} \tilde{H}_{m+n}(x)\tilde{H}_{m+k}(y)\tilde{H}_{n+k}(z)\frac{r^m s^n t^k}{m!n!k!} =
$$

$$
\frac{1}{\sqrt{1-r^2-s^2-t^2+2rst}} \times
$$

$$
\exp\left[\frac{2rst(x^2+y^2+z^2)-x^2(r^2+s^2)-y^2(r^2+t^2)-z^2(s^2+t^2)}{2(1-r^2-s^2-t^2+2rst)} + \frac{2xy(r-st)+2xz(s-rt)+2yz(t-rs)}{2(1-r^2-s^2-t^2+2rst)}\right] \quad (15)
$$

Foata and Garsia [8] extended Foata's proof [6] of the Mehler formula to give a combinatorial proof of the Kibble-Slepian formula. The left hand side of (14) was interpreted as the exponential generating function of the so-called n-*involutionary graphs*, while the right hand side could be written as the exponential of the series

$$
\frac{1}{2}\ln\frac{1}{\det R} + \frac{1}{2}\sum_{i,j}\left(\delta_{ij} - (R^{-1})_{ij}\right)z_i z_j \quad (16)
$$

where δ_{ij} is the Kronecker symbol. They showed that expression (16) is the generating function for the "connected components" of the n-involutionary graphs. Consequently, the exponential formula applies, proving (14).

1.2 The q-Mehler formula and its extensions

Throughout this paper, we shall use $(a;q)_n$ (or $(a)_n$ for short) to denote the q-shifted factorial:

$$
(a)_n = (a;q)_n := (1-a)(1-aq)\dots(1-aq^{n-1}).
$$

The q-analogue of an natural number n is denoted by $[n]_q$, and the well known Gaussian coefficient $G(n,k)$ is denoted by $\begin{bmatrix} n \\ k \end{bmatrix}_q$. They are defined as follows.

$$
\begin{aligned}
[0]_q &:= 0 \\
[n]_q &:= 1+q+\dots+q^{n-1}, n \geq 1 \\
\begin{bmatrix} n \\ k \end{bmatrix}_q &:= \frac{(q)_n}{(q)_{n-k}(q)_k} = \frac{(1-q^n)\dots(1-q^{n-k+1})}{(1-q^k)\dots(1-q)}, 0 \leq k \leq n.
\end{aligned}
$$

Most often, we shall drop the subscript q when there is no potential confusion.

A q-analogue of the Hermite polynomials, called the q-Hermite polynomials, was introduced by Rogers, who used them to prove Rogers-Ramanujan identities. Following [11], the q-Hermite polynomials can be defined by their generating function $H(x, t \mid q)$ as follows.

$$H(x,t \mid q) := \sum_{n=0}^{\infty} H_n(x \mid q)\frac{t^n}{(q)_n} = \prod_{k=0}^{\infty} \frac{1}{(1 - 2xtq^k + t^2q^{2k})}, \ |t| < 1. \tag{17}$$

The three term recurrence for $H_n(x \mid q)$ is

$$H_{n+1}(x \mid q) = 2xH_n(x \mid q) - (1 - q^n)H_{n-1}(x \mid q). \tag{18}$$

To get the corresponding version $\tilde{H}_n(x \mid q)$ of $\tilde{H}_n(x)$, we also have to normalize the $H_n(x \mid q)$. Define

$$\tilde{H}_n(x \mid q) := \frac{H_n(\frac{x}{2}\sqrt{1-q} \mid q)}{(1-q)^{n/2}}, \tag{19}$$

with the new three term recurrence:

$$\tilde{H}_{n+1}(x \mid q) = x\tilde{H}_n(x \mid q) - (1 + q + \cdots + q^{n-1})\tilde{H}_{n-1}(x \mid q). \tag{20}$$

As in the case of $\tilde{H}_n(x)$, there is a combinatorial interpretation for $\tilde{H}_n(x \mid q)$, due to Ismail, Stanton and Viennot [11]. As expected, this combinatorial interpretation gives $\tilde{H}_n(x \mid q)$ as a q-analogue of the matching polynomials. Notice that each $\alpha \in M_n$ can be viewed as an involution on $[1, n]$. Define a new statistic on α as follows.

$$s(\alpha) := \sum_{e \in \alpha} s(e)$$

where the sum goes over all edges e of α, and if $e = (i, j)$, $i < j$, then

$$s(e) := |\{k \mid i < k < j, \text{ and } \alpha(k) < j\}|$$

Pictorially, imagine putting n points $1, \ldots, n$ in this order on a horizontal line, then drawing all edges of α on the upper half plane. The statistic $s(e)$ for an edge e is the number of points k lying between i and j such that k is either a fixed point or an end-point of some edge $e' \in \alpha$, both of whose end-points are on the left of j (see Figure 1).

It is now an easy matter to prove

$$\tilde{H}_n(x \mid q) = \sum_{\alpha \in M_n} \tilde{w}(\alpha), \tag{21}$$

where

$$\tilde{w}(\alpha) = (-1)^{|\alpha|} x^{F(\alpha)} q^{s(\alpha)}. \tag{22}$$

We only need to verify that the right hand side of (21) satisfies the same recurrence (20) as $\tilde{H}_n(x \mid q)$. Also notice that when $q = 1$, $\tilde{H}_n(x \mid q)$ reduces to the matching polynomials.

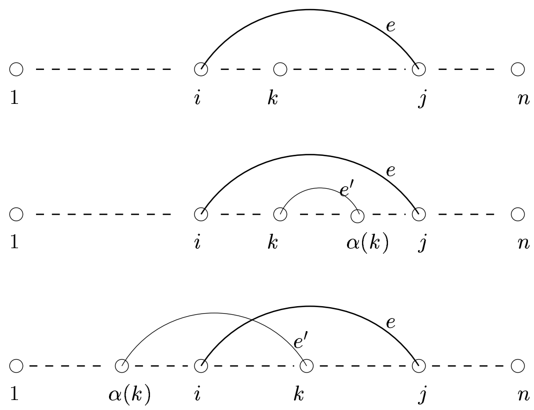


Figure 1: Illustration of $s(e)$, where $e = (i, j)$, $i < j$.

On the same line of reasoning as in the previous section, one would hope that (21) helps combinatorially discover the q-analogues of the Mehler formula and its extensions. This turned out to be not easy. There are several known equivalent forms of the q-Mehler formula. In terms of $H_n(x \mid q)$, it reads

$$\sum_{n=0}^{\infty} H_n(\cos\theta \mid q) H_n(\cos\varphi \mid q) \frac{t^n}{(q;q)_n} = \frac{(t^2)_\infty}{(te^{-i\theta-i\varphi}, te^{-i\theta+i\varphi}, te^{i\theta-i\varphi}, te^{i\theta+i\varphi})_\infty}, \tag{23}$$

or

$$\sum_{n=0}^{\infty} H_n(x \mid q) H_n(y \mid q) \frac{t^n}{(q)_n} = \frac{(t^2)_\infty}{\prod_{k=0}^{\infty} \left(1 - 4tq^k xy + 2t^2 q^{2k}(-1 + 2x^2 + 2y^2) - 4t^3 q^{3k} xy + t^4 q^{4k}\right)}. \tag{24}$$

On the other hand, let $h_n(x \mid q)$ be the generating function for the number of subspaces of $\mathbb{F}_q^n$:

$$h_n(x \mid q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k, \tag{25}$$

then it is not difficult to prove that

$$H_n(\cos\theta \mid q) = e^{-in\theta} h_n(e^{2i\theta} \mid q).$$

This gives another equivalent form of (23):

$$\sum_{n=0}^{\infty} h_n(x \mid q) h_n(y \mid q) \frac{t^n}{(q;q)_n} = \frac{(xyt^2)_\infty}{(t)_\infty (xt)_\infty (yt)_\infty (xyt)_\infty} \tag{26}$$

It is this form of the q-Mehler formula which has the only known combinatorial proof as was shown in [11], using the vector space interpretation as above. However, it does not seem to be possible to extend this proof to find a multilinear extension of the q-Mehler formula, using the same approach as with the regular Mehler formula. Firstly, we need a q-analogue of the exponential formula, which is not known in general. (A somewhat specialized q-analogue of the exponential formula was devised by Gessel [9], but we do not know how to use his method on linear spaces over finite fields.) Secondly, linear subspaces, although very useful in enumeration arguments, are difficult to be dealt with in bijective arguments. Hence, besides needing a q-analogue of the exponential formula, we also need a different combinatorial proof of q-Mehler formula which uses some easier-to-describe combinatorial objects.

1.3 A q-exponential formula

Gessel [9] gave a partial answer to the question raised near the end of the previous section. His paper was influential in the proof of the q-Mehler formula presented in this paper. He first gave a q-analogue of functional composition for Eulerian generating functions, which can be thought of as a q-analogue of exponential generating functions, then used this method to enumerate permutations by inversions and distribution of left-to-right maxima. The enumeration of permutations by inversions also gives rise to yet another variation of the q-Hermite polynomials. Any q-analogue of the exponential formula needs to address an important issue, namely the q-weight for each combinatorial object has to be well-behaved so that we can compose generating functions while preserving the weights. Gessel showed how to define a weight function on permutations so that his q-exponential formula could be used to enumerate certain permutations by inversions. Let us briefly describe his approach here.

Firstly, we define a q-analogue of the derivative:

$$\mathcal{D}f(t) = \frac{f(t) - f(qt)}{(1-q)t}. \tag{27}$$

Notice that

$$\begin{aligned} \mathcal{D}1 &= 0, \text{ and} \\ \mathcal{D}\frac{t^n}{n!_q} &= \frac{t^{n-1}}{(n-1)!_q}. \end{aligned}$$

Secondly, assuming $f(0) = 0$, a q-analogue ϕ_q of the map $\phi : f \to \frac{f^k}{k!}$ could be defined as $\phi_q(f) = f^{[k]}$, where $f^{[0]} = 1$ and for $k > 1$

$$\mathcal{D}f^{[k]} = \mathcal{D}f \cdot f^{[k-1]}, \text{ with } f^{[k]}(0) = 0.$$

An equivalent explicit form for $f^{[k]}$ can also be given. Suppose

$$f^{[k]}(t) = \sum_{n=0}^{\infty} f_{n,k} \frac{t^n}{n!_q},$$

then the $f_{n,k}$ satisfy

$$f_{n+1,k} = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} f_{i+1,1} f_{n-i,k-1} = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} f_{n-j+1,1} f_{j,k-1}$$

where $f_{n,k} = 0$ when $n < k$.

Next, let f be a function such that $f(0) = 0$ and g be a q-exponential generating function:

$$g(t) = \sum_{n=0}^{\infty} g_n \frac{t^n}{n!_q}.$$

We define a q-analogue of $g(f)$, denoted by $g[f]$, as follows.

$$g[f] = \sum_{n=0}^{\infty} g_n f^{[n]}. \tag{28}$$

It is easy to see that this q-functional composition satisfies the *chain rule*:

$$\mathcal{D}(g[f]) = (\mathcal{D}g)[f] \cdot \mathcal{D}f. \tag{29}$$

Now, let $e(t)$ be the q-analogue of the exponential function:

$$e(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!_q}.$$

Let f be a function such that $f(0) = 0$, and

$$f(t) = \sum_{n=0}^{\infty} f_n \frac{t^n}{n!_q}.$$

Let g be the q-analogue of e^f, namely $g(t) = (e[f])(t)$. If g's exponential form is

$$g(t) = \sum_{n=0}^{\infty} g_n \frac{t^n}{n!_q},$$

then equating coefficients from the chain rule (29) for $e[f]$, it is not difficult to show that

$$g_{n+1} = \begin{cases} 1 & \text{if } n = 0 \\ \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} g_{n-k} f_{k+1} & \text{if } n \geq 1 \end{cases}. \tag{30}$$

Moreover, using the fact that $\mathcal{D}g = g\mathcal{D}f$ and the definition of $\mathcal{D}$, we can write $g(t)$ in terms of $g(qt)$. Iterating this resulting recurrence, we obtain an infinite product form of $e[f]$ as follows.

$$e[f](t) = \prod_{n=0}^{\infty} \frac{1}{(1 - (1-q)q^n t \cdot \mathcal{D}f(q^n t))}. \tag{31}$$

This looks very close in form to a q-Mehler formula. For example, let us consider equation (24). To combinatorially prove a (24)-like identity for some variation $\bar{H}_n(x \mid q)$ of the q-Hermite polynomials we could attempt to do the following:

- Find a function f so that the right hand side of (24) is the same as the right hand side of (31),
- Simultaneously, find a combinatorial proof that $g_n = \bar{H}_n(x \mid q)\bar{H}_n(y \mid q)$ satisfies relation (30).
- Moreover, as we have mentioned earlier, we also need to pick $\bar{H}_n$ so that it enumerates better-behaved combinatorial objects than the vector subspaces, preferably a q-analogue of the involutionary graphs.

This idea is going to be the driven force behind our result, although what we will show is slightly more general than what was just described.

2 Main Result

In this section, we describe a Foata-style proof of q-Mehler formula for yet another variation of the q-Hermite polynomials.

2.1 Another variation of q-Hermite polynomials

The version of q-Hermite polynomials just mentioned, denoted by $\bar{H}_n(x \mid q)$ is a different form of the one described by Gessel [9]. In order to define $\bar{H}_n(x \mid q)$ combinatorially and to give motivations for defining it, we need some definition.

Let S_n denote the symmetric group on $[1, n]$ as usual. More generally, we use $Sym(N)$ to denote the set of all permutations on a set N of n distinct integers. Each word $\beta = i_1 \dots i_n$ where $\{i_1, \dots, i_n\} = N$ could be thought of as a permutation on N written in one line notation, i.e. $\beta \in Sym(N)$. The set N is called the *content* of β, and is denoted by $cont(\beta)$. Let $\phi : N \to [1, n]$ be the trivial one-to-one correspondence between N and $[1, n]$ which preserves order, then ϕ transforms each $\beta \in Sym(N)$ into $\phi(\beta) \in S_n$. The permutation $\phi(\beta) \in S_n$ is called the *reduced* permutation of β, and is denoted by $red(\beta)$.

A permutation $\beta \in Sym(N)$ is *basic* if β begins with the greatest element of $cont(\beta)$. In one line notation, each permutation $\pi = \pi_1 \dots \pi_n$ of S_n can be decomposed uniquely into blocks $\pi = \beta_1 \dots \beta_k$ where each block β_i is a basic permutation which begins with a left-to-right maximum. For example, $\pi = 53162784$ is decomposed uniquely into 4 blocks as follows.

$$\pi = 531\ \ 62\ \ 7\ \ 84.$$

We call this decomposition the *basic decomposition* of π.

A weight function w defined on permutations with values over some commutative algebra over the rationals is said to be *multiplicative* if it satisfies two conditions

(i) $w(\pi) = w(red(\pi))$.

(ii) If $\beta_1 \dots \beta_k$ is the basic decomposition of π, then $w(\pi) = w(\beta_1) \dots w(\beta_k)$.

From here on, we use B_n to denote the set of all basic permutations on $[1, n]$, D_n the set of all permutations on $[1, n]$ with only basic blocks of size at most 2, and $I(\beta)$ the number of inversions of a permutations $\beta \in Sym(N)$. Gessel proved the following simple but important theorem.

Theorem 2.1 (Gessel, 1982). *Suppose w is a multiplicative function on permutations. Let*

$$g_n = \sum_{\pi \in S_n} w(\pi) q^{I(\pi)}, \text{ and}$$

$$f_n = \sum_{\beta \in B_n} w(\beta) q^{I(\beta)}.$$

Then,

$$\sum_{n=0}^{\infty} g_n \frac{t^n}{n!_q} = e\left[\sum_{n=1}^{\infty} f_n \frac{t^n}{n!_q}\right]$$

Hence, in a sense, the basic permutations are the "connected components" of a permutation. An important corollary of this theorem is:

Corollary 2.2. *Suppose $\pi \in S_n$ has $b_i(\pi)$ basic blocks of length i. Let $w(\pi) = x_1^{b_1(\pi)} \dots x_n^{b_n(\pi)}$, where the x_i are indeterminates, then clearly w is multiplicative. Let $g_n = \sum_{\pi \in S_n} w(\pi) q^{I(\pi)}$, and $g(t \mid q) = \sum_{n=0}^{\infty} g_n \frac{t^n}{n!_q}$, then*

$$g(t \mid q) = \prod_{k=0}^{\infty} \frac{1}{1 - (1-q) q^k t X(q^{k+1} t)},$$

where $X(t) = \sum_{n=0}^{\infty} x_{n+1} t^n$.

A special case of this corollary gives us the promised new variation of the q-Hermite polynomials. Set $X(t) = x \cdot t^0 + 1 \cdot t^1$, so that

$$w(\pi) = \begin{cases} x^{b_1(\pi)} & \text{if } \pi \in D_n \\ 0 & \text{otherwise,} \end{cases}$$

and hence

$$g_n(x \mid q) = \sum_{\pi \in D_n} x^{b_1(\pi)} q^{I(\pi)}.$$

Theorem 2.1 now gives the q-exponential generating function for g_n as an infinite product:

$$\begin{aligned} g(x, t \mid q) = \sum_{n=0}^{\infty} g_n(x \mid q) \frac{t^n}{n!_q} &= \prod_{k=0}^{\infty} \frac{1}{1 - (1-q) q^k t X(q^{k+1} t)} \\ &= \prod_{k=0}^{\infty} \frac{1}{1 - (1-q) q^k t (x + q^{k+1} t)} \\ &= \prod_{k=0}^{\infty} \frac{1}{1 - 2uzq^k + z^2 q^{2k}}, \end{aligned} \tag{32}$$

where $u = \frac{-ix\sqrt{1-q}}{2\sqrt{q}}$, and $z = it\sqrt{(1-q)q}$. Comparing (32) with (17), then applying (19) give

$$g_n(x \mid q) = i^n q^{\frac{n}{2}} \tilde{H}_n\left(\frac{-ix}{\sqrt{q}} \mid q\right) \tag{33}$$

It is quite interesting that a proof of (33) can be provided combinatorially. First, we rewrite the right hand side of (33) using the matching interpretation (21) of $\tilde{H}_n(x \mid q)$ as follows.

$$\begin{aligned} i^n q^{n/2} \tilde{H}_n\left(\frac{-ix}{\sqrt{q}} \mid q\right) &= i^n q^{\frac{n}{2}} \sum_{\alpha \in M_n} (-1)^{|\alpha|} q^{s(\alpha)} \left(\frac{-ix}{\sqrt{q}}\right)^{F(\alpha)} \\ &= \sum_{\alpha \in M_n} \left(i^n (-1)^{|\alpha|} (-i)^{F(\alpha)}\right) x^{F(\alpha)} q^{\frac{n}{2} - \frac{F(\alpha)}{2} + s(\alpha)} \\ &= \sum_{\alpha \in M_n} x^{F(\alpha)} q^{|\alpha| + s(\alpha)} \end{aligned}$$

Now, (33) can be put in a combinatorial form as in the following proposition, whose proof will be bijective.

Proposition 2.3.

$$\sum_{\pi \in D_n} x^{b_1(\pi)} q^{I(\pi)} = \sum_{\alpha \in M_n} x^{F(\alpha)} q^{|\alpha| + s(\alpha)}.$$

Proof. We first describe a bijection $\varphi : D_n \longrightarrow M_n$, and then show that φ is also weight preserving. Given $\pi \in D_n$, each size-2 basic block $\pi_k \pi_{k+1}$ ($\pi_k > \pi_{k+1}$) gives rise to an

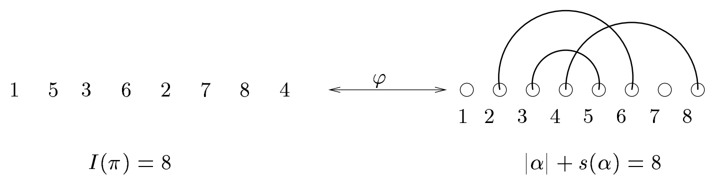


Figure 2: Illustration of the weight-preserving map from D_n onto M_n.

edge (π_{k+1}, π_k) of $\varphi(\pi)$. The rest are fixed points. Figure 2 shows the mapping when $\pi = 1\,5\,3\,6\,2\,7\,8\,4$.

To see that φ is weight-preserving, suppose $\alpha = \varphi(\pi)$. Firstly, under this bijection clearly $b_1(\pi) = F(\alpha)$. Secondly, for each edge $e = (i, j)$ of α with $j > i$, $s(e) + 1$ counts the number of inversions created by i and a number larger than i preceding it in π. Summing the quantity $s(e) + 1$ over all edges e of α, we get $s(\alpha) + |\alpha|$. □

We are now ready to define our new variation of q-Hermite polynomials combinatorially.

Definition 2.4. Let $\bar{H}_n(x \mid q)$ be defined by

$$\bar{H}_n(x \mid q) := \sum_{\alpha \in M_n} \bar{w}(\alpha), \tag{34}$$

where

$$\bar{w}(\alpha) := x^{F(\alpha)} q^{|\alpha| + s(\alpha)}. \tag{35}$$

The corresponding q-Mehler formula for these $\bar{H}_n$ can be written as

$$\sum_{n=0}^{\infty} \bar{H}_n(x \mid q) \bar{H}_n(y \mid q) \frac{t^n}{n!_q} = \frac{(q^2t^2)_\infty}{\prod_{k=0}^{\infty} \left[(1 - t^2 q^{2k+2})^2 - t(1-q)q^k \left((1 + t^2 q^{2k+2}) xy + t q^{k+1} (x^2 + y^2) \right) \right]} \tag{36}$$

2.2 Weighted $\mathbb{P}$-species

In this section, we note that the "connected components" of Theorem 2.1 do not have to be restricted to basic permutations. They could be any "combinatorial structures" defined on permutations as long as the weight function is multiplicative. To formally describe what this means, we will use the language of species as those in [2]. Our combinatorial structures on permutations shall be called $\mathbb{P}$-species (for permutation species).

We will be dealing with weighted species, thus we first need to make precise what a weighted set means.

Definition 2.5. Let $\mathbb{K} \subseteq \mathbb{C}$ be an integral domain and $\mathbb{A} = \mathbb{K}[\![q, t_1, t_2, \dots]\!]$ be a ring of formal power series or of polynomials over $\mathbb{K}$ on the variables $q, t_1, \dots$. An $\mathbb{A}$-*weighted set* is a pair (A, w) where A is a set and $w : A \to \mathbb{A}$ is a function associating a weight $w(a)$ to each element $a \in A$.

Definition 2.6. An $\mathbb{A}$-weighted set (A, w) is said to be *summable* if for each monomial $\mu = q^{n_0} t_1^{n_1} t_2^{n_2} \dots$, the number of elements $a \in A$ whose weight $w(a)$ contributes a non-zero coefficient to μ is finite.

Now, we are ready to define our weighted combinatorial structures on permutations.

Definition 2.7. An $\mathbb{A}$-*weighted* $\mathbb{P}$-*species* is a rule $\mathcal{F}$ which

(i) to each totally ordered set N, and each permutation $\sigma \in Sym(N)$, *associates* an $\mathbb{A}$-weighted set $(\mathcal{F}[N, \sigma], w)$,

(ii) to each increasing bijection $\gamma : N_1 \longrightarrow N_2$, and each permutation $\sigma \in S_{|N_1|}$ $(= S_{|N_2|})$, *associates* a weight-preserving bijection

$$\mathcal{F}[\gamma, \sigma] : (\mathcal{F}[N_1, \sigma_1], w) \longrightarrow (\mathcal{F}[N_2, \sigma_2], w),$$

where $\sigma_1 \in Sym(N_1)$ and $\sigma_2 \in Sym(N_2)$ are derived from σ in the natural way.

Moreover, these functions $\mathcal{F}[\gamma, \sigma]$ must also satisfy the *functorial properties*:

$$\mathcal{F}[Id_N, \sigma] = Id_{\mathcal{F}[N,\sigma]} \tag{37}$$

$$\mathcal{F}[\beta \circ \gamma, \sigma] = \mathcal{F}[\beta, \sigma] \circ \mathcal{F}[\gamma, \sigma] \tag{38}$$

Basically, the functorial properties say that the weighted sets $(\mathcal{F}[N, \sigma], w)$ depend only on the fact that N is totally ordered and on N's cardinality. When N has cardinality n, we shall use $\mathcal{F}[n, \sigma]$ to denote $\mathcal{F}[N, \sigma]$, and $\mathcal{F}[n]$ to denote

$$\bigcup_{\sigma \in S_n} \mathcal{F}[n, \sigma].$$

In words, $\mathcal{F}[n, \sigma]$ is the set of all structures of $\mathbb{P}$-species $\mathcal{F}$ on the permutation σ of a totally ordered set of size n, and $\mathcal{F}[n]$ is the set of all structures of $\mathbb{P}$-species $\mathcal{F}$ on a totally ordered set of cardinality n.

Definition 2.8. Let $\mathcal{F}$ be an $\mathbb{A}$-weighted $\mathbb{P}$-species with weight function w. The $\mathbb{P}$-*generating series* of $\mathcal{F}$ is the q-exponential formal power series $F_w(t \mid q)$ with coefficients in $\mathbb{A}$ defined by

$$F_w(t \mid q) := \sum_{n \geq 0} |\mathcal{F}[n]|_w \frac{t^n}{n!_q}, \tag{39}$$

where the q-inventory $|\mathcal{F}[n]|_w$ is

$$|\mathcal{F}[n]|_w := \sum_{\sigma \in S_n} \sum_{a \in \mathcal{F}[n,\sigma]} w(a) q^{I(\sigma)}. \tag{40}$$

To this end, the next step is to develop a general version of Gessel's theorem on $\mathbb{P}$-species. Theorem 2.1 was about partitioning permutations into basic blocks with a multiplicative weight function on the blocks. We shall generalize this notion by first defining the so-called permutation partition.

Definition 2.9. Given $\sigma \in S_n$, a *permutation partition* π of σ is a sequence of non empty words $\pi = (\sigma_1, \ldots, \sigma_k)$ such that

$$\sigma = \sigma_1 \sigma_2 \ldots \sigma_k$$

in one line notation, and that the largest elements of $\sigma_1, \ldots, \sigma_k$ form an increasing sequence.

We shall write $\pi \vdash \sigma$ for "π is a permutation partition of σ."

We are now ready to define a $\mathbb{P}$-species whose "connected components" are structures of another $\mathbb{P}$-species.

Definition 2.10. Let $\mathcal{F}_v$ be a weighted $\mathbb{P}$-species with weight function v. Define the $\mathbb{P}$-species $\mathcal{G}_w = \mathcal{E}(\mathcal{F})_w$ with weight function w as follows. For each totally ordered set N and $\sigma \in Sym(N)$, define

$$\mathcal{G}[N, \sigma] := \bigcup_{\pi \vdash \sigma} \mathcal{F}[N_1, \sigma_1] \times \cdots \times \mathcal{F}[N_k, \sigma_k], \tag{41}$$

where $\pi = (\sigma_1, \ldots, \sigma_k)$, and $N_i = cont(\sigma_i)$, for all $i = 1, \ldots, k$. Moreover, for each

$$G = (F_1, \ldots, F_k) \in \mathcal{F}[N_1, \sigma_1] \times \cdots \times \mathcal{F}[N_k, \sigma_k]$$

we associate

$$w(G) = v(F_1) \ldots v(F_k). \tag{42}$$

This is the analogue of the multiplicative property in Theorem 2.1. The fact that $\mathcal{E}(\mathcal{F})_w$ is a $\mathbb{P}$-species is easy to verify.

At last, we have all the notations needed for a generalization of Theorem 2.1.

Theorem 2.11. *Let $\mathcal{F}_v$ be a $\mathbb{P}$-species of structures with weight function v. Let $\mathcal{G}_w$ be the $\mathbb{P}$-species $\mathcal{E}(\mathcal{F})_w$ defined as above. Define a sequence $\{g_n\}_{n=0}^{\infty}$ by $g_0 = 1$ and*

$$g_n = |\mathcal{G}[n]|_w, \;\; n \geq 1.$$

Let $\{f_n\}_{n=0}^{\infty}$ be the sequence defined by $f_0 = 0$, and

$$f_{k+1} = |\mathcal{F}[k+1]|_v, \;\; \textit{for } k \geq 0.$$

Then,

$$1 + \sum_{n=1}^{\infty} g_n \frac{t^n}{n!_q} = e\left[\sum_{n=1}^{\infty} f_n \frac{t^n}{n!_q}\right] = \prod_{n=0}^{\infty} \frac{1}{(1-(1-q)q^n t \cdot \mathcal{D}F_v(q^n t))}. \tag{43}$$

namely

$$G_w(t \mid q) = e[F_v(t \mid q)]. \tag{44}$$

Proof. We only need to verify that the sequences g_n and f_n satisfy relation (30), namely when $n \geq 1$ we have

$$g_{n+1} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} g_{n-k} f_{k+1}.$$

Recall that each $G \in \mathcal{G}[n+1]$ is a sequence of structures of $\mathcal{F}$: $G = (F_1, \ldots, F_m)$. Let $\sigma_1, \ldots, \sigma_m$ be the corresponding permutations (or words) underlying $F_1, \ldots, F_m$. Let $N_i = cont(\sigma_i)$, for each $i = 1, \ldots, m$. Notice that $n+1 \in N_m$. Suppose $|F_m| = k+1$, $k \geq 0$, and let $V = N_m$. Let $K = V - \{n+1\}$, and let $\bar{V} = [1, n+1] - V$. For any two integer sets X and Y, let $I(X,Y)$ denote the number of inversions created by pairs of numbers in X and Y, namely

$$I(X,Y) = |\{(i,j) \mid i > j, i \in X, j \in Y\}|.$$

Note that $I(\bar{V}, V) = I(\bar{V}, K)$ by this definition, since $n+1 \in V$. Furthermore, let $G' \in \mathcal{G}[\bar{V}]$ be the structure of species $\mathcal{G}$ obtained from G by removing F_m. For each structure C of any $\mathcal{P}$-species, we shall use $\sigma(C)$ to denote the underlying permutation of C.

It is clear that

$$I(\sigma(G)) = I(\bar{V}, K) + I(\sigma(G')) + I(\sigma(F_m)),$$

and that

$$w(G) = w(G')v(F_m).$$

In order to form a $G \in \mathcal{G}[n+1]$, we can first pick a k-subset K of $[1, n]$ $(0 \le k \le n)$, form $V = K \cup \{n+1\}$, finally concatenate any pair of $G' \in \mathcal{G}[\bar{V}]$ and $F_m \in \mathcal{F}[V]$. Consequently, by definition and the multiplicativity of w we can write

$$\begin{aligned}
g_{n+1} &= \sum_{G \in \mathcal{G}[n+1]} w(G) q^{I(\sigma(G))} \\
&= \sum_{k=0}^{n} \sum_{K,|K|=k} \sum_{G' \in \mathcal{G}[\bar{V}]} \sum_{F_m \in \mathcal{F}[V]} q^{I(\bar{V},K)} \times w(G') q^{I(\sigma(G'))} \times v(F_m) q^{I(\sigma(F_m))} \\
&= \sum_{k=0}^{n} \sum_{K,|K|=k} \sum_{G' \in \mathcal{G}_{n-k}} \sum_{F_m \in \mathcal{F}_{k+1}} q^{I(\bar{V},K)} \times w(G') q^{I(\sigma(G'))} \times v(F_m) q^{I(\sigma(F_m))} \\
&= \sum_{k=0}^{n} \left(\sum_{K,|K|=k} q^{I([n]-K,K)} \right) \left(\sum_{G' \in \mathcal{G}_{n-k}} w(G') q^{I(\sigma(G'))} \right) \left(\sum_{F_m \in \mathcal{F}_{k+1}} v(F_m) q^{I(\sigma(F))} \right) \\
&= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} g_{n-k} f_{k+1}.
\end{aligned}$$

□

Example 2.12. Clearly Theorem 2.11 implies Theorem 2.1 and thus all other consequences of Theorem 2.1 as derived by Gessel [9].

Example 2.13. Take $v \equiv 1$ so that $w \equiv 1$ in Theorem 2.11, we obtain

$$1 + \sum_{n=1}^{\infty} g_n \frac{t^n}{n!_q} = \frac{(t;q)_\infty (tq;q)_\infty}{\prod_{n=0}^{\infty} \left(1 - 2tq^n + t^2 q^{2n+1}\right)}, \tag{45}$$

where,

$$g_n = \sum_{\sigma \in S_n} |\{\pi \mid \pi \vdash \sigma\}| q^{I(\sigma)}.$$

In fact, when $q \to 1$, g_n counts the number of sets of words on $[1, n]$ whose contents are disjoint and whose union of contents is exactly $[1, n]$. While, when $q \to 1$ the right hand side of (45) goes to $\exp\bigl(t/(1-t)\bigr)$. Thus, we could have proven easily identity (45) combinatorially when $q = 1$.

Following Gessel's line of derivation, we can generalize the previous example, as another corollary of Theorem 2.11, as follows.

Corollary 2.14. *Let $\pi = (\sigma_1, \ldots, \sigma_k)$ be any permutation partition of $\sigma \in S_n$. Let $b_i(\pi)$ be the number of words of size i of π. Define a weight function w for π by $w(\pi) = \prod_i x_i^{b_i(\pi)}$, and let*

$$g_n = \sum_{\sigma \in S_n} \sum_{\pi \vdash \sigma} w(\pi) q^{I(\sigma)}. \tag{46}$$

Then,

$$G_w(t \mid q) = \prod_{n=0}^{\infty} \frac{1}{\big(1-(1-q)q^n tX(q^n t)\big)}, \tag{47}$$

where

$$X(t) = \sum_{n=0}^{\infty} x_{n+1}[n+1]_q t^n.$$

Write $\pi \vdash_k \sigma$ if $\pi \vdash \sigma$ and all words of π are of size at most k. Set $X(t) = x + (1+q)t$, so that

$$w(\pi) = \begin{cases} x^{b_1(\pi)} & \text{if } \pi \vdash_2 \sigma \\ 0 & \text{otherwise,} \end{cases}$$

and hence

$$g_n(x \mid q) = \sum_{\sigma \in S_n} \sum_{\pi \vdash_2 \sigma} x^{b_1(\pi)} q^{I(\sigma)}.$$

Corollary 2.14 gives

$$\begin{aligned} G_w(x,t \mid q) = \sum_{n=0}^{\infty} g_n(x \mid q)\frac{t^n}{n!_q} &= \prod_{n=0}^{\infty} \frac{1}{1-(1-q)q^n tX(q^n t)} \\ &= \prod_{k=0}^{\infty} \frac{1}{1-2uzq^k + z^2 q^{2k}}, \end{aligned} \tag{48}$$

where $u = \frac{-it}{2}\sqrt{\frac{1-q}{1+q}}$ and $z = it\sqrt{1-q^2}$. We now get

$$g(x \mid q) = i^n (1+q)^{n/2} \tilde{H}\left(\frac{-ix}{\sqrt{1+q}} \middle| \, q \right) \tag{49}$$

and thus,

$$\sum_{\sigma \in S_n} \sum_{\pi \vdash_2 \sigma} x^{b_1(\pi)} q^{I(\sigma)} = \sum_{\alpha \in M_n} (1+q)^{|\alpha|} q^{s(\alpha)} x^{F(\alpha)}. \tag{50}$$

We already had a somewhat indirect combinatorial interpretation of (49). We leave the direct proof of (50) open for now.

2.3 A q-analogue of the bicolored n-involutionary graphs

In this section, we introduce a q-analogue of the bicolored n-involutionary graphs, then apply Theorem 2.11 for these graphs, one of whose corollaries will be the q-Mehler formula (36).

Definition 2.15. A graph $G = (N, E)$ is called an *ordered bicolored n-involutionary graph* if G satisfies the following conditions:

1. G has n vertices labeled by n distinct positive integers in N.

2. G has no multiple edges, but can have loops.

3. The n vertices of G line up on a horizontal line, so that we can speak of a vertex being on the left or right of another, and so that the vertices of G forms a permutation $\pi(G) = \pi_1\pi_2\dots\pi_n \in Sym(N)$.

4. Each edge of G is colored either red or blue.

5. Each vertex of G is incident to exactly 2 edges of different colors.

6. A non-loop edge of G can only connect some π_i to π_{i+1} unless it completes a cycle of G.

7. Let $C_1, \dots, C_m$ be the connected components of G from left to right. Let $L(C)$ denote the largest vertex number in a connected component C of G, then $\pi(G)$ must satisfy the condition that $L(C_1) < \cdots < L(C_m)$.

8. For each connected component C, the vertex numbered $L(C)$ has to be on the left of the blue edge incident to it.

9. If a connected component C is a cycle, then the vertex numbered $L(C)$ has to be the left most vertex among all vertices of C. It is not difficult to see that the connected components of G can only be in one of 5 forms as shown in Figure 3. In the figure, the bold lines represent blue edges and the thin lines represent red edges.

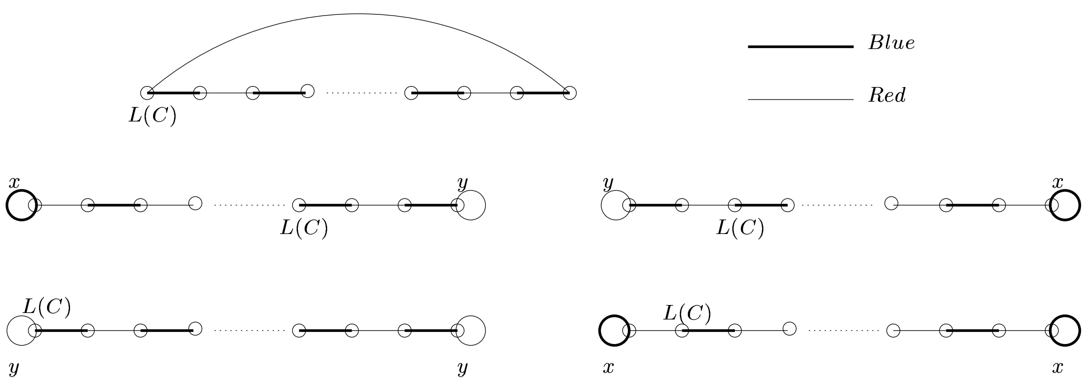


Figure 3: Possible connected component types of an ordered bicolored n-involutionary graph.

Let $\mathcal{G}_N$ denote the set of all ordered bicolored n-involutionary graphs on N, where N is an n-set of positive integers. Let $\mathcal{C}_N$ be the set of all graphs in $\mathcal{G}_N$ which have exactly one connected component. When $N = [1, n]$, $\mathcal{G}_n$ and $\mathcal{C}_n$ shall be used for convenience.

Let $\phi : N \to [1,n]$ be the trivial one-to-one correspondence between N and $[1,n]$ which preserves order. For each $G \in \mathcal{G}_N$, let $red(G)$ denote the graph obtained from G by renumbering each vertex v of G by $\phi(v)$. Conversely, we also use $N(G)$ to denote the set of vertices of G.

Definition 2.16. A weight function w defined on $\mathcal{G}_N$ with values over some commutative algebra over the rationals is said to be *multiplicative* if it satisfies the following conditions:

(i) $w(G) = w(red(G))$.

(ii) If $\gamma_1, \ldots, \gamma_k$ are the connected components of G (which are ordered bicolored involutionary graphs themselves), then $w(G) = w(\gamma_1) \ldots w(\gamma_k)$.

The following theorem is obviously a very special case of Theorem 2.11 applied to the ordered bicolored involutionary graphs.

Theorem 2.17. *Supposed w is a multiplicative function on $\mathcal{G}_n$. For $n \geq 0$, define a sequence $\{g_n\}_{n=0}^{\infty}$*

$$g_n = \sum_{G \in \mathcal{G}_n} w(G) q^{I(\pi(G))}.$$

Let $\{f_n\}_{n=0}^{\infty}$ be the sequence defined by $f_0 = 0$, and

$$f_{k+1} = \sum_{C \in \mathcal{C}_{k+1}} w(C) q^{I(\pi(C))}$$

for $k \geq 0$. Then,

$$\sum_{n=0}^{\infty} g_n \frac{t^n}{n!_q} = e\left[\sum_{n=1}^{\infty} f_n \frac{t^n}{n!_q}\right]$$

We are now ready to take the first step of the plan outlined at the end of section 1 by specializing w so that the right hand side of (31) is the same as the right hand side of (36), where the function f in (31) is

$$f(t) = \sum_{n=0}^{\infty} f_n \frac{t^n}{n!_q}.$$

Definition 2.18. Let G be a graph in $\mathcal{G}_N$. For each edge e (respectively vertex i) of G, let $C(e)$ (respectively $C(i)$) denote the connected component containing e (respectively i). Define a weight function θ on each edge e of G as follows.

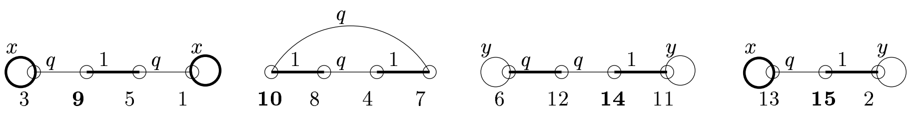
$$\theta(e) = \begin{cases} q & \text{if } e \text{ is a non-loop red edge} \\ q & \text{if } e \text{ is non-loop, blue and to the left of } L(C(e)) \\ 1 & \text{if } e \text{ is non-loop, blue and to the right of } L(C(e)) \\ y & \text{if } e \text{ is a red loop} \\ x & \text{if } e \text{ is a blue loop} \end{cases}$$

Let θ be a weight function defined on $\mathcal{G}_N$ by:

$$\theta(G) = \prod_{e \in E(G)} \theta(e),$$

then obviously θ is multiplicative.

We call an ordered bicolored n-involutionary graphs with the weight θ associated a *bicolored (q, n)-involutionary graph.* Figure 4 shows an example of such a graph. In the figure, the largest



$\pi = (3, 9, 5, 1, 10, 8, 4, 7, 6, 12, 14, 11, 13, 15, 2)$ $\qquad$ $I(\pi) = 33$

$$\theta(G)q^{I(\pi)} = q^7 x^3 y^3 \times q^{33}$$

Figure 4: An example of a bicolored (q, n)-involutionary graph.

vertex number $L(C)$ in each component C has been put in bold face.

Lemma 2.19. *Let θ be the function defined above, and $\{f_n\}_{n=0}^{\infty}$ be a sequence defined by $f_0 = 0$ and*

$$f_n = \sum_{C \in \mathcal{C}_n} \theta(C) q^{I(\pi(C))}, \text{ when } n \geq 1.$$

Moreover, let

$$f(t \mid q) := \sum_{n=0}^{\infty} f_n \frac{t^n}{n!_q}.$$

Then,

$$\mathcal{D}f(t \mid q) = \frac{(1 - t^2q^2)tq^2 + (1 + t^2q^2)xy + tq(x^2 + y^2)}{(1 - t^2q^2)(1 - t^2q^3)}. \tag{51}$$

Proof. Firstly, we claim that

$$f_{2k+1} = ([k] + [k+1])q^{2k}(2k)!_q xy.$$

To see this, let us consider Figure 3. The components in $\mathcal{C}_{2k+1}$ can only be the paths which start and end with different colored loops, and have largest vertex number $2k+1$. Summing $\theta(C)q^{I(\pi(C))}$ over all components C which start with a blue loop and end with a red loop we get the term

$$[k]q^{2k}(2k)!_q xy,$$

while the components which start red and end blue introduce the term

$$[k+1]q^{2k}(2k)!_q xy.$$

The details are easy to be verified and hence omitted here.

Secondly, we claim that

$$f_{2k+2} = q^{3k+2}(2k+1)!_q + [k+1]q^{2k+1}(2k+1)!_q(x^2+y^2).$$

Here, the term $q^{3k+2}(2k+1)!_q$ is from the cycle components, $[k+1]q^{2k+1}(2k+1)!_q x^2$ from the paths which start and end with a blue loop, and $[k+1]q^{2k+1}(2k+1)!_q y^2$ from the paths which start and end with a red loop.

By definition,

$$\begin{aligned} f(t \mid q) &= \sum_{n=0}^{\infty} f_n \frac{t^n}{n!_q} \\ &= \sum_{k=0}^{\infty} ([k] + [k+1])q^{2k}(2k)!_q xy \frac{t^{2k+1}}{(2k+1)!_q} + \\ &\quad \sum_{k=0}^{\infty} \left(q^{3k+2}(2k+1)!_q + [k+1]q^{2k+1}(2k+1)!_q(x^2+y^2) \right) \frac{t^{2k+2}}{(2k+2)!_q}. \end{aligned}$$

Hence,

$$\mathcal{D}f(t \mid q) = \sum_{k=0}^{\infty} ([k] + [k+1])q^{2k} xy t^{2k} + \sum_{k=0}^{\infty} q^{3k+2} t^{2k+1} + \\ \sum_{k=0}^{\infty} [k+1]q^{2k+1}(x^2+y^2)t^{2k+1} \quad (52)$$

Now, we calculate each term of (52) separately as follows.

$$\begin{aligned} xy\sum_{k=0}^{\infty}([k]+[k+1])q^{2k}t^{2k} &= xy(1+t^2q^2)\sum_{k=0}^{\infty}[k+1]q^{2k}t^{2k} \\ &= xy(1+t^2q^2)\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}q^{2(k-j)+3j}\right)t^{2k} \\ &= xy(1+t^2q^2)\sum_{i=0}^{\infty}t^{2i}q^{2i}\sum_{j=0}^{\infty}t^{2j}q^{3j} \\ &= \frac{(1+t^2q^2)xy}{(1-t^2q^2)(1-t^2q^3)}. \end{aligned} \tag{53}$$

Similarly,

$$\sum_{k=0}^{\infty}q^{3k+2}t^{2k+1} = \frac{tq^2}{1-t^2q^3}, \tag{54}$$

and

$$\begin{aligned} (x^2+y^2)\sum_{k=0}^{\infty}[k+1]q^{2k+1}t^{2k+1} &= (x^2+y^2)tq\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}q^{2k+j}\right)t^{2k} \\ &= (x^2+y^2)tq\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}q^{2(k-j)+3j}\right)t^{2(k-j)+2j} \\ &= (x^2+y^2)tq\sum_{i=0}^{\infty}t^{2i}q^{2i}\sum_{j=0}^{\infty}t^{2j}q^{3j} \\ &= \frac{tq(x^2+y^2)}{(1-t^2q^2)(1-t^2q^3)} \end{aligned} \tag{55}$$

Combining (53), (54) and (55) yields (51). ☐

Corollary 2.20. *Let θ be the function defined above, and $\{g_n\}_{n=0}^{\infty}$ be a sequence defined by*

$$g_n = \sum_{G\in\mathcal{G}_n}\theta(G)q^{I(\pi(G))}. \tag{56}$$

Then,

$$g(t\mid q) = \frac{(q^2t^2)_\infty}{\prod_{k=0}^{\infty}\left[(1-t^2q^{2k+2})^2 - t(1-q)q^k\left((1+t^2q^{2k+2})xy + tq^{k+1}(x^2+y^2)\right)\right]}$$

where

$$g(t\mid q) = \sum_{n=0}^{\infty}g_n\frac{t^n}{n!_q}.$$

Proof. This is straightforward from Theorem 2.17, Lemma 2.19 and equation (31). ☐

2.4 A bijective proof of the q-Mehler formula

Now, we have enough tools to show (36) bijectively. By Corollary 2.20, to prove (36) we are left to demonstrate that $\bar{H}_n(x \mid q)\bar{H}_n(y \mid q) = g_n$. We shall show this relation combinatorially as formally put in the following theorem.

Theorem 2.21. *Let $\bar{H}_n$ be defined combinatorially by equation* (34)*, and g_n by equation* (56)*. Then,*

$$\bar{H}_n(x \mid q)\bar{H}_n(y \mid q) = g_n.$$

Proof. We want to find a weight-preserving bijection φ which maps a pair $(\alpha, \alpha') \in M_{n+1} \times M_{n+1}$ to a graph $G \in \mathcal{G}_{n+1}$. Let (α, α') be a pair of matchings in $M_{n+1} \times M_{n+1}$, where the fixed points of α are weighted by x and of α' by y. As before, we view the vertices $1, \ldots, n+1$ of α and α' as lying on a horizontal line from left to right in that order, with the edges drawn on the upper half plane. Let $p_1, \ldots, p_a$ $(a \leq n+1)$ be the sequence of vertices of α starting from the right which are not left end-points of α's edges. Similarly, let $p'_1, \ldots, p'_{a'}$ be the corresponding sequence for α'. Notice that $p_1 = p'_1 = n+1$. Let $e_1, \ldots, e_{|\alpha|}$ (respectively $e'_1, \ldots, e'_{|\alpha'|}$) be the set of edges of α (respectively α') ordered by their right end-points starting from the right.

Our idea is to start from the right, look simultaneously at p_1 and p'_1, p_2 and p'_2, ... determine the "right place" to stop and build up the right most connected component of G based on the relative distribution of edges and points of α and α' seen so far. Then, remove certain points and edges from α and α' to get β and β' respectively, and re-apply the method to get the next (from the right) connected component of G, and so on.

Looking at p_1 and p'_1, p_2 and p'_2, ... there will roughly be 5 situations as follows.

1. At some $k+1$, all of p_i and p'_i, $1 \leq i \leq k+1$, are right end-points of edges in α and α' respectively, and $j = k+1$ is the least integer such that $s(e_j) = 0$.

2. We meet a fixed point p_{m+1} of α and then a fixed point p_{k+1} of α' where $m \leq k$. For this case to be disjoint from case 1, it is necessary that all edges e of α whose right end-points are on the right of p_{m+1} have $s(e) > 0$.

3. We meet a fixed point p'_{m+1} of α' strictly before a fixed point p_{k+1} of α.

4. Two fixed points of α are met before any fixed points of α'.

5. Two fixed points of α' are met before any fixed points of α.

Note that similar to case 2, the cases 3, 4, and 5 need to be defined so that they are disjoint from case 1. These cases determine our "right place" to stop as mentioned above.

Formally, we consider 5 cases as follows.

Case 1. There exists a k, $0 \leq k \leq \frac{(n-1)}{2}$, such that

(i) $j = k+1$ is the smallest integer where $s(e_j) = 0$. (i.e. $s(e_j) > 0$ for all $j \leq k$.)

(ii) For all $j = 1, \ldots, k+1$, e_j has right end-point p_j and e'_j has right end-point p'_j.

The situation is depicted in Figure 5. Let β (respectively β') be the matching obtained

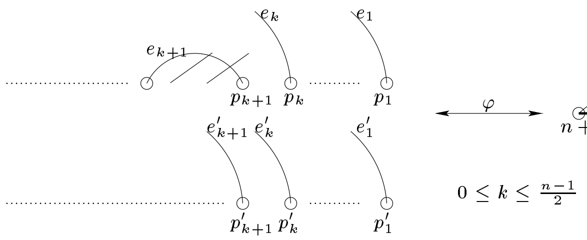


Figure 5: Illustration of case 1.

by removing $e_1, \ldots, e_{k+1}$ and their end-points (respectively $e'_1, \ldots, e'_{k+1}$ and their end-points) from α (respectively α'). We shall construct $G = \varphi(\alpha, \alpha')$ such that the last connected component of G is a cycle $C = (n+1, i_{2k+1}, \ldots, i_1)$, and that $\varphi(\beta, \beta')$ forms the rest of the components of G. Let $R = \{i_{2k+1}, \ldots, i_1\}$ be set of the rest of the points on the cycle as shown. To do this, we need to pick a permutation $\sigma = i_{2k+1} \ldots i_1 \in Sym(R)$, where $R = \{i_{2k+1}, \ldots, i_1\}$ is a set of distinct integers in $[1, n]$, such that the contribution w_C of this cycle C to the weight of G is exactly equal to the contribution w_E of $e_1, \ldots, e_{k+1}$ to the weight of α plus the contribution w'_E of $e'_1, \ldots, e'_{k+1}$ to the weight of α'. Notice that removing the edges e_j and e'_j does not have any effect on the total weights of the rest of edges of α and α'. Let $U = [1, n] - R$, and $I(U, R)$ be the number of inversions created by pairs of numbers in $U \times R$, namely the number of pairs $(u, r) \in U \times R$ such that $u > r$.

As each red edge on C is weighted q and each blue edge weighted 1, it is easy to see that

$$w_C = q^{I(U,R)+I(\sigma)+2k+1} \cdot q^{k+1} \quad (57)$$

$$w_E = q^{k+1} \cdot q^{\sum_{j=1}^{k+1} s(e_j)} \quad (58)$$

$$w'_E = q^{k+1} \cdot q^{\sum_{j=1}^{k+1} s(e'_j)} \quad (59)$$

Hence, we need to pick σ such that

$$I(U,R)+I(\sigma)=\sum_{j=1}^{k+1} s(e_j)+\sum_{j=1}^{k+1} s(e'_j)-k. \tag{60}$$

Observe that

$$\begin{aligned} s(e_{k+1}) &= 0, && (61)\\ 1\le s(e_j) &\le n+1-2j, \quad \forall j=1,\ldots,k, && (62)\\ 0\le s(e'_j) &\le n+1-2j, \quad \forall j=1,\ldots,k+1. && (63) \end{aligned}$$

Now, define a function f on $\{1,\ldots,2k+1\}$ by

$$f(t)=\begin{cases} n-t+2-s(e_j) & \text{if } t=2j, j=1,\ldots,k \\ n-t+1-s(e'_j) & \text{if } t=2j-1, j=1,\ldots,k+1. \end{cases}$$

Then, recursively determine $i_1,\ldots,i_{2k+1}$, element by element starting from i_1, working toward i_{2k+1} as follows.

$$i_t = \text{the } f(t)\text{th smallest number in } [1,n]-\{i_1,\ldots,i_{t-1}\}. \tag{64}$$

It is easy to check that $1\le f(t)\le n-(t-1)$ for all $t=1,\ldots,2k+1$ so that i_t is well defined. Moreover,

$$\begin{aligned} I(U,R)+I(\sigma) &= \sum_{t=1}^{2k+1} |\{j \mid j \text{ preceeds } i_t, j>i_t, j\neq n+1\}| \\ &= \sum_{t=1}^{2k+1}\big(n-(t-1)-f(t)\big) \\ &= \sum_{j=1}^{k}\big(n-(2j-1)-f(2j)\big)+\sum_{j=1}^{k+1}\big(n-(2j-2)-f(2j-1)\big) \\ &= \sum_{j=1}^{k+1} s(e_j)+\sum_{j=1}^{k+1} s(e'_j)-k, \end{aligned}$$

which is exactly (60).

Case 2. There exists a k, $0\le k\le \frac{n}{2}$, and an m, $0\le m\le k$ such that

(i) For all $j=1,\ldots,m$, p_j is the right end-point of e_j, and $s(e_j)>0$. Moreover, p_{m+1} is a fixed point, which is weighted by x. And, for all $j=m+2,\ldots,k+1$, p_j is the right end-point of e_{j-1}.

(ii) For all $j = 1, \ldots, k$, p'_j is the right end-point of e'_j. And, p'_{k+1} is a fixed point weighted by y.

The situation is depicted in Figure 6. This time, the last component C of G starts with a

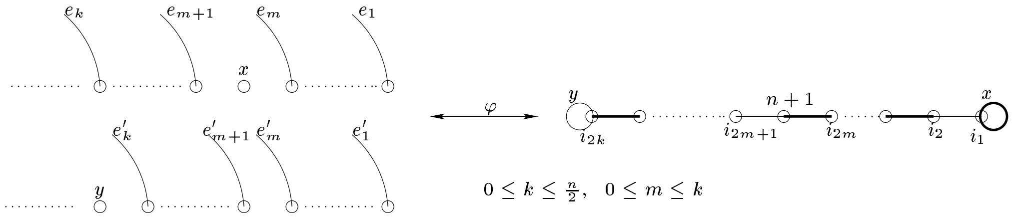


Figure 6: Illustration of case 2.

red loop and ends with a blue loop. The point $n+1$ is the $(2m+1)$st point from the right. Let σ, R, U be defined as in the previous case, then the corresponding w_C, w_E and w'_E are as follows.

$$\begin{aligned} w_C &= q^{I(U,R)+I(\sigma)+2m} \cdot q^{2k-m} \cdot xy && (65) \\ w_E &= q^k \cdot q^{\sum_{j=1}^k s(e_j)} \cdot x && (66) \\ w'_E &= q^k \cdot q^{\sum_{j=1}^k s(e'_j)} \cdot y. && (67) \end{aligned}$$

Hence, we need to pick σ so that

$$I(U,R) + I(\sigma) = \sum_{j=1}^{k} s(e_j) + \sum_{j=1}^{k} s(e'_j) - m. \quad (68)$$

For $t = 1, \ldots, 2k$, the corresponding $f(t)$ is:

$$f(t) = \begin{cases} n - t + 2 - s(e_j) & \text{if } t = 2j, j = 1, \ldots, m \\ n - t + 1 - s(e_j) & \text{if } t = 2j, j = m+1, \ldots, k \\ n - t + 1 - s(e'_j) & \text{if } t = 2j - 1, j = 1, \ldots, k. \end{cases} \quad (69)$$

As in the previous case, i_t is defined by (64). To show that i_t is well defined and that they satisfy (68), we only need to observe that

$$\begin{aligned} 1 \le s(e_j) &\le n + 1 - 2j, \quad j = 1, \ldots, m && (70) \\ 0 \le s(e_j) &\le n + 1 - (2j+1), \quad j = m+1, \ldots, k && (71) \\ 0 \le s(e'_j) &\le n + 1 - 2j, \quad j = 1, \ldots, k. && (72) \end{aligned}$$

Case 3. There exists a k, $1 \le k \le \frac{n}{2}$, and an m, $0 \le m \le k-1$, such that

(i) For all $j = 1, \ldots, k$, p_j is the right end-point of e_j, and p_{k+1} is a fixed point weighted x.

(ii) For all $j = 1, \ldots, m$, $s(e_j) > 0$.

(iii) For all $j = 1, \ldots, m$, p_j is the right end-point of e'_j, p_{m+1} is a fixed point weighted y, and for all $j = m+2, \ldots, k+1$ p_j is the right end-point of e'_{j-1}.

The situation is depicted in Figure 7. In this case, we have

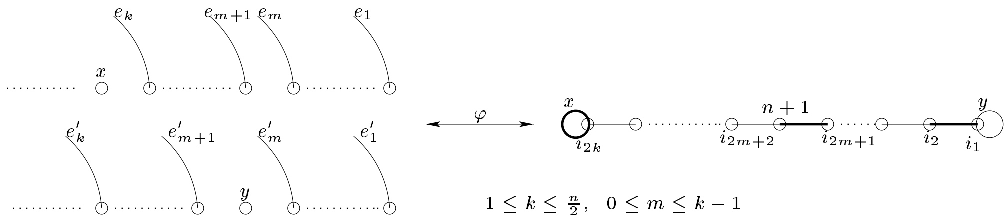


Figure 7: Illustration of case 3.

$$w_C = q^{I(U,R)+I(\sigma)+2m+1} \cdot q^{2k-(m+1)} \cdot xy \tag{73}$$

$$w_E = q^k \cdot q^{\sum_{j=1}^{k} s(e_j)} \cdot x \tag{74}$$

$$w'_E = q^k \cdot q^{\sum_{j=1}^{k} s(e'_j)} \cdot y. \tag{75}$$

Hence, we need to pick σ so that

$$I(U,R) + I(\sigma) = \sum_{j=1}^{k} s(e_j) + \sum_{j=1}^{k} s(e'_j) - m. \tag{76}$$

For $t = 1, \ldots, 2k$, the corresponding $f(t)$ is:

$$f(t) = \begin{cases} n - t + 2 - s(e_j) & \text{if } t = 2j, j = 1, \ldots, m \\ n - t + 1 - s(e'_j) & \text{if } t = 2j, j = m+1, \ldots, k \\ n - t + 1 - s(e'_j) & \text{if } t = 2j-1, j = 1, \ldots, m \\ n - t + 1 - s(e_j) & \text{if } t = 2j-1, j = m+1, \ldots, k. \end{cases} \tag{77}$$

As in the previous case, i_t is defined by (64). To show that i_t is well defined and that they satisfy (76), we only need to observe that

$$1 \leq s(e_j) \leq n + 1 - 2j, \quad j = 1, \ldots, m \tag{78}$$

$$0 \leq s(e_j) \leq n + 1 - 2j, \quad j = m+1, \ldots, k \tag{79}$$

$$0 \leq s(e'_j) \leq n + 1 - 2j, \quad j = 1, \ldots, m \tag{80}$$

$$0 \leq s(e'_j) \leq n + 1 - (2j+1), \quad j = m+1, \ldots, k. \tag{81}$$

Case 4. There exists a k, $0 \leq k \leq \frac{(n-1)}{2}$, and an m, $0 \leq m \leq k$, such that

(i) For all $j = 1, \ldots, m$, p_j is the right end-point of e_j, and $s(e_j) > 0$. Moreover, p_{m+1} and p_{k+2} are fixed points weighted x. For all $j = m+2, \ldots, k+1$, p_j is the right end-point of e_{j-1}.

(ii) For all $j = 1, \ldots, k+1$, p'_j is the right end-point of e'_j.

The situation is depicted in Figure 8. In this case, we have

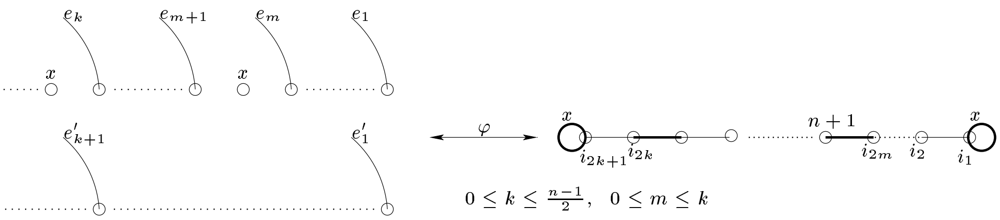


Figure 8: Illustration of case 4.

$$
\begin{aligned}
w_C &= q^{I(U,R)+I(\sigma)+2m} \cdot q^{2k+1-m} \cdot x^2 && (82)\\
w_E &= q^k \cdot q^{\sum_{j=1}^{k} s(e_j)} \cdot x^2 && (83)\\
w'_E &= q^{k+1} \cdot q^{\sum_{j=1}^{k+1} s(e'_j)}. && (84)
\end{aligned}
$$

Hence, we need to pick σ so that

$$I(U,R) + I(\sigma) = \sum_{j=1}^{k} s(e_j) + \sum_{j=1}^{k+1} s(e'_j) - m. \tag{85}$$

For $t = 1, \ldots, 2k+1$, the corresponding $f(t)$ is:

$$f(t) = \begin{cases} n - t + 2 - s(e_j) & \text{if } t = 2j,\ j = 1, \ldots, m \\ n - t + 1 - s(e_j) & \text{if } t = 2j,\ j = m+1, \ldots, k \\ n - t + 1 - s(e'_j) & \text{if } t = 2j-1,\ j = 1, \ldots, k+1. \end{cases} \tag{86}$$

As in the previous case, i_t is defined by (64). To show that i_t is well defined and that they satisfy (85), we only need to observe that

$$
\begin{aligned}
1 \leq s(e_j) &\leq n + 1 - 2j, \quad j = 1, \ldots, m && (87)\\
0 \leq s(e_j) &\leq n + 1 - (2j+1), \quad j = m+1, \ldots, k && (88)\\
0 \leq s(e'_j) &\leq n + 1 - 2j, \quad j = 1, \ldots, k+1. && (89)
\end{aligned}
$$

Case 5. There exists a k, $0 \leq k \leq \frac{(n-1)}{2}$, and an m, $0 \leq m \leq k$, such that

(i) For all $j = 1, \dots, m$, p'_j is the right end-point of e'_j. Moreover, p'_{m+1} and p'_{k+2} are fixed points weighted y. For all $j = m+2, \dots, k+1$, p'_j is the right end-point of e'_{j-1}.

(ii) For all $j = 1, \dots, k+1$, p_j is the right end-point of e_j.

(iii) For all $j = 1, \dots, m$, $s(e_j) > 0$.

The situation is depicted in Figure 9. In this case, we have

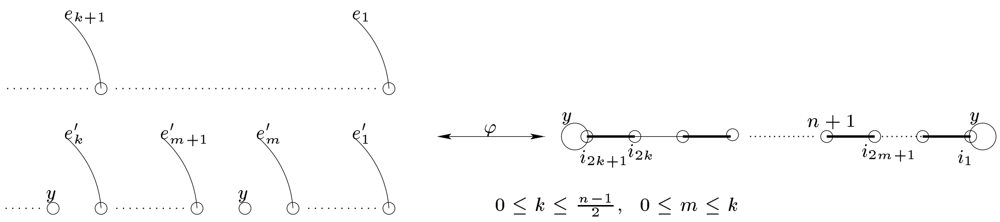


Figure 9: Illustration of case 5.

$$w_C = q^{I(U,R)+I(\sigma)+2m+1} \cdot q^{2k+1-(m+1)} \cdot y^2 \tag{90}$$

$$w_E = q^{k+1} \cdot q^{\sum_{j=1}^{k+1} s(e_j)} \tag{91}$$

$$w'_E = q^k \cdot q^{\sum_{j=1}^{k} s(e'_j)} \cdot y^2. \tag{92}$$

Hence, we need to pick σ so that

$$I(U,R) + I(\sigma) = \sum_{j=1}^{k+1} s(e_j) + \sum_{j=1}^{k} s(e'_j) - m. \tag{93}$$

For $t = 1, \dots, 2k+1$, the corresponding $f(t)$ is:

$$f(t) = \begin{cases} n - t + 2 - s(e_j) & \text{if } t = 2j,\, j = 1, \dots, m \\ n - t + 1 - s(e'_j) & \text{if } t = 2j,\, j = m+1, \dots, k \\ n - t + 1 - s(e'_j) & \text{if } t = 2j - 1,\, j = 1, \dots, m \\ n - t + 1 - s(e_j) & \text{if } t = 2j - 1,\, j = m+1, \dots, k+1. \end{cases} \tag{94}$$

As in the previous case, i_t is defined by (64). To show that i_t is well defined and that they satisfy (93), we only need to observe that

$$\begin{aligned}
1 \leq s(e_j) &\leq n+1-2j, \quad j = 1, \ldots, m & (95)\\
0 \leq s(e_j) &\leq n+1-2j, \quad j = m+1, \ldots, k+1 & (96)\\
0 \leq s(e'_j) &\leq n+1-2j, \quad j = 1, \ldots, m & (97)\\
0 \leq s(e'_j) &\leq n+1-(2j+1), \quad j = m+1, \ldots, k. & (98)
\end{aligned}$$

□

3 Discussions

We can show formula (36) in case $q = 1$ by showing

$$\begin{aligned}
g_{n+1} = &\left(\sum_{0 \leq 2k+1 \leq n} \binom{n}{2k+1} g_{n-2k-1} (2k+1)! \right) \\
&+ \left(\sum_{0 \leq 2k \leq n} \binom{n}{2k} g_{n-2k} (2k+1)! \right) xy \\
&+ \left(\sum_{0 \leq 2k+1 \leq n} \binom{n}{2k+1} g_{n-2k-1} (k+1)(2k+1)! \right) x^2 \\
&+ \left(\sum_{0 \leq 2k+1 \leq n} \binom{n}{2k+1} g_{n-2k-1} (k+1)(2k+1)! \right) y^2, \quad (99)
\end{aligned}$$

where g_n is *defined* to be $\bar{H}_n(x \mid 1)\bar{H}_n(y \mid 1)$, so that we don't need to prove the $q = 1$ version of Theorem 2.21. This relation has a very simple bijective proof. The series g_n in this case can be written as

$$g_n = \sum_{(\alpha, \alpha') \in M_n \times M_n} x^{F(\alpha)} y^{F(\alpha)}.$$

This time, we can think of the pair (α, α') as a pair of matching on the same set of n points $1, \ldots, n$. Color edges of α blue and α' red where the fixed points are the loops. The graph G formed by α and α' has the following properties: (a) each vertex is incident to exactly two edges (including loops) of different colors. Each component of G is either an even cycle, or a path which starts and ends with a loop. Each blue loop is weighted by x and red loop by y. Equation (99) follows readily by considering what type of component the vertex $n+1$ is located in. The first term is the result of $n+1$ being on an even cycle, the second term is when $n+1$ is on a path whose two loops have different colors, and the last two terms correspond to the case when $n+1$

is on a path whose two loops are of the same color. Unfortunately, this simple bijection does not generalize to the general q case, even after we have ordered the components of G in the way we did earlier. However, this does yield another simple proof of the Mehler formula.

One of the keys to our proof is showing Theorem 2.21 combinatorially, making this proof almost completely analogous to Foata's proof of the Mehler formula. However, for $n \geq 3$ we do not yet know the corresponding relationship.

Recall our initial motivation for finding a Foata-style proof of the q-Mehler formula, which was to combinatorially find multilinear extensions, especially a multilinear extension of the Kibble-Slepian formula would be very interesting to have. More work needs to be done to reach this goal. Several multilinear extensions have been found by Karande and Thakare [12], and Ismail and Stanton [10]. In particular, Karande and Thakare found two extensions as follows.

$$\sum_{k,m,n=0}^{\infty} h_{m+k}(x \mid q)h_{n+k}(y \mid q)h_{m+n}(z \mid q)\frac{r^m s^n t^k}{(q)_m(q)_n(q)_k} = \\ \frac{(xyt^2)_\infty (xyrs)_\infty}{(t)_\infty(xt)_\infty(yt)_\infty(xyt)_\infty(xr)_\infty(xzr)_\infty(ys)_\infty(yzs)_\infty}\times \\ \sum_{i,j=0}^{\infty}\frac{(xt)_i(yt)_j(xyt)_{i+j}r^i s^j}{(xyt^2)_{i+j}(q)_i(q)_j}\sum_{k=0}^{i+j}\begin{bmatrix} i+j \\ k \end{bmatrix}\frac{(xr)_i(ys)_j z^k}{(xyzrs)_k} \tag{100}$$

Define the polynomials $\phi_n(x, t \mid q)$ by

$$\phi_n(x,t \mid q) = \sum_{k=0}^{\infty}\begin{bmatrix} n \\ k \end{bmatrix}(x)_k t^k. \tag{101}$$

Then,

$$\sum_{m,n,k=0}^{\infty} h_m(x \mid q)h_n(y \mid q)h_k(z \mid q)h_{m+n+k}(w \mid q)\frac{r^m s^n t^k}{(q)_m(q)_n(q)_k} = \\ \frac{(xwr^2)_\infty}{(zwt)_\infty(xr)_\infty(r)_\infty(xwr)_\infty(wr)_\infty(wt)_\infty(yws)_\infty(ws)_\infty}\times \\ \sum_{i,j,k=0}^{\infty}\frac{(xwr)_{i+j+k}(wr)_{i+j+k}(wt)_{i+k}(yws)_j(ws)_j}{(xwr^2)_{i+j+k}(q)_i(q)_j(q)_k}\times \\ \phi_i(q^j yws, \frac{1}{y} \mid q)(ys)^i z^j t^{i+j}. \tag{102}$$

Ismail and Stanton found a different version of (102). To describe their result, we need a definition. Let

$${}_{k+1}\phi_k\left(\begin{matrix} a_1,\ldots,a_{k+1} \\ b_1,\ldots,b_k \end{matrix}\,\middle|\, q,x\right) := \sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_{k+1})_n}{(b_1)_n\cdots(b_k)_n}\frac{x^n}{(q)_n},$$

which is the q-analogue of the hypergeometric series ${}_{k+1}F_k$. Then, we have

$$\sum_{m,n=0}^{\infty} h_{m+n}(a \mid q) h_m(t_1/t_2 \mid q) h_n(t_3/t_4 \mid q) \frac{t_2^m t_4^n}{(q)_m (q)_n} = \frac{{}_4\phi_3\left(\left.{t_1,t_2,t_3,t_4 \atop q/a,0,0}\right| q,q\right)}{(a)_\infty \prod_{j=1}^4 (t_j)_\infty} + \frac{{}_4\phi_3\left(\left.{at_1,at_2,at_3,at_4 \atop aq,0,0}\right| q,q\right)}{(1/a)_\infty \prod_{j=1}^4 (at_j)_\infty} \tag{103}$$

By specializing variables and applying Gauss' theorem on ${}_2\phi_1$, this equation reduces to the q-Mehler formula (26). They actually found a general version of (103). Let

$$H(t_1,\ldots,t_k,a) := \frac{{}_k\phi_{k-1}\left(\left.{t_1,\ldots,t_k \atop q/a,0,\ldots,0}\right| q,q\right)}{(a)_\infty \prod_{j=1}^k (t_j)_\infty} + \frac{{}_k\phi_{k-1}\left(\left.{at_1,\ldots,at_k \atop aq,0,\ldots,0}\right| q,q\right)}{(1/a)_\infty \prod_{j=1}^k (at_j)_\infty} \tag{104}$$

Then,

$$\sum_{m_1,\ldots,m_k=0}^{\infty} h_{m_1+\cdots+m_k}(a \mid q) h_{m_1}(t_1/t_2 \mid q) \ldots h_{m_k}(t_{2k-1}/t_{2k} \mid q) \prod_{j=1}^{k} \frac{t_{2j}^{m_j}}{(q)_{m_j}} = H(t_1,\ldots,t_{2k},a), \tag{105}$$

and

$$\sum_{m_1,\ldots,m_k,n=0}^{\infty} h_{m_1+\cdots+m_k+n}(a \mid q) h_{m_1}(t_1/t_2 \mid q) \ldots h_{m_k}(t_{2k-1}/t_{2k} \mid q) \frac{t_{2k+1}^n}{(q)_n} \prod_{j=1}^{k} \frac{t_{2j}^{m_j}}{(q)_{m_j}} = H(t_1,\ldots,t_{2k},a). \tag{106}$$

However, equations (105) and (106) hold for $a < 0$ and $|t_i| < \min(1, -1/a)$, not as formal power series as the q-Mehler formula does.

It should be noted that there is no known "vector space proof" of the equations (100), (102), and (103). Equation (100) looks closest to the $n = 3$ case of the Mehler formula as in Example 1.3:

$$\sum_{m,n,k=0}^{\infty} \tilde{H}_{m+n}(x)\tilde{H}_{m+k}(y)\tilde{H}_{n+k}(z) \frac{r^m s^n t^k}{m!n!k!} = \frac{1}{\sqrt{1-r^2-s^2-t^2+2rst}} \times \exp\left[\frac{2rst(x^2+y^2+z^2) - x^2(r^2+s^2) - y^2(r^2+t^2) - z^2(s^2+t^2)}{2(1-r^2-s^2-t^2+2rst)} + \frac{2xy(r-st)+2xz(s-rt)+2yz(t-rs)}{2(1-r^2-s^2-t^2+2rst)}\right] \tag{107}$$

However, if we wish that every q-analogue of an exponential formula is an infinite product, then (100) is far from expectation.

Acknowledgements

I would like to thank Prof. Dennis Stanton at the school of Mathematics, University of Minnesota, for many helpful discussions and insightful suggestions, which made this work possible. This problem was suggested by him as part of my mathematics Masters thesis at the University of Minnesota.

References

[1] G. E. ANDREWS, R. ASKEY, AND R. ROY, *Special functions*, Cambridge University Press, Cambridge, 1999.

[2] F. BERGERON, G. LABELLE, AND P. LEROUX, *Combinatorial species and tree-like structures*, Cambridge University Press, Cambridge, 1998. Translated from the 1994 French original by Margaret Readdy, With a foreword by Gian-Carlo Rota.

[3] L. CARLITZ, *An extension of Mehler's formula*, Boll. Un. Mat. Ital. (4), 3 (1970), pp. 43–46.

[4] L. CARLITZ, *Some extensions of the Mehler formula*, Collect. Math., 21 (1970), pp. 117–130.

[5] T. S. CHIHARA, *An introduction to orthogonal polynomials*, Gordon and Breach Science Publishers, New York, 1978. Mathematics and its Applications, Vol. 13.

[6] D. FOATA, *A combinatorial proof of the Mehler formula*, J. Combinatorial Theory Ser. A, 24 (1978), pp. 367–376.

[7] ———, *Some Hermite polynomial identities and their combinatorics*, Adv. in Appl. Math., 2 (1981), pp. 250–259.

[8] D. FOATA AND A. M. GARSIA, *A combinatorial approach to the Mehler formulas for Hermite polynomials*, in Relations between combinatorics and other parts of mathematics (Proc. Sympos. Pure Math., Ohio State Univ., Columbus, Ohio, 1978), Amer. Math. Soc., Providence, R.I., 1979, pp. 163–179.

[9] I. M. GESSEL, *A q-analog of the exponential formula*, Discrete Math., 40 (1982), pp. 69–80.

[10] M. E. H. ISMAIL AND D. STANTON, *On the Askey-Wilson and Rogers polynomials*, Canad. J. Math., 40 (1988), pp. 1025–1045.

[11] M. E. H. ISMAIL, D. STANTON, AND G. VIENNOT, *The combinatorics of q-Hermite polynomials and the Askey-Wilson integral*, European J. Combin., 8 (1987), pp. 379–392.

[12] B. K. KARANDE AND N. K. THAKARE, *On certain q-orthogonal polynomials*, Indian J. Pure Appl. Math., 7 (1976), pp. 728–736.

[13] W. F. KIBBLE, *An extension of a theorem of Mehler's on Hermite polynomials*, Proc. Cambridge Philos. Soc., 41 (1945), pp. 12–15.

[14] J. D. LOUCK, *Extension of the Kibble-Slepian formula for Hermite polynomials using boson operator methods*, Adv. in Appl. Math., 2 (1981), pp. 239–249.

[15] D. SLEPIAN, *On the symmetrized Kronecker power of a matrix and extensions of Mehler's formula for Hermite polynomials*, SIAM J. Math. Anal., 3 (1972), pp. 606–616.

[16] D. STANTON, *Orthogonal polynomials and combinatorics*, in Special Functions 2000: Current Perspective and future directions, Kluwer, 2000, pp. 389–410.