On Even Triangulations of 2-Connected Embedded Graphs

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Abstract

Recently, Hoffmann and Kriegel proved an important combinatorial theorem [4]: Every 2-connected bipartite plane graph G has a triangulation in which all vertices have even degree (it's called an *even triangulation*). Combined with a classical Whitney's Theorem, this result implies that every such a graph has a 3-colorable plane triangulation. Using this theorem, Hoffmann and Kriegel significantly improved the upper bounds of several art gallery and prison guard problems. A complicated $O(n^2)$ time algorithm was obtained in [4] for constructing an even triangulation of G. Hoffmann and Kriegel conjectured that there is an $O(n^{3/2})$ time algorithm for solving this problem.

In this paper, we develop a simple proof of the above theorem. Our proof reveals and relies on a natural correspondence between even triangulations of G and certain orientations of G. Based on this new proof, we obtain a very simple $O(n)$ time algorithm for finding an even triangulation of G. We also extend our proof to show the existence of even triangulations for similar graphs on high genus surface.

Key words: plane graph, even triangulations, graph coloring, high genus graph.

1 Introduction

Let $G = (V, E)$ be a 2-connected bipartite plane graph. A *triangulation* of G is a plane graph obtained from G by adding new edges into the faces of G so that all of its faces are triangles. A triangulation G' of G is called *even* if all vertices of G' have even degree. Recently, Hoffmann and Kriegel proved an important combinatorial theorem [3, 4]:

Theorem 1 *Every 2-connected bipartite plane graph has an even triangulation.*

Combined with the following classical Whitney's Theorem:

Theorem 2 *A plane triangulation is 3-colorable iff all of its vertices have even degree.*

*Research supported in part by NSF Grant CCR-9912418.

Theorem 1 implies that every 2-connected bipartite plane graph has a 3-colorable plane triangulation. An elegant proof of Theorem 2 can be found in [9].

In addition to its importance in Graph Theory, Hoffmann and Kriegel showed that, based on Theorem 1, the upper bounds of several *art gallery* and *prison guard* problems (a group of problems extensively studied in Computational Geometry, see [10, 11]) can be significantly improved [4]. Theorem 1 was proved in [4] by showing that a linear equation system derived from the input graph G has a solution. An even triangulation of G is found by solving this linear equation system. By using the *generalized nested dissection technique* of Lipton, Rose and Tarjan [8], Hoffmann and Kriegel obtained a complicated algorithm with $O(n^2)$ runtime. This algorithm is the bottleneck of the algorithms for solving the art gallery and prison guard application problems discussed in [4]. By showing another linear equation system derived from G has a solution, Hoffmann and Kriegel discovered a relationship between all even triangulations of G and certain closed walks in the dual graph of G.

It was conjectured in [4] that an even triangulation of G can be found in $O(n^{1.5})$ time. In this paper, we give a very different proof of Theorem 1 without relying on the linear equation system derived in [4]. Our new proof is based on the discovery of a natural one-to-one correspondence between the set of all even triangulations of G and the set of certain orientations of G and its dual graph G^*. This new proof leads to a *very simple* $O(n)$ time algorithm for constructing an even triangulation of G. Thus, all algorithms for solving the art gallery and prison guard problems discussed in [4] can be uniformly improved to linear time. As a by-product, we also obtain a simple formula for calculating the number of distinct even triangulations of G.

A natural question is whether these theorems hold true for similar graphs on high genus surfaces. We show that our new proof of Theorem 1 and the algorithm for constructing even triangulations of 2-connected bipartite plane graphs also apply to similar graphs on high genus surfaces. However, because a crucial property that is true for plane graphs does not hold for graphs on high genus surfaces, we can only prove a lower bound on the number of distinct even triangulations of such graphs. The problem of determining the exact value of this number remains an open problem.

The rest of the paper is organized as follows. In Section 2, we introduce the definitions and preliminary results in [3, 4]. In Section 3, we provide new proofs of Theorem 1 and another key theorem in [3, 4]. In Section 4, we present our algorithm for finding even triangulations of plane graphs. In Section 5, we investigate the problem for graphs on high genus surfaces.

2 Preliminaries

In this section, we give definitions and preliminary results. All definitions are standard and can be found in [1]. Let $G = (V, E)$ be a graph with $n = |V|$ vertices and $m = |E|$ edges. The *degree* of a vertex $v \in V$, denoted by $deg_G(v)$, is the number of edges incident to v. If G is clearly understood, we simply write $deg(v)$ for $deg_G(v)$. For a subset $V_1 \subseteq V$, $G - V_1$ denotes the graph obtained from G by deleting the vertices in V_1 and their incident edges. A vertex v of a connected graph G is called a *cut vertex* if $G - \{v\}$ is disconnected. G is *2-connected* if it has no cut vertices. $G = (V, E)$ is *bipartite* if its vertex set V can be partitioned into two subsets V_1 and V_2 such that no two vertices in V_1 are adjacent and no two vertices in V_2 are adjacent. A *k-coloring* of G is a coloring of V by k colors so that no two vertices with same color are adjacent to each other. Note that G is bipartite iff it's 2-colorable.

A *cycle* C of G is a sequence of distinct vertices $u_1, u_2, \ldots, u_k$ such that $(u_i, u_{i+1}) \in E$ for $1 \le i < k$ and $(u_k, u_1) \in E$. We also use C to denote the set of the edges in it. If C contains k edges, it is called a k-cycle. A 3-cycle is also called a *triangle*. A *closed walk* is similarly defined except that it allows repeated vertices (but not repeated edges).

A *plane* graph G is a graph embedded in the plane without edge crossings (i.e. an embedded planar graph). The embedding of a plane graph G divides the plane into a number of regions. All regions are called *faces*. The unbounded region is called the *exterior face*. Other regions are called *interior faces*. The *degree* of a face is the number of edges on its boundary.

The dual graph $G^* = (V^*, E^*)$ of a plane graph G is defined as follows: For each face f of G, V^* contains a vertex v_f. For each edge e in G, G^* has an edge $e^* = (v_{f_1}, v_{f_2})$ where f_1 and f_2 are the two faces of G with e on their common boundary. e^* is called the *dual edge* of e. The mapping $e \Leftrightarrow e^*$ is a one-to-one correspondence between E and E^*.

A *diagonal* of a plane graph G is an edge which does not belong to E and connects two vertices of a facial cycle. G is *triangulated* if it has no diagonals. (Namely all of its facial cycles, including the exterior face, are triangles). A *triangulation* of G is obtained from G by adding a set of diagonals such that the resulting graph is plane and triangulated. An *even triangulation* is a triangulation in which all vertices have even degree. In this case, we also call the set of diagonals added into G an even triangulation of G. The overloading of the term “even triangulation” makes our discussion easier. The meaning of “even triangulation” is always clear from context.

For a 2-connected bipartite plane graph G, all of its facial cycles have even length. In order to triangulate G, we first add diagonals into the faces of G so that all facial cycles of the resulting graph are 4-cycles. This can be easily done in linear time. Thus, without loss of generality, G always denotes a 2-connected bipartite plane graph all of whose facial cycles have length 4. Such a graph will be called a 2-connected maximal bipartite plane graph (2MBP graph for short). By Euler's formula, a 2MBP graph with n vertices always has $n-2$ faces and $m = 2n-4$ edges. We denote the faces of G by $Q(G) = \{q_1, q_2, \ldots, q_{n-2}\}$. When G is clearly understood, we simply use Q to denote $Q(G)$.

Since G is bipartite, we can fix a 2-coloring of G with colors 0 and 1. For any face $q_i \in Q$, the set of the four vertices on the boundary of q_i is denoted by V_{q_i} and we set $Q_v = \{q_i \in Q | v \in V_{q_i}\}$. Since every facial cycle of G is a 4-cycle, every face $q_i \in Q$ has two diagonals: the diagonal joining the two 0-colored (1-colored, respectively) vertices in V_{q_i} is called the 0-diagonal (1-diagonal, respectively). Thus a triangulation of G is nothing but choosing for each face q_i either the 0-diagonal or the 1-diagonal and adding it into q_i. We associate each face $q_i \in Q$ with a $\{0,1\}$-valued variable x_i, and each triangulation T of G with a vector $\vec{x} = (x_1, x_2, \ldots, x_{n-2}) \in GF(2)^{n-2}$, where:

$$T \text{ contains the 0-diagonal of the face } q_i \Longleftrightarrow x_i = 1$$

This mapping defines a one-to-one correspondence between the set of triangulations of G and $GF(2)^{n-2}$. We choose $GF(2)$ because most calculations in this section are interpreted in $GF(2)$.

Observation: If $c(v)$ is the color of a vertex $v \in V_{q_i}$, then the term $x_i + c(v) \pmod 2$ describes the increase of the degree of v after adding the diagonal of q_i which corresponds to the value x_i into the face q_i. Based on this observation, Hoffmann and Kriegel [3, 4] proved that a vector $\vec{x} = (x_i)_{1 \le i \le n-2} \in GF(2)^{n-2}$ represents an even triangulation of G iff $\vec{x}$ is a solution of the following linear equation system (with n equations and $n-2$ variables) over $GF(2)$:

$$\sum_{q_i \in Q_v} x_i = deg(v) + |Q_v|c(v) \pmod 2 \quad (\forall v \in V) \tag{1}$$

In [4], Hoffmann and Kriegel showed that Equation (1) always has a solution, and hence proved Theorem 1. Therefore, the term "a vector $\vec{x}$ representing an even triangulation of G" and the term "a solution vector of Equation (1)" can be used interchangeably.

To obtain all solutions of Equation (1) (i.e. all even triangulations of G), [4] introduced the concept of *straight walk*. For a 2MBP graph G, its dual graph G^* is 4-regular and connected. Consider a walk S in G^*. Since G^* is 4-regular, at every vertex of S, we have four possible choices to continue the walk: go back, turn left, go straight, or turn right. A closed walk of G^* consisting of only straight steps at every vertex is called a *straight walk* or an *S-walk*. The edge set of G^* can be uniquely partitioned into S-walks. We use $\mathcal{S}(G^*) = \{S_1, \ldots, S_k\}$ to denote this partition, where each S_i ($1 \leq i \leq k$) is an S-walk of G^*. Each vertex of G^* (i.e. each face of G) occurs either twice on one S-walk or on two different S-walks. If a face f occurs on one S-walk twice, it is called a *1-walk face*. If f occurs on two different S-walks, it is called a *2-walk face*.

Figure 1 shows a 2MBP graph and its dual graph G^*. The edges of G are represented by solid lines. The edges of G^* are represented by dotted lines. The vertices of G^* (i.e. the faces of G) are represented by small circles. $\mathcal{S}(G^*)$ contains 3 S-walks S_1, S_2 and S_3. The face q_1 is a 1-walk face since it occurs on S_3 twice. The face q_2 is a 2-walk face since it occurs on both S_2 and S_3. (The S-walks in this figure are directed. Its meaning will be discussed in the next section).

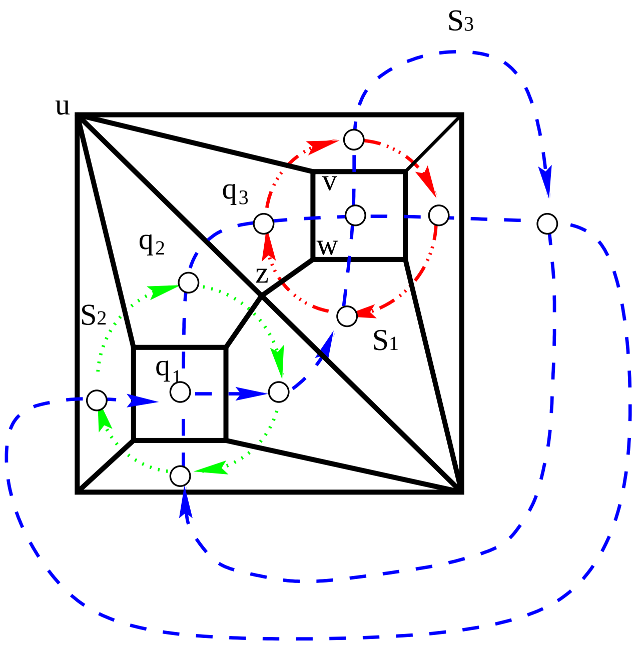


Figure 1: A 2MBP graph G, the dual graph G^* and one of its S-orientation $\mathcal{O}$.

Let S be an S-walk of G^*. Take an even triangulation of G. If we flip the diagonals of every face on S, it's easy to see that we get another even triangulation of G. (If a face f occurs on S twice, its diagonal is flipped twice and hence remains unchanged.)

Theorem 3 *Let G be a 2MBP graph and $\mathcal{S}(G^*) = \{S_1, \ldots, S_k\}$ be the S-walks of G^*. The following statements hold:*

1. *If T is an even triangulation of G and the diagonals of T are flipped along an S-walk S_i, we obtain another even triangulation of G.*

2. *If T_1 and T_2 are two even triangulations of G, these is a collection of S-walks such that by flipping the diagonals of T_1 along these S-walks we obtain T_2.*

The statement 1 of Theorem 3 was proved in [4]. The statement 2 of Theorem 3 was stated in [4]. Hoffmann and Kriegel mentioned that the statement 2 can be proved by showing another linear equation system derived from G has a solution. However, since "the arguments are much more involved", its proof was omitted in [4], and was given in the technical report [3].

3 Even Triangulations of 2-Connected Maximal Bipartite Plane Graphs

In this section, we provide new proofs of Theorems 1 and 3. First, we introduce a few definitions. An *orientation* of an undirected graph is an assignment of directions to its edges.

Definition 1 Let G be a 2MBP graph.

1. A *G-orientation* of G is an orientation of G such that every facial cycle of G is decomposed into two directed paths, one in clockwise direction and another in counterclockwise direction, each of length 2.

2. Let $\mathcal{G}_1, \mathcal{G}_2$ be two G-orientations of G. If every edge of G has reverse direction in $\mathcal{G}_1$ and $\mathcal{G}_2$, we say $\mathcal{G}_2$ is the *reverse* of $\mathcal{G}_1$. In this case, $\mathcal{G}_1$ and $\mathcal{G}_2$ are called a *G-orientation pair* of G.

3. Fix a G-orientation $\mathcal{G}$ of G. For each face q of G, the starting and the ending vertices of the two directed paths on the boundary of q are called the *primary* vertices of q. The diagonal of q connecting the two primary vertices is called its *primary diagonal* (with respect to $\mathcal{G}$). The other diagonal of q is called its *secondary diagonal* (with respect to $\mathcal{G}$).

Note that, for a G-orientation pair $\mathcal{G}_1, \mathcal{G}_2$ and any face q of G, the primary diagonal of q with respect to $\mathcal{G}_1$ is the same as its primary diagonal with respect to $\mathcal{G}_2$. If $\mathcal{G}_1, \mathcal{G}_2$ do not form a G-orientation pair, then for some faces q of G, the primary diagonal of q with respect to $\mathcal{G}_1$ is different from that with respect to $\mathcal{G}_2$.

Definition 2 Let G be a 2MBP graph and G^* be its dual graph with $\mathcal{S}(G^*) = \{S_1, \ldots, S_k\}$.

1. An *S-orientation* of G^* is an orientation of G^* such that every S-walk S_i $(1 \le i \le k)$ is a directed closed walk.

2. Let $\mathcal{O}_1, \mathcal{O}_2$ be two S-orientations of G^*. If every S_i $(1 \le i \le k)$ has reverse direction in $\mathcal{O}_1$ and $\mathcal{O}_2$, we say $\mathcal{O}_2$ is the *reverse* of $\mathcal{O}_1$. In this case, $\mathcal{O}_1$ and $\mathcal{O}_2$ are called an *S-orientation pair* of G^*.

3. Fix an S-orientation $\mathcal{O}$ of G^*. For each face q of G, if an S-walk S_i of G^* steps out of (into, respectively) q through an edge e on the boundary of q, then e is called an *out-edge* (*in-edge*, respectively) of q (with respect to $\mathcal{O}$).

We can assign two different directions to each S-walk. So, if G^* has k S-walks, it has 2^k distinct S-orientations and 2^{k-1} distinct S-orientation pairs.

Consider an arbitrary S-orientation $\mathcal{O}$ of G^*. It is easy to check that every face q of G has two out-edges and two in-edges with respect to $\mathcal{O}$, and the two in-edges of q are always incident to a common vertex on the boundary of q. Thus there are always two non-adjacent vertices on the boundary of q that are incident to both in-edges and out-edges of q. For example, consider the face q_3 in Figure 1. An S-orientation $\mathcal{O}$ of G^* is shown in the figure. With respect to $\mathcal{O}$, the edge (u, v) is an out-edge of q_3 and the edge (w, z) is an in-edge of q_3. The vertices u and w are incident to both in-edges and out-edges of q_3.

Next we show that there exists a natural one-to-one mapping between the set of S-orientations of G^* and the set of G-orientations of G which preserves the pair relation.

Definition 3 Let G be a 2MBP graph and $\mathcal{O}$ be an S-orientation of G^*. We define an orientation of G from $\mathcal{O}$ as follows. Let e be any edge of G and e^* be its dual edge in G^*. Let q_1 and q_2 be the two faces of G with e on their boundaries. Suppose that e^* is directed from q_2 to q_1 in $\mathcal{O}$. When traveling e^* from q_2 to q_1, we direct e from right to left. (In other words, e is directed counterclockwise on the boundary of q_2 and clockwise on the boundary of q_1). This orientation of G will be called the *orientation induced* from $\mathcal{O}$ and denoted by $\pi(\mathcal{O})$.

An example of the induced orientation is shown in Figure 2 (a). The S-walks S_i, S_j, S_m pass through the faces q_1, q_2 in the shown directions. The induced directions of the edges on the boundaries of q_1 and q_2 are also shown.

Lemma 1 *Let G be a 2MBP graph. If $\mathcal{O}$ is an S-orientation of G^*, then $\pi(\mathcal{O})$ is a G-orientation of G. If $\mathcal{O}$ and $\mathcal{O}'$ form an S-orientation pair, then $\pi(\mathcal{O})$ and $\pi(\mathcal{O}')$ form a G-orientation pair of G.*

Proof: Let $\mathcal{O}$ be an S-orientation of G^*. For any face q of G, its two in-edges with respect to $\mathcal{O}$ are incident and hence form a directed path of length 2, in clockwise direction, in $\pi(\mathcal{O})$. Its two out-edges with respect to $\mathcal{O}$ are also incident and hence form a directed path of length 2, in counterclockwise direction, in $\pi(\mathcal{O})$. Thus $\pi(\mathcal{O})$ is a G-orientation of G.

Let $\mathcal{O}$ and $\mathcal{O}'$ be an S-orientation pair of G^*. Let $\mathcal{G} = \pi(\mathcal{O})$ and $\mathcal{G}' = \pi(\mathcal{O}')$. The direction of each S-walk in $\mathcal{O}'$ is the reverse of that in $\mathcal{O}$. So for any face q of G, its two *in-edges* with respect to $\mathcal{O}$ are the two *out-edges* with respect to $\mathcal{O}'$. Therefore, the direction of each edge e in $\mathcal{G}'$ is the reverse of that in $\mathcal{G}$. Hence $\mathcal{G}$ and $\mathcal{G}'$ form a G-orientation pair. $\square$

Let $\mathcal{O}$ be an S-orientation of G^* and $\mathcal{G} = \pi(\mathcal{O})$. For any face q of G, the primary diagonal d of q with respect to $\mathcal{G}$ is the diagonal connecting the two vertices on the boundary of q that

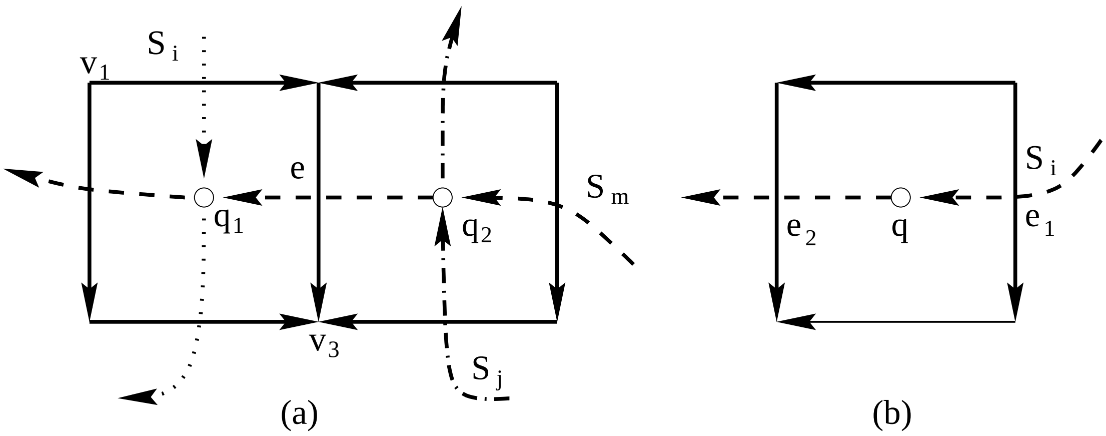


Figure 2: (a) An S-orientation $\mathcal{O}$ induces a G-orientation $\mathcal{G}$; (b) A G-orientation $\mathcal{G}$ derives an S-orientation $\mathcal{O}$.

are incident to both the in-edges and the out-edges of q with respect to $\mathcal{O}$. We also call d the primary diagonal of q with respect to $\mathcal{O}$. For example, consider the face q_1 in Figure 2 (a). The vertices v_1 and v_3 are incident to both in-edges and out-edges of q_1. Thus the primary diagonal of q_1 with respect to $\mathcal{O}$ is (v_1, v_3).

Note that if q is a 1-walk face that occurs on an S-walk S_i twice and if the direction of S_i is reversed, the primary diagonal of q remains unchanged. On the other hand, consider a 2-walk face q that occurs on two S-walks S_i and S_j. If both S_i and S_j reverse directions, the primary diagonal of q remains unchanged. If only one of the two S-walks reverses direction, then the primary and the secondary diagonals of q swap.

Definition 4 Let G be a 2MBP graph and $\mathcal{G}$ be a G-orientation of G. We define an orientation of G^* as follows. Consider any edge e^* in G^*. Let e be the edge in G corresponding to e^*. When traveling e along its direction in $\mathcal{G}$, we direct e^* from the face q_2 on the left to the face q_1 on the right. This orientation of G^* will be called the orientation *derived from* $\mathcal{G}$, and denoted by $\delta(\mathcal{G})$.

Lemma 2 *Let G be a 2MBP graph. If $\mathcal{G}$ is a G-orientation G , then $\delta(\mathcal{G})$ is an S-orientation of G^*. If $\mathcal{G}, \mathcal{G}'$ form a G-orientation pair, then $\delta(\mathcal{G})$ and $\delta(\mathcal{G}')$ form an S-orientation pair.*

Proof: Let $\mathcal{S}(G^*) = \{S_1, S_2, \cdots, S_k\}$ be S-walks of G^*. Let $\mathcal{G}$ be a G-orientation of G. We want to show that every S-walk S_i is a directed closed walk in the orientation $\mathcal{O} = \delta(\mathcal{G})$ of G^*. It is enough to show that any two consecutive edges on S_i are assigned consistent directions in $\mathcal{O}$.

Consider any face q on S_i. Let e_1 and e_2 be the two edges on the boundary of q that are walked through by S_i. Note that e_1 and e_2 are opposite on the boundary of q. Let e_1^* and e_2^* be the dual edges corresponding to e_1 and e_2, respectively. Without lose of generality, assume e_1^* is directed into q. We need to show e_2^* is directed out of q (see Figure 2 (b)).

By the definition of $\delta(\mathcal{G})$, e_1 is clockwise on the boundary of q. Because $\mathcal{G}$ is a G-orientation, e_1 and e_2 must have different direction, and hence e_2 is directed counterclockwise on the boundary of q. By the definition of $\delta(\mathcal{G})$, e_2^* is directed out of q in $\mathcal{O}$ as to be shown. Since this is true for any face q on S_i, S_i is indeed a directed closed walk in $\mathcal{O}$. Hence $\mathcal{O}$ is an S-orientation of G^*.

Let $\mathcal{G}$ and $\mathcal{G}'$ be a G-orientation pair. For each edge e of G, the direction of e in $\mathcal{G}'$ is the reverse of that in $\mathcal{G}$. Hence, the direction of its dual edge e^* in $\mathcal{O}' = \delta(\mathcal{G}')$ is the reverse of that in $\mathcal{O} = \delta(\mathcal{G})$. Therefore $\mathcal{O}$ and $\mathcal{O}'$ form an S-orientation pair of G^*. □

By Definitions 3 and 4, it's straightforward to verify that π and δ are inverse mappings to each other. Hence we have:

Theorem 4 *For a 2MBP G, the mapping π (and δ) is a one-to-one correspondence between the set of G-orientations of G and the set of S-orientations of G^*, which preserves the pair relation.*

The following theorem describes how to obtain an even triangulation from a G-orientation of G.

Theorem 5 *Let G be a 2MBP graph.*

1. *Let $\mathcal{G}$ be a G-orientation of G. Then adding the primary diagonals into each face of G results an even triangulation of G, which is called the even triangulation* determined by *$\mathcal{G}$, and denoted by $\mathcal{T}(\mathcal{G})$.*
2. *Let $\mathcal{G}$ and $\mathcal{G}'$ be two G-orientations of G. If $\mathcal{G}, \mathcal{G}'$ form a G-orientation pair of G, they determine the same even triangulation of G, i.e. $\mathcal{T}(\mathcal{G}) = \mathcal{T}(\mathcal{G}')$. If $\mathcal{G}, \mathcal{G}'$ do not form a G-orientation pair of G, they determine different even triangulations of G, i.e. $\mathcal{T}(\mathcal{G}) \neq \mathcal{T}(\mathcal{G}')$.*

Proof: Statement 1: Let G' be the graph resulting from adding the primary diagonals (with respect to $\mathcal{G}$) into the faces of G. Consider any vertex v of G. Let $e_1, e_2, \cdots, e_t$ be the edges in G incident to v in clockwise direction. Thus $deg_G(v) = t$. Let q_i $(1 \leq i \leq t)$ be the face incident to v and with e_i and e_{i+1} on its boundary (where $e_{t+1} = e_1$).

We call e_j a *go-in* (*go-out*, respectively) edge of v if e_j is directed into (out of, respectively) v in $\mathcal{G}$. A face q_i is called a *gap face* of v if one of the two edges e_i and e_{i+1} is a go-in edge of v and another is a go-out edge of v. q_i is called a *good face* of v if e_i, e_{i+1} are both go-in edges of v or both go-out edges of v. Denote the number of gap faces of v by $gap(v)$. Since a gap face is always between a block of consecutive go-out edges and a block of consecutive go-in edges around v, it is easy to see that $gap(v)$ is always an even number.

The primary diagonal (with respect to $\mathcal{G}$) of each face q is the diagonal connecting the starting and the ending vertices of the two directed paths on the boundary of q. Thus v is incident to the primary diagonal of the face q_i iff q_i is a good face of v. Hence $deg_{G'}(v) = deg_G(v) + t - gap(v) = 2t - gap(v)$ is always even. Therefore $G' = \mathcal{T}(\mathcal{G})$ is an even triangulation of G.

Figure 3 (a) shows an example of this construction. The solid lines are edges in G. The dotted lines are primary diagonals which are added into G'. The black dots indicate the gap faces. We have $deg_{G'}(v) = 2 \times 6 - 4 = 8$.

Statement 2: Suppose $\mathcal{G}$ and $\mathcal{G}'$ form a G-orientation pair of G. For any face q of G, the primary diagonal of q with respect to $\mathcal{G}$ is the same as that with respect to $\mathcal{G}'$. So $\mathcal{T}(\mathcal{G})$ is the same as $\mathcal{T}(\mathcal{G}')$.

Suppose $\mathcal{G}, \mathcal{G}'$ do not form a G-orientation pair. Then they have different sets of primary diagonals. So $\mathcal{T}(\mathcal{G})$ is different from $\mathcal{T}(\mathcal{G}')$. □

If $\mathcal{G}$ is induced from an S-orientation $\mathcal{O}$ of G^*, we also call $\mathcal{T}(\mathcal{G})$ the even triangulation *determined by* $\mathcal{O}$, and denote it by $\mathcal{T}(\mathcal{O})$.

Based on the discussion above, we have the following:

New Proof of Theorem 1: Let $\mathcal{S}(G^*) = \{S_1, S_2, \cdots, S_k\}$ be the S-walks of G^*. Arbitrarily assign a direction to each S_i. This gives an S-orientation $\mathcal{O}$ of G^*. By Lemma 1, we get an G-orientation $\mathcal{G} = \pi(\mathcal{O})$. By Theorem 5, we get an even triangulation $\mathcal{T}(\mathcal{G})$ of G. □

Theorem 5 states that every G-orientation pair determines an even triangulation of G. Next we show that every even triangulation of G is determined by a G-orientation pair of G.

Theorem 6 *Let G be a 2MBP graph and $\mathcal{S}(G^*) = \{S_1, S_2, \cdots, S_k\}$ be the S-walks of G^*.*

1. *Every even triangulation G' of G is determined by a G-orientation pair of G.*
2. *G has exactly 2^{k-1} distinct even triangulations.*

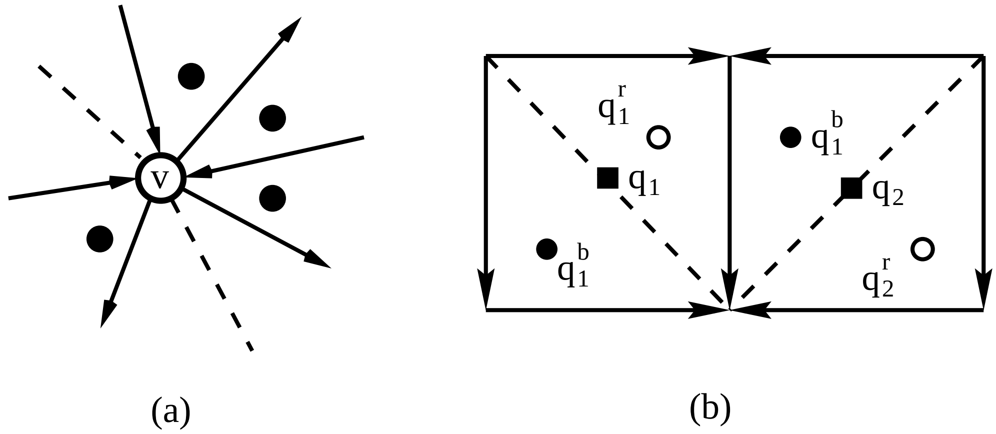


Figure 3: (a) Adding primary diagonals with respect to a G-orientation $\mathcal{G}$ results even degree at vertex v; (b) An even triangulation of G induces a G-orientation $\mathcal{G}$ on faces q_1 and q_2.

Proof: Statement 1: Let G' be any even triangulation of G. We want to show there exists a G-orientation $\mathcal{G}$ of G such that $G' = \mathcal{T}(\mathcal{G})$.

Let G'^* be the dual graph of G'. For each face q of G, the diagonal of q from G' splits q into two faces, which will be called the *subfaces* of G. So q contains two subfaces.

Since each vertex has even degree in G', each facial cycle of G'^* is of even length. Thus G'^* is a bipartite plane graph. So we can color its vertices by 2 colors. In other words, we can color the subfaces of G by red and black so that no two red (black, respectively) subfaces are adjacent. Note that each face q of G contains one red and one black subface.

Consider any edge e of G. Let q_1 and q_2 be the two faces of G with e on their common boundary. Let q_1^r and q_1^b be the red and the black subfaces contained in q_1, respectively. Let q_2^r and q_2^b be the red and the black subfaces contained in q_2, respectively. Note that exactly one red subface and one black subface from the four subfaces $q_1^r, q_1^b, q_2^r, q_2^b$ have e on their boundaries.

We direct e clockwise on its neighboring red subface. (Hence, e is directed counterclockwise on its black neighboring subface). This way, each edge e of G is consistently assigned a direction. Let $\mathcal{G}$ denote this orientation of G.

Consider any face q of G. Let q^r and q^b be the red and black subfaces contained in q, respectively. Two boundary edges of q are the boundary edges of q^r. Hence they are directed clockwise and form a directed path of length 2 in $\mathcal{G}$. The other two boundary edges of q are the boundary edges of q^b. Hence they are directed counterclockwise and form another directed path of length 2 in $\mathcal{G}$. Thus $\mathcal{G}$ is indeed a G-orientation of G.

Figure 3 (b) shows an example of this construction. Two faces q_1 and q_2 are shown in the figure by black squares. Each of them contains two subfaces. The red subfaces are indicated by empty circles. The black subfaces are indicated by solid circles. The edges of G are directed as described above.

For each face q of G, the diagonal d of q from the even triangulation G' is the diagonal which separates the red and the black subfaces of q. So it is the diagonal of q connecting the starting and the ending vertices of the two directed paths on the boundary of q. Hence d is the primary diagonal of q with respect to $\mathcal{G}$. Therefore, the even triangulation $\mathcal{T}(\mathcal{G})$ determined by $\mathcal{G}$ is exactly G'.

Statement 2: It follows directly from the following facts:

- We can assign two directions to each S-walk S_i ($1 \leq i \leq k$). So G^* has 2^k different S-orientations and 2^{k-1} different S-orientation pairs. By Theorem 4, there are exactly 2^{k-1} different G-orientation pairs of G.
- Each G-orientation pair determines a distinct even triangulation of G.
- Every even triangulation of G is determined by a G-orientation pair.

□

In order to provide a new proof of Theorem 3, we need the following:

Lemma 3 *Let G be a 2MBP and $\mathcal{S}(G^*) = \{S_1, \cdots, S_k\}$ be the S-walks of G^*. Let $\mathcal{O}_1$ be an S-orientation of G^*. Let $\mathcal{O}_2$ be the S-orientation of G^* obtained from $\mathcal{O}_1$ by reversing the direction of an S-walk S_i.*

Then $\mathcal{T}(\mathcal{O}_2)$ can be obtained from $\mathcal{T}(\mathcal{O}_1)$ by flipping the diagonals of all faces on S_i. (If q occurs on S_i twice, its diagonal is flipped twice and hence remains unchanged).

Proof: It directly follows from the following facts:

- If a face q does not occur on S_i, its primary diagonal with respect to $\mathcal{O}_1$ is the same as that with respect to $\mathcal{O}_2$.
- If a face q occurs on S_i once, its primary diagonal with respect to $\mathcal{O}_1$ is its secondary diagonal with respect to $\mathcal{O}_2$.
- If a face q occurs on S_i twice, its primary diagonal with respect to $\mathcal{O}_1$ is the same as that with respect to $\mathcal{O}_2$.

$\square$

We are now ready to give the following:

New Proof of Theorem 3:

Statement 1: Let T be an even triangulation of a 2MBP G. Then T is determined by an S-orientation $\mathcal{O}$ of G^* by Theorem 6, i.e. $T = \mathcal{T}(\mathcal{O})$. Let $\mathcal{O}'$ be the S-orientation obtained from $\mathcal{O}$ by reversing the direction of an S-walk S_i. Let T' be the even triangulation $\mathcal{T}(\mathcal{O}')$ of G determined by $\mathcal{O}'$. By Lemma 3, T' is obtained from T be flipping the diagonals of the faces on S_i.

Statement 2: Let T_1 and T_2 be any two even triangulations of G. By Theorem 6, $T_1 = \mathcal{T}(\mathcal{O}_1)$ is determined by an S-orientation $\mathcal{O}_1$ and $T_2 = \mathcal{T}(\mathcal{O}_2)$ is determined by another S-orientation $\mathcal{O}_2$. Let $S_{i_1}, S_{i_2}, \cdots, S_{i_t}$ $(t \leq k)$ be those S-walks whose directions in $\mathcal{O}_1$ and $\mathcal{O}_2$ are reversed. By repeated applications of Lemma 3, if we start with T_1 and flip the diagonals of the faces on the S-walks S_{i_j} $(1 \leq j \leq t)$ one by one, we get T_2. $\square$

4 Algorithm

Based on the discussion in Section 3, we obtain the following:

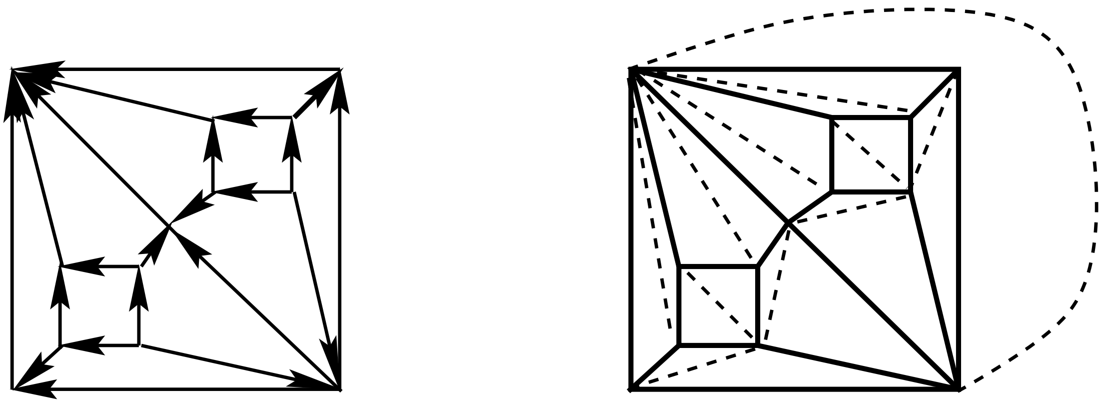
Algorithm 1: Even Triangulation
Input: A 2-connected bipartite plane graph G.
Output: An even triangulation of G.

1. Add diagonals into G, if necessary, to make all facial cycles length 4. For simplicity, we still use G to denote the resulting graph.
2. Construct the dual graph G^*.
3. Travel the edges of G^* to trace the S-walks of G^*. Each S-walk is assigned a direction when being traveled.
4. For each face q_i of G, identify its in-edges, out-edges and the primary diagonal. Add the primary diagonal into q_i.

Figure 4 shows the even triangulation constructed by Algorithm 1 for the graph G in Figure 1.

Theorem 7 *Given a 2-connected bipartite plane graph G, Algorithm 1 finds an even triangulation of G in $O(n)$ time.*

Proof: The correctness of Algorithm 1 directly follows from Theorem 5. All steps of Algorithm 1 can be easily implemented using elementary graph algorithm techniques (for example see [2]), each in $O(n)$ time. $\square$



(a) (b)

Figure 4: (a) A G-orientation $\mathcal{G}$ of G corresponding to the S-orientation $\mathcal{O}$ in Figure 1; (b) An even triangulation of G determined by $\mathcal{G}$.

5 Even Triangulations of 2-Connected Graphs on High Genus Surfaces

In this section, the term *surface* describes a closed, connected, orientable surface without boundary. Informally, it describes a sphere with g ($g \geq 0$) handles where g is the *genus* of the surface. We denotes it by $\mathcal{F}_g$. A graph G is said to be *embedded* on $\mathcal{F}_g$ if G is drawn on $\mathcal{F}_g$ such that no two edges cross. The drawing of G divides $\mathcal{F}_g$ into a number of connected regions. Each region is called a *face* of G. In this section, G always denotes a graph that can be embedded on $\mathcal{F}_g$ and each of its faces is an open disc on $\mathcal{F}_g$. (Such a graph G cannot be embedded on $\mathcal{F}_{g-1}$ without edge crossings. Although G can be embedded on $\mathcal{F}_{g+1}$, some of its faces are not open discs on $\mathcal{F}_{g+1}$. See [6, 7]). Note that a plane graph is just a graph embedded on $\mathcal{F}_0$.

As for embedded planar graphs, an *even triangulation* G' of G is obtained from G by adding diagonals into the faces of G such that every face of G' is a triangle and every vertex of G' has even degree. In this section, we investigate the problem of determining the existence and the number of even triangulations of G.

If all of the facial cycles of G are of even length, we call G a *2-connected even face graph on* $\mathcal{F}_g$ (or 2EFg graph for short). Obviously, if G is a 2EFg graph, we can add diagonals into its faces (if necessary) to make all of its facial cycles length 4. We call such a graph a *2-connected maximal even face graph on* $\mathcal{F}_g$ (or 2MEFg graph for short). Note that 2MEFg graphs are the counterpart of 2MBP graphs on high genus surfaces.

Our question is: Given a 2MEFg graph G, does G always have an even triangulation? If so, how many?

According to Euler's formula, if G is a 2MEFg graph with n vertices, m edges and f faces, we have $n + f - m = 2 - 2g$. Since each face of G is a 4-cycle, we have $4f = 2m$ and $f = n + 2g - 2$. The faces of G will be denoted by $Q(G) = \{q_1, q_2, \dots, q_{n+2g-2}\}$.

We can construct $\mathcal{F}_g$ from a $4g$ regular polygon on plane (see [6, 7]). For example, to construct a torus $\mathcal{F}_1$ from a square (see Figure 5 (a)), we glue the side S_i with the side S_i^{-1}

$(i = 1, 2)$ along their directions without twisting. The resulting surface is $\mathcal{F}_1$. Thus the points p and p^{-1} in the figure become the same point on $\mathcal{F}_1$. The four corner points of the square become the same point on $\mathcal{F}_1$. In the figure, the line segment between the point v_1 and the point p^{-1} and the line segment between the point p and the point v_3 form a continuous line segment on $\mathcal{F}_1$ between v_1 and v_3. Figure 5 (a) shows a 2MEFg graph G on $\mathcal{F}_1$ with 6 vertices (indicated by black dots), 12 edges (indicated by solid lines) and 6 faces. All the faces are 4-cycles. However, the three edges $(v_1, v_2), (v_2, v_3), (v_3, v_1)$ form a (non-facial) cycle of length 3. So G is *not* bipartite.

This example demonstrates a crucial difference between graphs on the plane and graphs on $\mathcal{F}_g$:

- If every face of a plane graph G is an even cycle, then G is bipartite.
- In contrast, even if every face of a graph G embedded on $\mathcal{F}_g$ $(g > 0)$ is an even cycle, G is not necessarily bipartite.

As we will see, this difference makes it much harder to determine the number of distinct even triangulations of G.

Let G be a 2MEFg graph. Since G is not necessarily bipartite, we cannot color the vertices of G by two colors 0 and 1. So Equation (1) does not make sense any more. Hence we cannot rely on the existence of the solutions of Equation (1) to show the existence of even triangulations of G. So the methods developed in [3, 4] cannot be used here at all.

On the other hand, most concepts and some theorems in Sections 3 and 4 can be easily adapted to handle graphs G on $\mathcal{F}_g$. Specifically, we can still construct the dual graph G^* of G, where each vertex of G^* is a face of G, and each edge of G^* corresponds to an edge in G. Obviously G^* can be embedded on $\mathcal{F}_g$. Moreover, since $\mathcal{F}_g$ is an orientable surface, it has 2 continuous normal vectors, each remains consistent while traveling on $\mathcal{F}_g$. If we fix one normal vector of $\mathcal{F}_g$, we can still define the notions of clockwise, counterclockwise, right and left directions on $\mathcal{F}_g$.

Thus the edge set of G^* can still be partitioned into a set of S-walks. The concepts of 1-walk face, 2-walk face, S-orientation, S-orientation pair, G-orientation, G-orientation pair, the mappings π and δ, in-edge and out-edge, primary diagonal can all be defined as in Section 3 without any change. Moreover, the 2MEFg graph versions of Lemma 1, Lemma 2, Theorem 4 and Theorem 5 are still valid (whose proofs require no change). In particular, we state the following 2MEFg graph version of Theorem 5:

Theorem 8 *Let G be a 2MEFg graph on $\mathcal{F}_g$.*

1. *Let $\mathcal{G}$ be a G-orientation of G. Then adding the primary diagonals to each face of G results an even triangulation of G, which is called the even triangulation determined by $\mathcal{G}$, and denoted by $\mathcal{T}(\mathcal{G})$.*
2. *Let $\mathcal{G}$ and $\mathcal{G}'$ be two G-orientations of G. If $\mathcal{G}, \mathcal{G}'$ form a G-orientation pair of G, they determine the same even triangulation of G, i.e. $\mathcal{T}(\mathcal{G}) = \mathcal{T}(\mathcal{G}')$. If $\mathcal{G}, \mathcal{G}'$ do not form a G-orientation pair of G, they determine different even triangulations of G, i.e. $\mathcal{T}(\mathcal{G}) \neq \mathcal{T}(\mathcal{G}')$.*

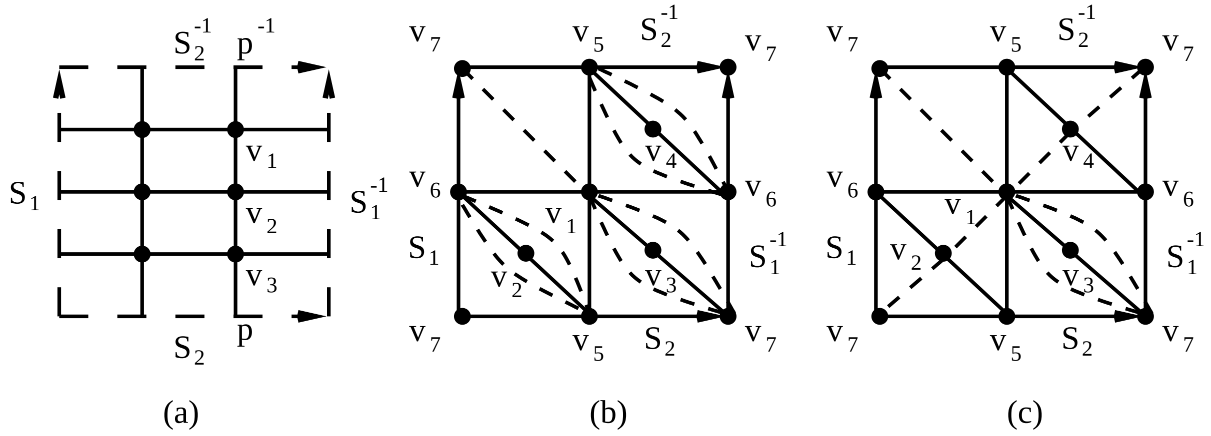


Figure 5: (a) A 2MEFg graph G_1 on $\mathcal{F}_1$ with odd cycles; (b) An even triangulation of a 2MEFg graph G on $\mathcal{F}_1$ determined by its unique S-orientation pair; (c) An even triangulation of G not determined by any S-orientation.

Because of Theorem 8, Algorithm 1 still correctly constructs an even triangulation of a 2EFg graph G in linear time. So the 2EFg graph version of Theorem 7 is also valid.

Given any even triangulation T of a 2MEFg graph G, if we flip the diagonals of every face on an S-walk S of G^*, we get another even triangulation. (If a face q occurs twice on S, its diagonal is flipped twice and hence remains unchanged). Thus the statement 1 of Theorem 3 is still true.

The proof of the statement 1 of Theorem 6 relies on the assumption that if all facial cycles of G are even cycles then G is bipartite. Since this assumption is not true for 2MEFg graphs, this proof is no longer valid. In fact, as we will show later, the two statements of Theorem 6 and the statement 2 of Theorem 3 are *false* for 2MEFg graphs.

We have the following weaker version of the statement 2 of Theorem 6.

Theorem 9 *Let G be a 2MEFg graph on $\mathcal{F}_g$ and $\mathcal{S}(G^*) = \{S_1, \ldots, S_k\}$ be all S-walks of G^*. Then G has at least $\max(2^{k-1}, 2^{2g-2})$ distinct even triangulations.*

Proof: Since we can assign two directions to each S-walk S_i ($1 \le i \le k$), G^* has 2^{k-1} S-orientation pairs. Hence G has 2^{k-1} distinct G-orientation pairs. By Theorem 8, G has at least 2^{k-1} distinct even triangulations.

Now fix an even triangulation T of G. We associate each face q_i of G with a $\{0,1\}$-valued variable x_i, and each triangulation T' of G with a vector $\vec{x} = (x_1, x_2, \ldots, x_{n+2g-2})$ in $GF(2)^{n+2g-2}$, where:

$$T \text{ and } T' \text{ contain different diagonals of the face } q_i \iff x_i = 1$$

It is easy to see that (a) T' is an even triangulation of G iff its corresponding vector $\vec{x}$ is a solution of the following linear Equation (2) over $GF(2)$; and (b) distinct even triangulations correspond to distinct solutions of Equation (2) and vice versa:

$$\sum_{q_i \in Q_v} x_i = 0 \pmod 2 \quad (\forall v \in V) \tag{2}$$

Since Equation (2) has n equations and $n+2g-2$ variables, it has at least $2g-2$ free variables. Thus, Equation (2) has at least 2^{2g-2} distinct solutions (see [5]) and G has at least 2^{2g-2} distinct even triangulations. □

Figures 5 (b) and (c) show an example that the number of even triangulations of G can be strictly bigger than $\max(2^{k-1}, 2^{2g-2})$. These figures show a 2MEFg graph G on $\mathcal{F}_1$ (after gluing the side S_i with S_i^{-1} for $i = 1, 2$). G has 7 vertices (indicated by black dots and labeled as $v_1, \cdots, v_7$) and 14 edges (indicated by solid lines). It can be verified that G^* has a single S-walk on $\mathcal{F}_1$ and hence an unique S-orientation pair. Figures 5 (b) and (c) show two distinct even triangulations of G (where the added diagonals are indicated by dotted lines). The even triangulation shown in Figure 5 (b) is determined by the unique S-orientation pair of G^*. So the even triangulation shown in Figure 5 (c) is *not* determined by any S-orientation of G^*. Therefore the statement 2 of Theorem 3 and Theorem 6 are not valid for a 2MEFg graph G.

We raise the following open problem to conclude this paper:

Open Problem: What properties of a 2MEFg graph G determine all its even triangulations and how many are they?

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