Canonical Ordering Tree and Its Applications in Graph Drawing *

Huaming Zhang Xin He
Department of Computer Science and Engineering
State University of New York at Buffalo
Buffalo, NY 14260, USA
Email: huazhang@cse.buffalo.edu, xinhe@cse.buffalo.edu

Abstract

We study the properties of Schnyder's realizers and canonical ordering trees of plane graphs. Based on these newly discovered properties, we obtain compact drawings of two styles for any plane graph G with n vertices. First we show that G has a *visibility representation* with height at most $\lceil \frac{15n}{16} \rceil$. This improves the previous best bound of $(n-1)$. Second, we show that every plane graph G has a *straight-line grid embedding* on an $(n-\Delta_0-1)\times(n-\Delta_0-1)$ grid, where Δ_0 is the number of cyclic faces of G with respect to its *minimum realizer*. This improves the previous best bound of $(n-1)\times(n-1)$.

We also study the properties of the *regular edge labeling* of 4-connected plane triangulation. Based on these properties, we show that every such a graph has a *canonical ordering tree* with at most $\lceil \frac{n+1}{2} \rceil$ leaves. This improves the previously known bound of $\lfloor \frac{2n+1}{3} \rfloor$. We show that every 4-connected plane graph has a visibility representation with height at most $\lceil \frac{3n}{4} \rceil$. All drawings discussed in this paper can be obtained in linear time.

1 Introduction

The concepts of *canonical ordering* and *canonical ordering tree* of plane triangulations and tri-connected plane graphs played crucial roles in designing several graph-drawing algorithms [8, 9, 10, 13, 15, 16, 17]. Recently, Chang et. al. generalized these concepts to arbitrary connected plane graphs [5], which leads to improvements in several graph-drawing algorithms [4, 5].

A *visibility representation* (VR for short) of a plane graph G is a representation, where the vertices of G are represented by non-overlapping horizontal segments (called *vertex segments*), and each edge of G is represented by a vertical line segment touching the vertex segments of its end vertices. The problem of computing a compact VR is important not only in algorithmic graph theory, but also in practical applications such as VLSI layout. A VR of a 2-connected plane graph G can be obtained from an *st-numbering* of G and the corresponding st-numbering of its dual [24, 27]. Using this approach, the height of the VR is bounded by $(n-1)$ and the width of the VR is bounded by $(2n-5)$ [24, 27].

Some work have been done to reduce the width of the VR. Kant and He showed that every 4-connected plane graph has a VR with width at most $(n-1)$ [18]. Kant proved that every plane graph has a VR with width at most $\lfloor \frac{3n-6}{2} \rfloor$ [17]. Very recently, Lin et. al. reduced the width to $\lfloor \frac{22n-42}{15} \rfloor$ by choosing the best st-numbering from three st-numberings derived from a Schnyder's realizer of G [22]. However, the height of the VR of a general plane graph remains to be the trivial bound of $(n-1)$. In this paper we prove that every plane graph G has a VR with height at most $\lceil \frac{15n}{16} \rceil$.

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Finding a *straight-line grid embedding* of a plane graph G is another extensively studied problem. It is known for long time that such an embedding exists. However, the drawings produced by earlier algorithms require grids of exponential size [9]. A breakthrough was achieved in [9, 10]: It was shown that such an embedding can be done on a $(2n-4) \times (n-2)$ grid. The grid size was reduced to $(n-1) \times (n-1)$ in [26]. For a 4-connected plane triangulation G with at least four exterior vertices, the size of the grid can be reduced to $(n/2) \times (n/2)$ [12, 23]. It is known that there exists a plane graph whose straight line embedding requires a grid of size at least $\lceil 2n/3 \rceil \times \lceil 2n/3 \rceil$. It has been conjectured that every plane graph G has such an embedding on a $\lceil 2n/3 \rceil \times \lceil 2n/3 \rceil$ grid. However, the best known bound remains to be $(n-1) \times (n-1)$ given in [26]. In this paper, we show that every plane graph has a straight-line grid embedding on an $(n - \Delta_0 - 1) \times (n - \Delta_0 - 1)$ grid, where Δ_0 is the number of clockwise cyclic faces of G with respect to its *minimum realizer* $\mathcal{R}_0$. While Δ_0 can be 0 in the worst case, it is the largest among all realizers of G. Our work is the first result reducing the grid size below $(n-1)$ by a non-trivial parameter.

In many applications of canonical ordering tree T, the number of leaves of T is a crucial parameter. For example, in the *floor-planning* algorithm in [4] and the *2-visibility drawing* algorithm in [5] for plane graphs, the width of both drawings are bounded by the number of leaves of the canonical ordering tree used. Thus, the problems of finding more compact drawings are reduced to finding a canonical ordering tree with fewer leaves. However, it is known that there exists a plane graph for which every canonical ordering tree has at least $\lfloor \frac{2n+1}{3} \rfloor$ leaves. Thus the width of the drawings produced by the algorithms in [4, 5] is bounded by $\lfloor \frac{2n+1}{3} \rfloor$.

We study properties of *regular edge labeling* (REL for short) of a 4-connected plane triangulation G. The properties and applications of REL have been studied in [11, 18]. Our study reveals the interconnections between the REL, the canonical ordering tree, and Schnyder's realizer. This leads to the finding of a canonical ordering tree with at most $\lceil \frac{n+1}{2} \rceil$ leaves for a 4-connected plane triangulation. Based on this, we prove that every 4-connected plane graph G has a VR with height at most $\lceil \frac{3n}{4} \rceil$. All drawings discussed in this paper can be obtained in linear time.

The present paper is organized as follows. Section 2 introduces preliminaries. Section 3 presents the construction of a VR with height at most $\lceil \frac{15n}{16} \rceil$. In Section 4, we present the algorithm for compact straight-line grid embedding. Section 5 explores the properties of REL of a 4-connected plane triangulation G and proves G has a canonical ordering tree with at most $\lceil \frac{n+1}{2} \rceil$ leaves. Section 6 studies the connection between the REL and the st-numbering and proves a 4-connected plane graph has a VR with height at most $\lceil \frac{3n}{4} \rceil$. Section 7 concludes the paper.

2 Preliminaries

In this section, we give definitions and preliminary results. $G = (V, E)$ denotes a graph with $n = |V|$ vertices and $m = |E|$ edges. The *degree* $deg(v)$ of a vertex v is the number of edges incident to v. A *planar graph* G is a graph which can be embedded on the plane without edge crossings. A *plane graph* is a planar graph with a fixed embedding. An embedding of a plane graph divides the plane into a number of regions, called *faces*. The unbounded region is the *exterior face*. Other regions are *interior faces*. The vertices and the edges on the exterior face are called *exterior vertices* and *exterior edges*. Other vertices and edges are *interior vertices* and *interior edges*. A *path* P of G is a sequence of distinct vertices $u_1, u_2, \ldots, u_k$ such that $(u_i, u_{i+1}) \in E$ for $1 \le i < k$. We also use P to denote the set of the edges in it. Each u_i for $1 < i < k$ is called *an internal vertex* of P. Furthermore, if $(u_k, u_1) \in E$, then $u_1, u_2, \ldots, u_k$ is called a *cycle*. We normally use C to denote a cycle and the set of the edges in it. If C contains k vertices, it is a k-cycle. A *triangle* (*quadrangle*, resp.) is a 3-cycle (4-cycle, resp.) A cycle C of G divides the plane into its interior and exterior regions. If C contains at least one vertex in its interior region, C is called a *separating cycle.* If all facial cycles of G are triangles, G is a *plane triangulation.* A graph G is *2-connected* if for any vertex u in G, $G - \{u\}$ is connected. G is *4-connected* if for any three distinct vertices u_1, u_2, u_3 in G, $G - \{u_1, u_2, u_3\}$ is connected. A 4-connected plane triangulation does not have separating triangles except the boundary cycle of its exterior face. G is called a *directed graph* (digraph for short) if each edge of G is assigned a direction. We abbreviate the words "counterclockwise" and "clockwise" as ccw and cw respectively.

The dual graph $G^* = (V^*, E^*)$ of a plane graph G is defined as follows: For each face F of G, G^* has a vertex v_F. For each edge e in G, G^* has a *dual edge* $e^* = (v_{F_1}, v_{F_2})$ where F_1 and F_2 are the two faces of G with e on their common boundaries.

Let G be a 2-connected plane digraph with two specified exterior vertices s and t. G is called an *st-plane graph* if it is acyclic with s as the only source and t as the only sink. The properties of st-plane graphs were studied in [20, 24]. In particular, for every face f of G, its boundary cycle consists of two directed paths. The path on its left (right, resp.) side is called the *left (right, resp.) path* of f. There is exact one source (sink, resp.) vertex on the boundary of f, it is called the source (sink, resp.) of f.

An *orientation* of a graph G is a digraph obtained from G by assigning a direction to each edge of G. We will use G to denote both the resulting digraph and the underlying undirected graph unless otherwise specified. (Its meaning will be clear from the context.) For a plane graph G and two vertices s, t, an orientation of G is called an *st-orientation* if the resulting digraph is an st-plane graph.

Let G be a 2-connected plane graph and s, t two distinct exterior vertices of G. An *st-numbering* of G is a one-to-one mapping $\xi : V \to \{1, 2, \cdots, n\}$, such that $\xi(s) = 1$, $\xi(t) = n$, and each vertex $v \neq s, t$ has two neighbors u, w with $\xi(u) < \xi(v) < \xi(w)$, where u (w, resp.) is called a *smaller neighbor (bigger neighbor, resp.)* of v. Given an st-numbering ξ of G, we can orient G by directing each edge in E from its lower numbered end vertex to its higher numbered end vertex. The resulting orientation is called the *orientation derived from* ξ which, obviously, is an st-orientation of G. On the other hand, if $G = (V, E)$ has an st-orientation $\mathcal{O}$, we can define an 1-1 mapping $\xi : V \to \{1, \cdots, n\}$ by topological sort. It is easy to see that ξ is an st-numbering and the orientation derived from ξ is $\mathcal{O}$.

Lempel et. al. [20] showed that for every 2-connected plane graph G and any two exterior vertices s and t, there exists an st-numbering. The following lemma was given in [24, 27]:

Lemma 1 *Let G be a 2-connected plane graph. Let ξ be an st-numbering of G. A VR of G can be obtained from ξ in linear time. The height of the VR is the length of the longest directed path in the st-orientation of G derived from ξ.*

Let G be a plane triangulation with $n \geq 3$ vertices and $m = 3n - 6$ edges. Let $v_1, v_2, \cdots, v_n$ be an ordering of the vertices of G where v_1, v_2, v_n are the three exterior vertices of G in ccw order. Let G_k be the subgraph of G induced by $v_1, v_2, \cdots, v_k$ and H_k the exterior face of G_k. Let $G - G_k$ be the subgraph of G obtained by removing $v_1, v_2, \cdots, v_k$.

Definition 1 [10] *An ordering $v_1, \cdots, v_n$ of a plane triangulation G is* canonical *if the following hold for every $k = 3, \cdots, n$:*

1. *G_k is biconnected, and its exterior face H_k is a cycle containing the edge (v_1, v_2).*
2. *The vertex v_k is on the exterior face of G_k, and its neighbors in G_{k-1} form a subinterval of the path $H_{k-1} - (v_1, v_2)$ with at least two vertices. Furthermore, if $k < n$, v_k has at least one neighbor in $G - G_k$. (Note that the case $k = 3$ is degenerated, and $H_2 - (v_1, v_2)$ is regarded as the edge (v_1, v_2) itself.)*

Fig. 1 illustrates a canonical ordering of a plane triangulation. A canonical ordering of G can be viewed as an order in which G is reconstructed from a single edge (v_1, v_2) step by step. At step k, when

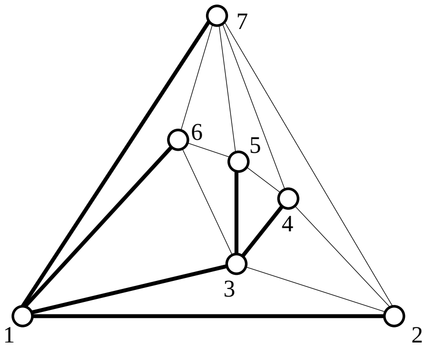


Figure 1: A canonical ordering tree of a plane triangulation G.

v_k is added to construct G_k, let $c_l, c_{l+1}, \cdots, c_r$ be the lower ordered neighbors of v_k from left to right on the exterior face of G_{k-1}. We call (v_k, c_l) the *left edge* of v_k, (v_k, c_r) the *right edge* of v_k, and the edges (c_p, v_k) with $l < p < r$ the *internal edges* of v_k. The collection T of the left edges of the vertices v_j for $3 \leq j \leq n$ plus the edge (v_1, v_2) is a spanning tree of G and is called a *canonical ordering tree* of G [10, 14]. The tree drawn in thick lines in Fig. 1 is a canonical ordering tree of G.

Let T be a rooted spanning tree of a plane graph G. Two distinct vertices of G are *unrelated* with respect to T if neither of them is an ancestor of the other in T. An edge of G is *unrelated* with respect to T if its end vertices are unrelated. While traveling T in ccw (cw, resp.) preorder (postorder, resp.), if each vertex of G is assigned a number from $\{1, 2, \cdots, n\}$ according to the order being visited, the resulting order is called the *ccw (cw, resp.) preordering (postordering, resp.)* of G with respect to T.

In [5], the concept of canonical ordering tree was generalized to any connected plane graph, which they call *orderly spanning tree*, as follows:

Definition 2 *Let G be a plane graph and T a spanning tree of G. Let $v_1, v_2, \cdots, v_n$ be the ccw preordering of the vertices of G with respect to T.*

(1) A vertex v_i of G is orderly *with respect to T if the neighbors of v_i in G form the following four blocks in ccw order around v_i:*

$B_1(v_i)$: the parent of v_i,

$B_2(v_i)$: the unrelated neighbors v_j of v_i with $j < i$,

$B_3(v_i)$: the children of v_i, and

$B_4(v_i)$: the unrelated neighbors v_j of v_i with $j > i$.

where each block could be empty.

(2) T is called an orderly spanning tree *of G if v_1 is an exterior vertex, and each v_i $(1 \leq i \leq n)$ is orderly with respect to T.*

Let T be an orderly spanning tree of a plane triangulation G. Let $v_1, v_2, \cdots, v_n$ be the ccw preordering of the vertices of G with respect to T. It is easy to see that v_1, v_2, v_n are the exterior vertices in ccw order, and for any interior vertex v_i $(3 \leq i \leq n-1)$, both $B_2(v_i)$ and $B_4(v_i)$ are nonempty.

The following is another related concept called Schnyder's realizer [25, 26]:

Definition 3 *Let G be a plane triangulation with three exterior vertices v_1, v_2, v_n in ccw order. A* realizer *$\mathcal{R}$ of G is a partition of the interior edges of G into three sets T_1, T_2, T_n of directed edges such that the following hold:*

- *For each $i \in \{1, 2, n\}$, the interior edges incident to v_i are in T_i and directed toward v_i.*

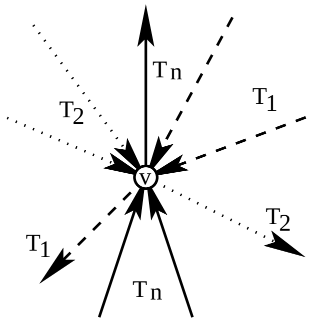


Figure 2: Edge directions around an interior vertex v.

- *For each interior vertex v of G, v has exactly one edge leaving v in each of T_1, T_2, T_n. The ccw order of the edges incident to v is: leaving in T_1, entering in T_n, leaving in T_2, entering in T_1, leaving in T_n and entering in T_2 (see Fig. 2). Each entering block could be empty.*

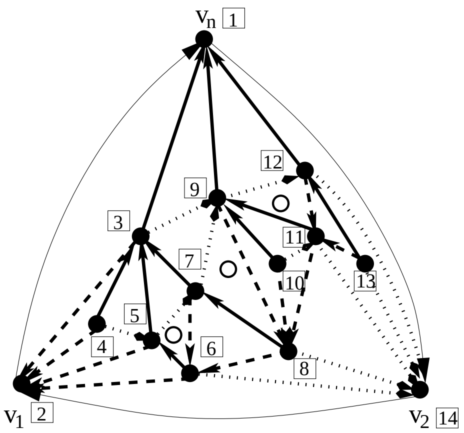


Figure 3: A plane triangulation G and the minimum realizer of G.

It was shown in [25, 26] that every plane triangulation G has a realizer $\mathcal{R}$, and each T_i ($i \in \{1, 2, n\}$) is a tree rooted at the vertex v_i containing all interior vertices of G. Fig. 3 shows a realizer of a plane triangulation G. Three trees T_1, T_2, T_n are drawn as dashed line, dotted line, and thick solid line respectively. (Ignore the small boxes containing integers and the directions of the exterior edges for now. Their meaning will be explained later.)

The following lemma summarizes the results related to the above concepts. They were shown in [4, 21, 22, 25]:

Lemma 2 *Let G be a plane triangulation with three exterior vertices v_1, v_2, v_n in ccw order.*

1. *If T is an orderly spanning tree of G, then the ccw preordering of the vertices of G with respect to T is a canonical ordering of G. Hence T is also a canonical ordering tree of G. A canonical ordering tree of G is also an orderly spanning tree of G.*

2. *Let $\{T_1, T_2, T_n\}$ be a realizer of G, where T_i is rooted at v_i for each $i = 1, 2, n$. Then each T_i plus both exterior edges of G incident to v_i is an orderly spanning tree of G.*

3. *A canonical ordering tree T of G with at most $\lfloor\frac{2n+1}{3}\rfloor$ leaves is obtainable in $O(n)$ time. And there is a plane triangulation G for which every canonical ordering tree has at least $\lfloor\frac{2n+1}{3}\rfloor$ leaves.*

4. *A realizer of G can be obtained from an orderly spanning tree T in $O(n)$ time, where one of the tree in the realizer is obtained from T by removing the two edges on the exterior face.*

The following lemma shows the connection between canonical ordering trees and st-numberings.

Lemma 3 *Let G be a plane triangulation and T be a canonical ordering tree (or equivalently, an orderly spanning tree) of G rooted at v_1. Then the ccw preordering $v_1, v_2, \cdots, v_n$ of the vertices of G with respect to T is an st-numbering of G.*

Proof: First observe that v_1, v_2, v_n are the exterior vertices of G in ccw order. For v_1 and v_2, v_n is their bigger neighbor. For v_2 and v_n, v_1 is their smaller neighbor. For any other vertex v_i, $3 \leq i \leq n-1$, both $B_2(v_i)$ and $B_4(v_i)$ are nonempty. The parent of v_i in T and all vertices in $B_2(v_i)$ are smaller neighbors of v_i. The children of v_i (if any) in T and all vertices in $B_4(v_i)$ are bigger neighbors of v_i. $\square$

For example, consider the tree T_n (rooted at v_n) shown in Fig. 3. By Lemma 2 (1) and (2), the union of T_n and the two exterior edges (v_1, v_n) and (v_2, v_n) is a canonical ordering tree of G. Denote it by $T_n{}'$. The ccw preordering of the vertices of G with respect to $T_n{}'$ are shown in integers inside the small boxes. It is an st-numbering of G by Lemma 3.

Let G be a plane triangulation with three exterior vertices v_1, v_2, v_n in ccw order. Let $\mathcal{R} = \{T_1, T_2, T_n\}$ be a realizer of G. We direct the exterior edges as $v_1 \to v_n$, $v_n \to v_2$, $v_2 \to v_1$. The resulting orientation is called the *orientation induced by* $\mathcal{R}$ and denoted by $G(\mathcal{R})$. Note that $G(\mathcal{R})$ is **never** an st-orientation because it is always cyclic.

Let $\Delta(\mathcal{R})$ denote the number of interior cyclic faces of $G(\mathcal{R})$. Let η_1, η_2, η_n be the number of internal (i.e. non-leaf) vertices in T_1, T_2, T_n, respectively. The following results were proved in [1, 2, 22]:

Lemma 4 *Let G be a plane triangulation with three exterior vertices v_1, v_2, v_n in ccw order.*

1. *Let $\mathcal{R} = \{T_1, T_2, T_n\}$ be any realizer of G. Then $\eta_1 + \eta_2 + \eta_n - \Delta(\mathcal{R}) = n - 1$.*

2. *There is an unique realizer $\mathcal{R}_0$ of G such that all interior cyclic faces of $G(\mathcal{R}_0)$ are directed in cw direction. $\mathcal{R}_0$ can be obtained in $O(n)$ time.*

The realizer $\mathcal{R}_0$ stated in the statement (2) of Lemma 4 is called the *minimum realizer* of G [1, 2]. Among all realizers of G, $G(\mathcal{R}_0)$ has the largest number of interior cyclic faces in cw direction. $\mathcal{R}_0$ plays a central role in our algorithms for compact VR representation and straight-line grid embedding.

For example, the realizer shown in Fig. 3 is actually the minimum realizer $\mathcal{R}_0$ of G. The orientation of G drawn in the figure is the orientation of $G(\mathcal{R}_0)$. It has three cw interior cyclic faces, marked by empty circles. We will call such faces the cyclic faces of $G(\mathcal{R}_0)$. (Note that although the exterior face is cyclic in $G(\mathcal{R}_0)$, we do not count it in $\Delta(\mathcal{R}_0)$ since it is not an interior face.)

In the induced orientation $G(\mathcal{R}_0)$:

- $\mathcal{F}_{3cw}$ denotes the set of interior faces of $G(\mathcal{R}_0)$ with three cw edges and let $\Delta_0 = |\mathcal{F}_{3cw}|$.

- $\mathcal{F}_{2cw}$ denotes the set of interior faces of $G(\mathcal{R}_0)$ with two cw edges and one ccw edge and let $\alpha_0 = |\mathcal{F}_{2cw}|$.

- $\mathcal{F}_{1cw}$ denotes the set of interior faces of $G(\mathcal{R}_0)$ with one cw edge and two ccw edges and let $\beta_0 = |\mathcal{F}_{1cw}|$.

Note that $\mathcal{F}_{3cw}, \mathcal{F}_{2cw}, \mathcal{F}_{1cw}$ form a partition of the set of the interior faces of $G(\mathcal{R}_0)$. The following theorem is needed in later sections.

Theorem 1 *1.* $\Delta_0 + \alpha_0 = n - \Delta_0 - 1$.

2. $\Delta_0 \le \lfloor \frac{n-1}{2} \rfloor$.

3. G has a canonical ordering tree with at least $\lceil \frac{n+1}{2} \rceil$ *leaves.*

Proof: (1) The number of interior faces of $G(\mathcal{R}_0)$ is $\Delta_0 + \alpha_0 + \beta_0 = 2n - 5$.

Since every edge (including the exterior edges) of $G(\mathcal{R}_0)$ is a cw edge of exactly one interior face, we have: $3\Delta_0 + 2\alpha_0 + \beta_0 = 3n - 6$. This gives $2\Delta_0 + \alpha_0 = n - 1$, which implies (1).

(2) Since $2\Delta_0 = n - \alpha_0 - 1 \le n - 1$, we have $\Delta_0 \le \lfloor \frac{n-1}{2} \rfloor$.

(3) Let $T_i{}'$ ($i \in \{1, 2, n\}$) be the canonical ordering tree obtained by adding the two exterior edges incident to v_i to T_i. Let $\eta_i{}'$ be the number of internal vertices of the tree $T_i{}'$, which obviously is the same as the number of the internal vertices of T_i. By Lemma 4 (1), we have $\eta_1{}' + \eta_2{}' + \eta_n{}' - \Delta_0 = n - 1$. Let l_i ($i \in \{1, 2, n\}$) be the number of leaves of $T_i{}'$. Then $\eta_i{}' = n - l_i$. Thus $l_1 + l_2 + l_n = 2n + 1 - \Delta_0$. Because $\Delta_0 \le \lfloor \frac{n-1}{2} \rfloor$, we have $l_1 + l_2 + l_n \ge \lceil \frac{3n+3}{2} \rceil$. Hence at least one of $l_i \ge \lceil \frac{n+1}{2} \rceil$. □

3 Compact Visibility Representation

In this section we present our theorem on the height of the compact VR. Let G be a plane triangulation with exterior vertices v_1, v_2, v_n in ccw order. Let T be an orderly spanning tree of G rooted at v_n. Let ρ be the ccw preordering of the vertices of G with respect to T. In this ordering, $\rho(v_n) = 1, \rho(v_1) = 2$ and $\rho(v_2) = n$. For any vertex v other than v_1, v_2, v_n, v has a nonempty set $B_2(v)$ of smaller neighbors and a nonempty set $B_4(v)$ of bigger neighbors.

We will construct two vertex numberings of G according to T simultaneously. The first vertex numbering ξ_T of G is defined as follows:

Step 1: Traveling from the leftmost unassigned leaf of T by **ccw postordering** with respect to T. (The first visited vertex is v_1.) Stop assigning numbers when we reach either the next leaf of T, or the root v_n. When we reach v_n, the numbering process is complete. When we reach a leaf, do not assign a number to it at this moment. Continue to step 2 if there are leaves remaining to be traveled.

Step 2: Traveling from the rightmost unassigned leaf of T by **cw postordering** with respect to T. (Initially, it is v_2.) Stop assigning numbers when we reach either the next leaf of T, or the root v_n. When we reach v_n, we are done. When we reach a leaf, do not assign a number to it at this moment. Loop back to step 1 if there are leaves remaining to be traveled.

Fig. 4 (a) illustrates this numbering process using the canonical ordering tree $T = T_n{}'$ (shown as thick solid lines), which is obtained from the minimum realizer in Fig. 3 by applying Lemma 2 (2). The integers inside the small boxes are the numbers assigned to the vertices. In this process, we visit v_1 first in step 1, then move to step 2 to visit v_2. Then we loop back to step 1 to visit the vertex numbered 3, followed by step 2 to visit vertices numbered 4 and 5, and so on, until we reach v_n. Note that we always have $\xi_T(v_1) = 1$, $\xi_T(v_2) = 2$ and $\xi_T(v_n) = n$.

The second vertex numbering ξ'_T is defined similarly, except that the order of step 1 and step 2 are swapped. Fig. 4 (b) shows the numbering ξ'_T for the graph G in Fig. 3 by using the canonical ordering tree $T_n{}'$. Note that we always have $\xi'_T(v_2) = 1$, $\xi'_T(v_1) = 2$ and $\xi'_T(v_n) = n$.

Next we prove the following technical lemma:

Lemma 5 *Let T be a canonical ordering tree of a plane triangulation G. Then ξ_T, ξ'_T are two st-numberings of G. The st-orientation of G derived from ξ_T (ξ'_T, resp.) are denoted by G_T (G'_T, resp.).*

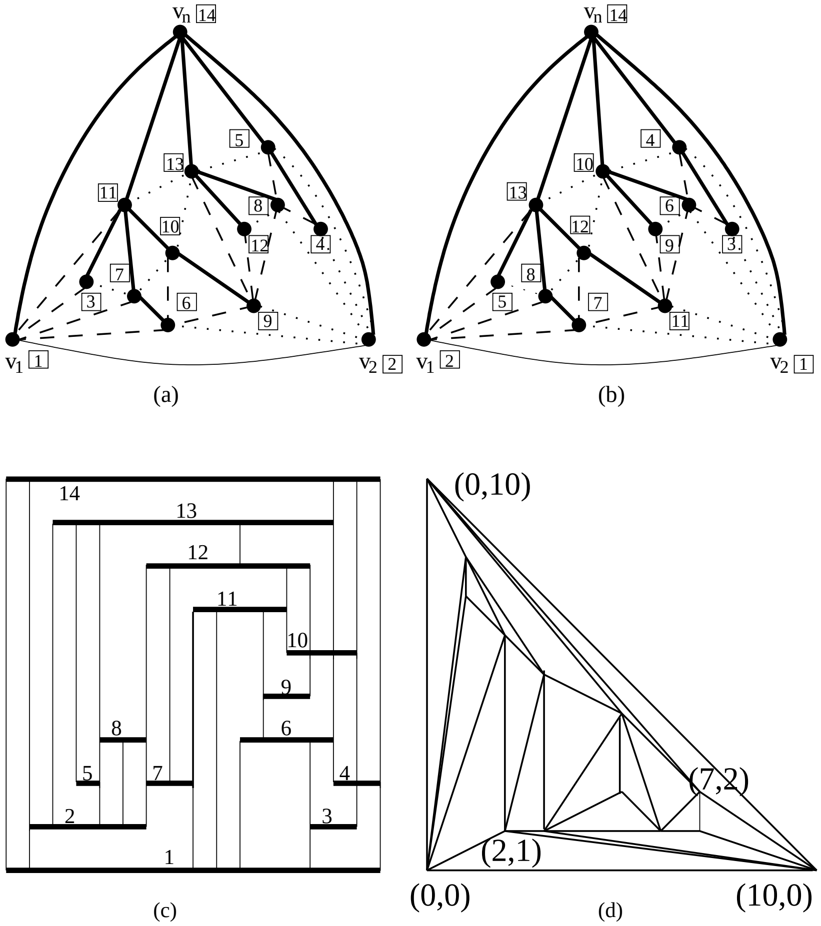


Figure 4: Two drawings of the graph G in Fig. 3

Proof: We show that ξ_T is an st-numbering of G. For each vertex other than the root v_n, its parent in T is assigned a bigger number by ξ_T. For each vertex v other than v_1, we need to show that at least one of its neighbor is assigned a smaller number by ξ_T. There are four cases.

Case 1: v is not a leaf of T. Its children in T are smaller neighbors of v.

Case 2: v is v_2. v_1 is its smaller neighbor.

Case 3: v is a leaf other than v_1 and is numbered in step 1. All the vertices in the nonempty set $B_2(v)$ are traveled before v by ξ_T. Hence, they are smaller neighbors of v.

Case 4: v is a leaf other than v_2 and is numbered in step 2. All the vertices in the nonempty set $B_4(v)$ are traveled before v by ξ_T. Hence, they are smaller neighbors of v.

Therefore, ξ_T is an st-numbering. Similarly, we can show ξ'_T is also an st-numbering. $\Box$

Let v', v'' (u', u'', resp.) be the $(2i-1)$-th and $2i$-th leaf traveled by ξ_T in step 1 (step 2, resp.). Then, u', u'' (v', v'', resp.) are the $(2i-1)$-th and $2i$-th leaf traveled by ξ'_T in its own step 1 (step 2, resp.). These four leaves will be called the *i-th block of leaves of T*. For example, in Fig. 4 (b), the vertices $v' = 2, v'' = 5, u' = 1, u'' = 3$ are the first block of leaves of T'_n. The order of the four leaves in the i-th block traveled by ξ_T (ξ'_T, resp.) is v', u', v'', u'' (u', v', u'', v'', resp.). Though it is possible that some non-leaf vertices of T are traveled by ξ_T (ξ'_T, resp.) between the four leaves.

Lemma 6 *Let T be a canonical ordering tree of a plane triangulation G. Let P be a directed path in G_T, P' a directed path in G'_T. Then for any block B of leaves of T, one of P and P' cannot pass through* *all the four vertices in B.*

Proof: If P does not pass through all the four vertices in B, we are done. Let's assume that P passes through all the four vertices in B. Let $v'_1, \cdots, v'_{t_1}$ (can be degenerated to empty set) be all the vertices traveled by ξ_T between v' and u'. (See Fig. 5 for an illustration. Some edges on P are drawn in solid lines.) Then $v'_1, \cdots, v'_{t_1}$ are all the vertices traveled between v' and u'' by ξ'_T. Let $u'_1, \cdots, u'_{t_2}$ (can be empty set) be all the vertices traveled by ξ_T between u' and v''. Then $u', u'_1, \cdots, u'_{t_2}$ are on the unique path, denoted by Q_1, of T from u' to the root v_n. Let $v''_1, \cdots, v''_{t_3}$ (can be empty set) be all the vertices traveled by ξ_T between v'' and u''. Then $v'', v''_1, \cdots, v''_{t_3}$ are on the unique path of T from v'' to the root v_n, denote this unique path by P_1.

Since P passes through both u' and v'', and $u'_1, \cdots, u'_{t_2}$ are all the vertices traveled by ξ_T between u' and v'', there is a vertex $w \in \{u', u'_1, \cdots, u'_{t_2}\}$ such that the edge (w, v'') is in P, and $\xi_T(v'') > \xi_T(w) \geq \xi_T(u')$. Denote the unique path from w to v_n in T by Q_2. Note that Q_2 is sub-path of Q_1.

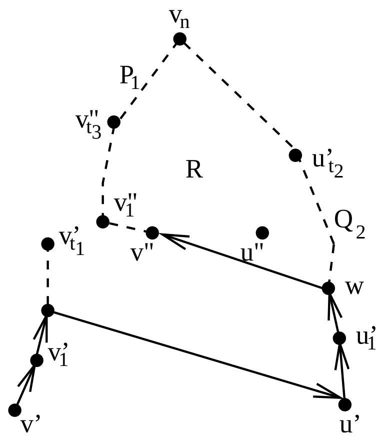


Figure 5: The proof of Lemma 6.

Let R be the closed region enclosed by the edge (w, v'') and the paths P_1 and Q_2. Now consider the position of u'' with respect to R. Because Q_1 is the path connecting u' to the root v_n in T and u'' is a leaf of T, u'' cannot be on Q_1. Therefore, u'' cannot be on Q_2. Similarly, u'' cannot be on P_1. So u'' is either inside or outside of R.

Suppose u'' is outside of R. Consider the unique path in T from u'' to the root v_n. There are two possibilities: (1) It intersects with P_1 from left. Then ξ_T travels u'' before v'', which is impossible. (2) It intersects with Q_2 from right. Then ξ_T travels u'' before u', which is again impossible. Therefore, u'' must be inside R.

Similarly, it can be shown that v' and all v'_j $(1 \leq j \leq t_1)$ are outside of R. Therefore none of v' or v'_j $(1 \leq j \leq t_1)$ can be a neighbor of u''.

Consider the path P'. If v' is not on P', we are done. Assume v' is on P'. If P' passes through u'' after it passes through v', it has to reach u'' from one of $v', v'_1, \cdots, v'_{t_1}$. (This is because $v'_1, \cdots, v'_{t_1}$ are all the vertices traveled between v' and u'' by ξ'_T.) However, since none of $v', v'_j, 1 \leq j \leq t_1$ is a neighbor of u'', this is impossible. Thus P' cannot pass through all the four vertices in B. □

Theorem 2 *Every plane graph G with n vertices has a VR with height at most $\lceil \frac{15n}{16} \rceil$, which can be obtained in linear time.*

Proof: Without loss of generality, we assume G is a plane triangulation. By Theorem 1 (3), from the minimal realizer $\mathcal{R}_0$ of G, we can obtain a canonical ordering tree T of G with at least $l \geq \lceil \frac{n+1}{2} \rceil$

leaves. Thus T has at least $\lfloor \frac{n}{8} \rfloor$ disjoint blocks of leaves. T induces two st-orientations G_T and G'_T by Lemma 5. This can be done in $O(n)$ time.

Let P be a longest directed path in G_T, P' a longest directed path in G'_T. By Lemma 6, for any block B of leaves of T, one of P or P' has to bypass at least one leaf in B. Thus P and P' together have to bypass at least $\lfloor \frac{n}{8} \rfloor$ leaves. So one of them has to bypass at least $\lfloor \frac{n}{16} \rfloor$ leaves. Hence its length is at most $\lceil \frac{15n}{16} \rceil$. Thus the length of the longest directed path in one of G_T, G'_T is at most $\lceil \frac{15n}{16} \rceil$.

We apply the VR algorithm in Lemma 1 to one of G_T, G'_T whose longest directed path is shorter. This results in a VR of G with height at most $\lceil \frac{15n}{16} \rceil$. The running time is $O(n)$. □

For example, in Fig. 4 (b), the longest directed path in the orientation derived from the given st-numbering passes through the vertices numbered $1, 3, 4, 6, 9, 10, 11, 12, 13, 14$. However, in the st-numbering shown in Fig. 3, the longest directed path passes through all vertices. Fig. 4 (c) shows the VR of the graph G, using the st-numbering in Fig. 4 (b).

4 Compact Straight-line Grid Embedding

In this section we present our theorem on the size of a compact straight-line grid embedding. Without loss of generality, we assume G is a plane triangulation.

Define an order relation $<_{lex}$ on two pairs of numbers (x_1, x_2) and (y_1, y_2) as follows: $(x_1, x_2) <_{lex} (y_1, y_2)$ iff either $x_1 < y_1$, or $x_1 = y_1$ and $x_2 < y_2$. The following definition and lemma were given in [25, 26].

Definition 4 *A* weak barycentric representation *of a graph $G = (V, E)$ is an injective mapping $\pi : v \in V \to (\pi_1(v), \pi_2(v), \pi_n(v)) \in R^3$ that satisfies the following conditions:*

1. *$\pi_1(v) + \pi_2(v) + \pi_n(v) = 1$ for all vertices v.*

2. *For each edge $(x, y) \in E$ and each vertex $z \notin \{x, y\}$, there is a $k \in \{1, 2, n\}$ such that $(\pi_k(x), \pi_{k-1}(x)) <_{lex} (\pi_k(z), \pi_{k-1}(z))$ and $(\pi_k(y), \pi_{k-1}(y)) <_{lex} (\pi_k(z), \pi_{k-1}(z))$ (when $k = 1$ or n, $k-1$ denotes n or 2, resp.).*

Lemma 7 *Let $\pi : v \in V \to (\pi_1(v), \pi_2(v), \pi_n(v))$ be a weak barycentric representation of G. Then given any noncolinear points $\mathbf{a}, \mathbf{b}, \mathbf{c}$ on the plane, the mapping $f : v \in V \to \pi_1(v)\mathbf{a} + \pi_2(v)\mathbf{b} + \pi_n(v)\mathbf{c}$ defines a straight-line embedding of G in the plane spanned by $\mathbf{a}, \mathbf{b}, \mathbf{c}$.*

Let $G = (V, E)$ be a plane graph with three exterior vertices v_1, v_2, v_n in ccw order. Let $\mathcal{R}_0 = \{T_1, T_2, T_n\}$ be the minimum realizer of G. Let $\mathcal{F} = \mathcal{F}_{3cw} \cup \mathcal{F}_{2cw}$ be the set of interior faces of $G(\mathcal{R}_0)$ that have either three or two cw edges on their boundaries. Thus $|\mathcal{F}| = \Delta_0 + \alpha_0$.

For each vertex $v \in G$, let $P_1(v)$ ($P_2(v)$ and $P_3(v)$, resp.) be the path in T_1 (T_2 and T_n, resp.) from v to the root v_1 (v_2 and v_n, resp.). $P_1(v), P_2(v), P_n(v)$ divides G into three regions $R_1(v), R_2(v)$ and $R_n(v)$, where $R_i(v)$ denotes the closed region bounded by the paths $P_j(v), P_k(v)$ and the exterior edge (v_j, v_k) $(i \neq j, k)$.

For each interior vertex v of G, let $\pi_1(v)$ ($\pi_2(v)$ and $\pi_n(v)$, resp.) be the number of faces of $G(\mathcal{R}_0)$ in $R_1(v)$ ($R_2(v)$ and $R_n(v)$, resp.) that belong to $\mathcal{F}$. Note that since the two interior faces of $G(\mathcal{R}_0)$ incident to the exterior edges (v_1, v_2) and (v_1, v_n) are not in $R_1(v)$ and these two faces belong to $\mathcal{F}$, we have $\pi_1(v) \leq \Delta_0 + \alpha_0 - 2$. Similarly, we have $\pi_2(v), \pi_n(v) \leq \Delta_0 + \alpha_0 - 2$.

For the three exterior vertices v_1, v_2 and v_n, define $\pi_i(v_i) = \Delta_0 + \alpha_0$ and $\pi_j(v_i) = 0$ for $j \neq i$. Note that for all vertices v, $\pi_1(v) + \pi_2(v) + \pi_n(v) = \Delta_0 + \alpha_0$.

Lemma 8 *If $u \in R_i(v) - P_{i+1}(v)$, then $\pi_i(u) < \pi_i(v)$ (when $i = 2$ or n, $i + 1$ denotes n or 1 resp.).*

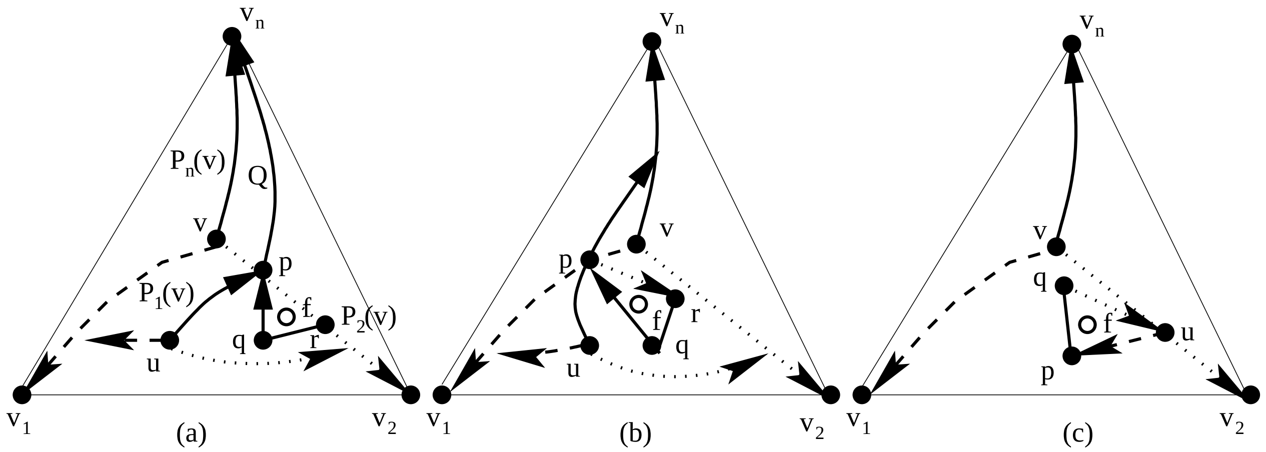


Figure 6: The proof of Lemma 8.

Proof: We only prove the case $i = n$. The other cases are symmetric. We need to show that $u \in R_n(v) - P_1(v)$ implies $\pi_n(u) < \pi_n(v)$.

Case 1: u is in the interior of the region $R_n(v)$. Let Q be the path in T_n from u to the root v_n of T_n. Q must intersect either $P_1(v)$ or $P_2(v)$ at a vertex p.

Case 1 (a): Suppose p is on $P_2(v)$. Let r be the parent of p in T_2. Let (q, p) be the edge incident to p that is next to the edge (r, p) in cw order around p. (See Fig. 6 (a).) Let f be the face of G bounded by the edges $(p, r), (r, q), (q, p)$. Note that f is in $R_n(v)$ but not in $R_n(u)$. By the property of the realizer, the edge (q, p) must be in T_n and directed toward p. The edge (p, r) is in T_2 and directed toward r. So f has at least two cw edges on its boundary, and hence is in $\mathcal{F}$. Thus, $\pi_n(u) \leq \pi_n(v) - 1 < \pi_n(v)$.

Case 1 (b): Suppose p is on $P_1(v)$ (but $p \neq v$). By the property of the realizer, there must be an edge (p, r) in T_2 directed from p to r and this edge must be in the region $R_n(v)$. (See Fig. 6 (b).) Let (q, p) be the edge incident to p that is next to the edge (r, p) in cw order around p. Let f be the face of G bounded by the edges $(p, r), (r, q), (q, p)$. Note that f is in $R_n(v)$ but not in $R_n(u)$. By the property of the realizer, the edge (q, p) must be in T_n and directed toward p. The edge (p, r) is in T_2 and directed toward r. So f has at least two cw edges on its boundary, and hence is in $\mathcal{F}$. Thus, $\pi_n(u) \leq \pi_n(v) - 1 < \pi_n(v)$.

Case 2: u is on the path $P_2(v)$. By the property of the realizer, there exists an edge (u, p) in T_1 directed toward p and this edge must be in the region $R_n(v)$. Let (u, q) be the edge incident to u that is next to the edge (u, p) in cw order around u. Let f be the face bounded by the edges $(u, p), (p, q), (q, u)$. (See Fig. 6 (c).) Note that f is in $R_n(v)$ but not in $R_n(u)$. By the property of the realizer, the edge (q, u) must be in T_2 and directed toward u. The edge (u, p) is in T_1 and directed toward p. So f has at least two cw edges on its boundary and hence is in $\mathcal{F}$. Thus, $\pi_n(u) \leq \pi_n(v) - 1 < \pi_n(v)$. □

Lemma 9 *Let u and v be two distinct vertices of G. If v is an interior vertex of G and $u \in R_i(v)$, then $(\pi_i(u), \pi_{i-1}(u)) <_{lex} (\pi_i(v), \pi_{i-1}(v))$ (when $i = 1$ or n, $i - 1$ denotes n or 2 resp.).*

Proof: We only prove the case $u \in R_n(v)$. The other cases are symmetric. We need to show $(\pi_n(u), \pi_2(u)) <_{lex} (\pi_n(v), \pi_2(v))$.

If $u \in R_n(v) - P_1(v)$, then by Lemma 8 we have $\pi_n(u) < \pi_n(v)$, which implies $(\pi_n(u), \pi_2(u)) <_{lex} (\pi_n(v), \pi_2(v))$. Otherwise $u \in P_1(v)$. This implies $u \in R_2(v) - P_n(v)$. By Lemma 8 with $i = 2$, we have: $\pi_2(u) < \pi_2(v)$. Since $R_n(u) \subset R_n(v)$, we have $\pi_n(u) \leq \pi_n(v)$. Thus $(\pi_n(u), \pi_2(u)) <_{lex} (\pi_n(v), \pi_2(v))$ as to be shown. □

Theorem 3 *The mapping $\pi : v \in V \to \frac{1}{\Delta_0+\alpha_0}(\pi_1(v), \pi_2(v), \pi_n(v))$ is a weak barycentric representation.*

Proof: The first condition of the barycentric representation is clearly satisfied, which implies the injectivity of π. We need to verify the second condition in the Definition 4. Consider an edge (x, y) of G and a vertex $z \neq x, y$. If z is an exterior vertex v_i, then $\pi_i(z) = \Delta_0 + \alpha_0 > \pi_i(x), \pi_i(y)$, which implies $(\pi_i(x), \pi_{i-1}(x)) <_{lex} (\pi_i(z), \pi_{i-1}(z))$ and $(\pi_i(y), \pi_{i-1}(y)) <_{lex} (\pi_i(z), \pi_{i-1}(z))$.

Otherwise, z is an interior vertex and $x, y \in R_i(z)$ for some $i \in \{1, 2, n\}$. By Lemma 9, we have $(\pi_i(x), \pi_{i-1}(x)) <_{lex} (\pi_i(z), \pi_{i-1}(z))$ and $(\pi_i(y), \pi_{i-1}(y)) <_{lex} (\pi_i(z), \pi_{i-1}(z))$. □

Theorem 4 *Every plane triangulation G with n vertices has a straight line grid embedding on a grid of size $(n - \Delta_0 - 1) \times (n - \Delta_0 - 1)$ where Δ_0 is the number of cyclic faces of G with respect to its minimum realizer $\mathcal{R}_0$. The embedding can be constructed in linear time.*

Proof: Pick three points $\mathbf{a} = (0, 0)$, $\mathbf{b} = (\Delta_0 + \alpha_0, 0)$ and $\mathbf{c} = (0, \Delta_0 + \alpha_0)$. By Theorem 3 and Lemma 7, the mapping $v \to (\pi_2(v), \pi_n(v))$ is a straight-line grid embedding on a grid of size $(\Delta_0 + \alpha_0) \times (\Delta_0 + \alpha_0)$. By Theorem 1 (1), this equals to the size stated in the theorem.

To construct the embedding, we need to construct the minimal realizer $\mathcal{R}_0$ of G, which can be done in linear time [2]. The calculation of the coordinates of the embedding can be done in $O(n)$ time by using the techniques given in [26]. □

As an example, Fig. 4 (d) shows a straight-line grid embedding of G on a grid of size $(n - \Delta_0 - 1) \times (n - \Delta_0 - 1)$. It is obtained by using the minimal realizer $\mathcal{R}_0$ of G in Fig. 3. For example, the coordinate of the vertex numbered 6 in Fig. 3 is $(\pi_2(6), \pi_n(6)) = (2, 1)$.

5 Canonical Ordering Tree with Fewer Leaves for 4-Connected Plane Triangulations

In order to obtain a canonical ordering tree for a 4-connected plane triangulation with fewer leaves, we need another concept and a related lemma from [11] as follows:

Definition 5 *Let G' be a plane graph with four vertices v_W, v_S, v_E, v_N in ccw order on its exterior face. A* regular edge labeling (REL for short) *of G' is a partition of the interior edges of G' into two subsets S_1, S_2 of directed edges such that the following hold:*

1. *For each interior vertex v, the edges incident to v appear in ccw order around v as follows: a set of edges in S_1 leaving v; a set of edges in S_2 entering v; a set of edges in S_1 entering v; a set of edges in S_2 leaving v. Each set is nonempty.*

2. *All interior edges incident to v_N are in S_1 and entering v_N. All interior edges incident to v_W are in S_2 and leaving v_W. All interior edges incident to v_S are in S_1 and leaving v_S. All interior edges incident to v_E are in S_2 and entering v_E. Each block is not empty.*

Lemma 10 *Let G' be a plane graph with four vertices on its exterior face. G' has a REL if and only if the following conditions hold: (1) Every interior face of G' is a triangle and the exterior face of G' is a quadrangle; (2) G' has no separating triangles.*

A graph satisfying the two conditions in the above lemma will be called a *proper triangulated plane* (PTP for short) graph.

Let G be a 4-connected plane triangulation with three exterior vertices v_W, v_E, v_N in ccw order. Delete the edge (v_W, v_E). Denote the new exterior vertices by v_S and the resulting plane graph by G'.

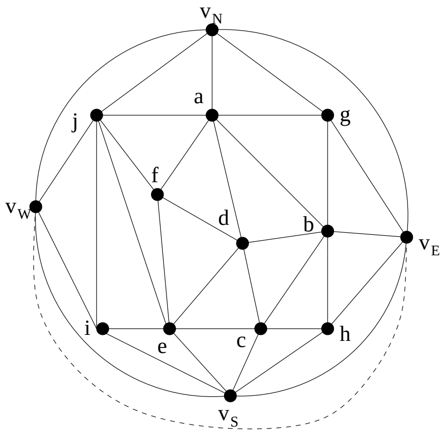
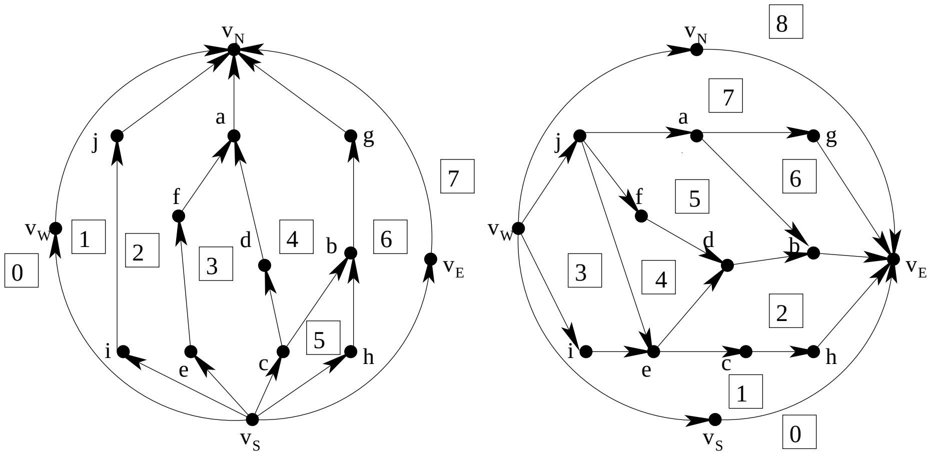


Figure 7: A 4-connected plane triangulation G and the PTP graph G' after deleting (v_W, v_E).



(a) S−N Net G_1 (b) W−E Net G_2

Figure 8: The S-N net G_1 and the W-E net G_2 for the PTP graph G' in Fig. 7.

(See Fig. 7 for an example.) G' does not have separating triangles, and it has four exterior vertices v_W, v_S, v_E, v_N in ccw order on its own exterior face. Thus, G' is a PTP graph and has a REL (S_1, S_2) according to Lemma 10.

We investigate the properties of the REL of G'. Denote by G_1 the directed subgraph of G' induced by S_1 and the four exterior edges directed as $v_S \to v_W, v_W \to v_N, v_S \to v_E, v_E \to v_N$. Let E_1 be the edge set of G_1. (E_1 is the union of S_1 and the four exterior edges.) Then G_1 is an st-plane graph with source v_S and sink v_N. Similarly, let G_2 be the directed subgraph of G' induced by S_2 and the four exterior edges directed as $v_W \to v_S, v_S \to v_E, v_W \to v_N, v_N \to v_E$. Let E_2 be the edge set of G_2. Then G_2 is an st-plane graph with source v_W and sink v_E. We will call G_1 the S-N net and G_2 the W-E net of G' derived from the REL (S_1, S_2). For example, Fig. 8 shows a REL with its derived S-N and W-E nets for the PTP graph G' shown in Fig. 7. (Ignore the small boxes containing integers for now. Their meaning will be explained later.)

Consider the S-N net G_1. For each edge $e \in E_1$, let $left(e)$ ($right(e)$, resp.) denote the face of G_1 on the left (right, resp.) of e. Define the dual graph of G_1, denoted by G_1^* as follows. The node set of G_1^* is the set of the interior faces of G_1 plus two exterior faces f_W and f_E, (which are obtained by "splitting" the exterior face of G_1). For each edge $e \in E_1$, there is a corresponding edge e^* in G_1^* directed from the face $left(e)$ to the face $right(e)$. G_1 is an st-plane graph, so G_1^* is also an st-plane graph with f_W as the only source and f_E as the only sink [20, 24]. For each face f in G_1, define the *upper left (upper right, resp.)* edge of f to be the last edge of the left (right, resp.) path of f, the *lower left (lower right, resp.)* edge of f to be the first edge of the left (right, resp.) path of f. Note that these four edges are distinct. (This is because (S_1, S_2) is a REL, so each of the two directed paths on the boundary of f contains at least two edges.)

Similarly, we can define the dual G_2^* for G_2 with the word "left" ("right", resp.) replaced by the word "below" ("above", resp.). G_2^* is also an st-plane graph with source f_S and sink f_N. For each face f in G_2, we similarly define the *left above, left below, right above*, and *right below* edge of f. These four edges are also distinct.

For $i = 1, 2$, assume G_i has k_i faces. Thus G_i^* has $k_i + 1$ nodes. By using topological sort, we can assign a distinct number from $0, 1, \cdots, k_1$ to each node of G_1^* such that, for any edge e in E_1, the number assigned to the face $left(e)$ is smaller than the number assigned to the face $right(e)$. The faces f_W and f_E are numbered by 0 and k_1, respectively. Similarly, we can assign a distinct number from $0, 1, \cdots, k_2$ to each node of G_2^* such that, for any edge e in E_2, the number assigned to the face $below(e)$ is smaller than the number assigned to the face $above(e)$. Such an assignment is called a *consistent numbering* of $G_i, i = 1, 2$ [11]. They can be computed in linear time by using topological sort.

Fix a consistent numbering of G_1. Let f_r $(0 \leq r \leq k_1)$ denote the face numbered by r. We define *the i-th S-N separation path SN_i* to be the directed path in G_1 such that the faces numbered by $0, 1, \cdots, i-1$ are on its left and the other faces are on its right. (These paths were called path system in [11].) G_1 has exactly k_1 S-N separation paths and all of them are directed from v_S to v_N. Note that SN_{r+1} is obtained from SN_r by deleting the left path of the face f_r and adding the right path of f_r.

The *j-th W-E separation path WE_j* of G_2 is defined similarly. G_2 has exactly k_2 W-E separation paths and all of them are directed paths from v_W to v_E [11].

For example, the faces in Fig. 8 are consistently numbered for G_1 and G_2 (indicated by the integers in small boxes). The 4-th W-E separation path WE_4 in G_2 consists of the vertices v_W, j, e, d, b, v_E. Note that there are 4 vertices (i, c, h, v_S) below it and 4 vertices (f, a, g, v_N) above it.

In order to prove our main theorem, we need the following technical lemma:

Lemma 11 *Let G' be a PTP graph with n vertices. Let G_1 be the S-N net and G_2 be the W-E net derived from a REL (S_1, S_2) of G'. Suppose G_1 has k_1 faces and G_2 has k_2 faces. Then:*

(1) $k_1 + k_2 = n + 1$.

(2) The i-th S-N separation path SN_i has at least $i - 1$ vertices on its left and at least $k_1 - i$ vertices on its right in G_1. The j-th W-E separation path WE_j has at least $j - 1$ vertices below it and at least $k_2 - j$ vertices above it in G_2.

Proof: (1) We apply Euler's formula to the graph G_1 and G_2. Because G_1, G_2 have the same vertex set as G, we have $|V| + k_1 - 2 = |E_1|$ and $|V| + k_2 - 2 = |E_2|$. Adding these two identities, we have: $2|V| + k_1 + k_2 - 4 = |E_1| + |E_2|$. Since E_1 and E_2 double use four exterior edges, we have $|E_1| + |E_2| - 4 = 3n - 7$. So $k_1 + k_2 = n + 1$.

(2) We prove that there are at least $i - 1$ vertices on the left of SN_i. The statement that there are at least $k_1 - i$ vertices on the right of SN_i can be proved by symmetry.

Denote the set of the vertices on the left of SN_i by S_i. We show that $|S_i| \geq i - 1$ by induction. When $i = 1$, it is trivially true. Let's assume it is true when $i = r$, which means $|S_r| \geq r - 1$. Consider S_{r+1}. The only face of G_1 enclosed by paths SN_r and SN_{r+1} is the face f_r numbered by r. The left path P of the face f_r is on SN_r, but not on SN_{r+1}. P contains at least one internal vertex v that is neither the source nor the sink of f_r. Since v is on SN_r, it is not on the left of SN_r. So $v \notin S_r$. However, $v \in S_{r+1}$. Thus S_{r+1} contains S_r plus at least one extra vertex v. By induction hypotheses, $|S_r| \geq r - 1$. So $|S_{r+1}| \geq r$. This completes the induction.

The statement for the W-E separation paths can be proved similarly. □

For example, in Fig. 8 (a), when we expand from S_3 to S_4, the left path P of the face numbered 3 is the path (v_S, e, f, a). The internal vertices e, f of P are not in S_3, but they are in S_4.

Next, we show how to get a canonical ordering tree from the S-N net and the W-E net respectively. We prove that one of the two canonical ordering trees has at most $\lceil \frac{n+1}{2} \rceil$ leaves.

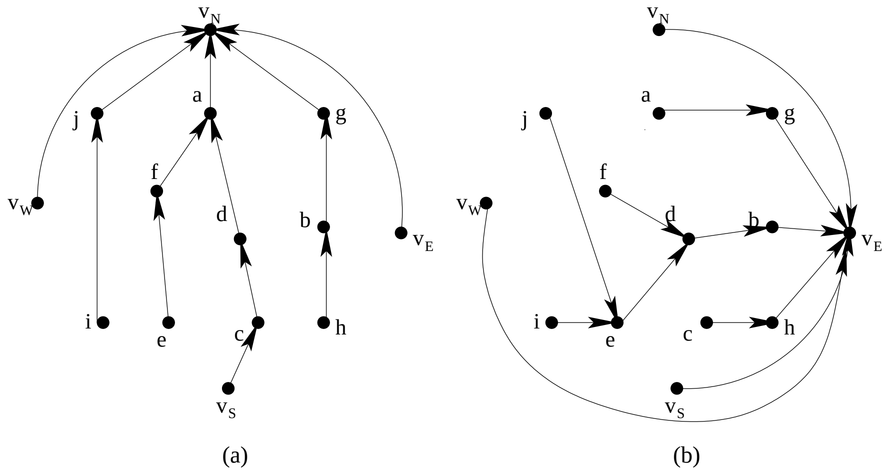


Figure 9: (a) A canonical ordering tree of G from the S-N net in Fig. 8 (a), (b) A canonical ordering tree of G from the W-E net in Fig. 8 (b).

Theorem 5 *Let G be a 4-connected plane triangulation with three exterior vertices v_W, v_E, v_N in ccw order. Delete the edge (v_W, v_E). Denote the new exterior vertex by v_S and the resulting graph by G'. Let (S_1, S_2) be a REL of G'. Let G_1 be the S-N net and G_2 be the W-E net derived from (S_1, S_2). Then the following statements hold:*

1. *For each vertex $v \neq v_N, v_S$ in G_1, select an outgoing edge in G_1. For v_S, select an outgoing edge not leading to v_W or v_E. Then the set T_1 of the selected edges is a canonical ordering tree of G.*
2. *For each vertex $v \neq v_W, v_E$ in G_2, select an outgoing edge in G_2. For v_W, select the edge (v_W, v_E). Then the set T_2 of the selected edges is a canonical ordering tree of G.*
3. *One of T_1, T_2 has at most $\lceil \frac{n+1}{2} \rceil$ leaves. It can be obtained in linear time.*

Proof: Fig. 9 (a) shows the tree described in the statement (1) for the S-N net in Fig. 8 (a). Fig. 9 (b) shows the tree described in the statement (2) for the W-E net in Fig. 8 (b).

(1) Since G_1 is an acyclic digraph, T_1 is acyclic and each vertex, except v_N, has exactly one outgoing edge in T_1. So T_1 is a directed tree rooted at v_N.

Claim: all edges in $G - T_1$ are unrelated with respect to T_1.

Suppose, on the contrary, there exists an edge $e = (v_j, v_i)$ in $G - T_1$ that is related with respect to T_1. Without loss of generality, suppose v_i is an ancestor of v_j in T_1. Let P be the unique path in T_1 from v_j to v_i. Because the edge set E of G is the union of three disjoint edge subsets S_1, S_2 and $\{(v_W, v_S), (v_S, v_E), (v_W, v_N), (v_E, v_N), (v_W, v_E)\}$, there are three possible cases.

Case 1: e is in S_1. Because there is no directed cycle in G_1, e is directed from v_j to v_i in G_1. Let R be the region in G_1 enclosed by the path P and the edge e. Let f be the face of G_1 that is inside R and contains e on its boundary. Then v_j and v_i are the source and the sink of f respectively. Thus f has one side consisting of only one edge $e = (v_j, v_i)$. This is impossible.

Case 2: e is in S_2. Then e is inside an interior face f of G_1. So the path P is on one side of f. Without loss of generality, assume P is on the left side of f. If v_i is the sink of f, then there is another path P' on the right side of f ending at v_i. Thus, $e \in S_2$ is sandwiched between two incoming edges of G_1 incident to v_i. This is impossible. Similarly, v_j cannot be the source of f. Thus v_j and v_i are internal vertices of the left path of f. From the definition of REL, the edge $e = (v_j, v_i)$ must be directed as outgoing from v_j and outgoing from v_i in G_2. This is impossible.

Case 3: e is in $\{(v_W, v_S), (v_S, v_E), (v_W, v_N), (v_E, v_N), (v_W, v_E)\}$. (v_W, v_N) and (v_E, v_N) are in T_1, so $e \neq (v_W, v_N)$ and $e \neq (v_E, v_N)$. Thus e has to be one of (v_W, v_E), (v_W, v_S) or (v_S, v_E). Both ending vertices of these three edges are leaves of T_1, none of them can be an ancestor of the other. So e cannot be related with respect to T_1. Thus this case is also impossible.

This completes the proof of the claim. It remains to show that every vertex v of G is orderly with respect to T_1.

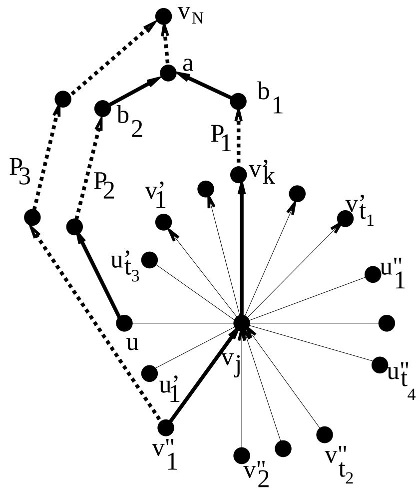


Figure 10: The vertex v_j is orderly in tree T_1

Let $v_1, v_2, \cdots, v_n$ be the ccw preordering of the vertices of G with respect to T_1. Obviously, v_N, v_W, v_E are numbered $1, 2, n$ respectively (see Fig. 9 (a)). It's easy to verify that they are orderly with respect to T_1.

Consider any other vertex v_j, $j \neq 1, 2, n$ in T_1. Let $(v_j, v'_1), \cdots, (v_j, v'_{t_1})$ be the outgoing edges from v_j in cw order in S_1; $(v''_1, v_j), (v''_2, v_j), \cdots, (v''_{t_2}, v_j)$ the incoming edges to v_j in ccw order in S_1; $(u'_1, v_j), \cdots, (u'_{t_3}, v_j)$ the incoming edges to v_j in cw order in S_2; and $(v_j, u''_1), \cdots, (v_j, u''_{t_4})$ the outgoing edges from v_j in cw order in S_2. Suppose the edge (v_j, v'_k) is selected into T_1. (see Fig. 10 for an illustration. The edges in S_1 are arrowed. The edges in S_2 are not arrowed.)

First, observe that each vertex v''_p $(2 \leq p \leq t_2 - 1)$ has only one outgoing edge (v''_p, v_j) in S_1. Hence all edges (v''_p, v_j) $(2 \leq p \leq t_2 - 1)$ must be selected and included in T_1.

If (v''_1, v_j) is in T_1, let $B_2(v_j) = \{u'_1, \cdots, u'_{t_3}, v'_1, \cdots, v'_{k-1}\}$. Otherwise, if (v''_1, v_j) is not in T_1, let $B_2(v_j) = \{v''_1, u'_1, \cdots, u'_{t_3}, v'_1, \cdots, v'_{k-1}\}$. $B_2(v_j)$ is a consecutive block of neighbors of v_j. We want to show that the ccw preordering number of every vertex in $B_2(v_j)$ is smaller than j. Let P_1 be the unique path in T_1 connecting v_j to the root v_N. Consider a vertex u in $B_2(v_j)$. Let P_2 be the unique path in T_1 from u to v_N. Let a be the lowest common vertex of P_1 and P_2. Let b_1 and b_2 be the children of a that are on P_1 and P_2, respectively. Starting at v''_1, we trace an unique path P_3 in G_1 as follows: for each vertex v we travel to its leftmost outgoing edge in G_1. Continue this until reaching v_N.

Let R be the closed region enclosed by the edge (v''_1, v_j), the paths P_1 and P_3. Since the vertex u is inside R, P_2 is inside R. Thus P_2 is on the left of P_1 and on the right of P_3. So the edge (a, b_2) appears before the edge (a, b_1) in ccw order around a. This implies that the ccw preordering number of u with respect to T_1 is smaller than j as to be shown.

Similarly, we can define the vertex set $B_4(v_j)$ on the other side of v_j, and prove that the ccw preordering number of every vertex in $B_4(v_j)$ is bigger than j. Thus, v_j is orderly with respect to T_1. Since this is true for any vertex v_j, T_1 is an orderly spanning tree of G, hence also a canonical ordering tree of G by Lemma 2 (1).

(2) The proof is similar to (1).

(3) Assume G_1 has k_1 faces and G_2 has k_2 faces. The canonical ordering tree T_1 can also be obtained by deleting edges from G_1. Because deleting one edge destroys one face of G_1 and T_1 is connected, the number of edges that must be deleted from G_1 to make T_1 is exactly the number of faces of G_1 minus 1, namely $k_1 - 1$. For each leaf $v \neq v_S$ of T_1, all (at least one) incoming edges to v in G_1 are deleted. So T_1 has at most $k_1 - 1$ leaves other than v_S. Adding the leaf v_S, the number of leaves of T_1 is at most k_1.

Similar argument also apply to the canonical ordering tree T_2. But since we add the edge (v_W, v_E) into T_2 which results in one more leaf, the number of leaves of T_2 is at most the number of faces of G_2 plus 1, namely k_2+1. The sum of the numbers of the leaves in T_1 and T_2 is thus at most $k_1+k_2+1 = n+2$ by Lemma 11 (1). Hence one of T_1 and T_2 has at most $\lfloor \frac{n+2}{2} \rfloor = \lceil \frac{n+1}{2} \rceil$ leaves.

A REL of G' can be obtained in $O(n)$ time [18]. Obviously a canonical ordering tree T with at most $\lceil \frac{n+1}{2} \rceil$ leaves can be obtained in $O(n)$ time from a REL by the construction above. □

6 Compact Visibility Representation of 4-Connected Plane Graphs

In this section, we prove the following:

Theorem 6 *Every 4-connected plane graph G with n vertices has a VR with height at most $\lceil \frac{3n}{4} \rceil$, which can be obtained in linear time.*

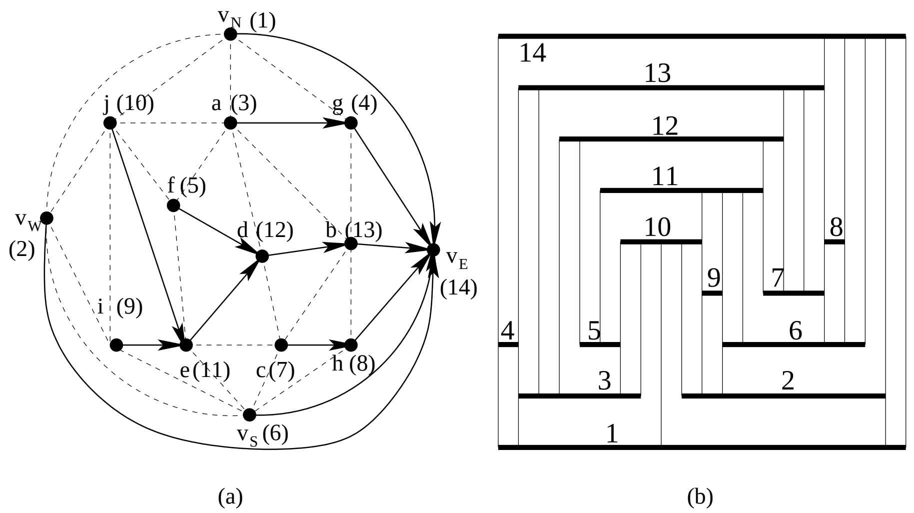


Figure 11: An st-numbering and its corresponding VR of the graph G in Fig. 7.

Proof: Without loss of generality, assume G is a 4-connected plane triangulation with three exterior vertices v_W, v_E, v_N in ccw order. Delete the edge (v_W, v_E) and denote the new exterior vertex by v_S and the resulting PTP graph by G'. Let (S_1, S_2) be a REL of G'. Let G_1, G_2 be the S-N and W-E nets derived from (S_1, S_2). Assume G_1 has k_1 faces and G_2 has k_2 faces. By Lemma 11 (1), $k_1 + k_2 = n + 1$. Assign consistent numbers to faces of G_1 and G_2. There are two cases to consider:

Case 1: $k_2 \geq \frac{n+1}{2}$. We use G_2 to construct a canonical ordering tree of G. Let P be the $\lfloor \frac{k_2+1}{2} \rfloor$th W-E separation path. For the vertex v_W, select the edge (v_W, v_E). For each vertex on P other than v_W, select its outgoing edge in P. For any other vertex v, select one outgoing edge from v in G_2. The set of selected edges is an orderly spanning tree T of G by Theorem 5 (2) and Lemma 2 (1). With respect to the ccw preordering of T, every interior vertex v has nonempty neighbor sets $B_2(v)$ and $B_4(v)$.

Next, we construct an st-numbering of G using T. We preassign 2 to v_W. Then we assign numbers $1, 3, \cdots, n$ to other vertices of G in the **ccw postordering** of T, starting from the vertex v_N until we reach the vertices on P. Do not assign numbers to vertices on P at this moment. Then we jump to the vertex v_S and assign the remaining numbers in the **cw postordering** of T, starting from v_S, until we reach the vertices on P. Keep assigning the remaining numbers to the vertices on P until reaching the root v_E, which is assigned n.

Fig. 11 (a) shows an example. We choose the W-E net G_2 in Fig. 8 (b) and the 4-th W-E separation path (v_W, j, e, d, b, v_E). The edges of the chosen canonical ordering tree are drawn in solid lines. Other edges are drawn in dashed lines. The resulting st-numbering is shown in integers in the parentheses.

Let R_1 denote the closed region enclosed by P and the edge (v_W, v_E). Let R_2 denote the closed region enclosed by P and the path (v_W, v_N, v_E).

We now show that this numbering is indeed an st-numbering of G. For any vertex v other than the root v_E, its parent in T is a bigger neighbor of v. Thus we only need to show that each vertex $v \neq v_N$ has at least one smaller neighbor. There are several cases.

(1) $v = v_W$: v_N is a smaller neighbor of v.

(2) $v = v_S$: v_W is a smaller neighbor of v.

(3) $v = v_E$: Every neighbor is a smaller neighbor of v.

(4) v is inside R_2: v is an interior vertex of G. Thus all the vertices in the nonempty set $B_2(v)$ are assigned smaller numbers than v.

(5) $v \neq v_S$ is inside R_1: v is an interior vertex of G. Thus all the vertices in the nonempty set $B_4(v)$ are assigned smaller numbers than v.

(6) v is an internal vertex of the path P: All the vertices in the nonempty sets $B_2(v)$ and $B_4(v)$ are assigned smaller numbers than v.

Therefore this numbering is indeed an st-numbering of G.

Consider any directed path P' in the resulting st-orientation. If P' passes through a vertex u inside R_1 and a vertex v inside R_2, then P' has to intersect the path P at a vertex v', which should be numbered between u and v. However, for any vertex v' on P, either $v' = v_W$, then it is numbered smaller than both u and v, or $v' \neq v_W$, then it is numbered bigger than both u and v. So this is impossible. Thus, P' has to bypass either all vertices inside R_1, or all vertices inside R_2. Therefore it has to bypass at least $\lfloor \frac{n}{4} \rfloor$ vertices by Lemma 11 (2). So the length of any directed path is at most $\lceil \frac{3n}{4} \rceil$.

Applying the VR algorithm in Lemma 1 to this st-numbering gives a VR of G with height at most $\lceil \frac{3n}{4} \rceil$. It can be obtained in $O(n)$ time.

Fig. 11 (b) shows a VR of the graph G using the st-numbering in Fig. 11 (a).

Case 2: $k_1 \geq \frac{n+1}{2}$. It is similar to Case 1, except that, instead of using G_2, we use G_1 to construct the required st-numbering of G. □

7 Conclusion and Open Problems

In this paper, we presented a new method to construct an st-numbering of a plane triangulation G by using the minimum realizer of G. This guarantees that the length of the longest path in the resulting

st-orientation is at most $\lceil \frac{15n}{16} \rceil$. This results a VR of G with height at most $\lceil \frac{15n}{16} \rceil$. It deserves further study whether this bound can be reduced.

One interesting aspect is that our method prefers a canonical ordering tree with more leaves, which is to the contrary of most applications of the canonical ordering trees. We proved that every plane triangulation has a canonical ordering tree with at least $\lceil \frac{n+1}{2} \rceil$ leaves. It deserves further study to improve this bound.

We also gave a more compact straight-line grid embedding of plane graphs, also by using the minimum realizer. It is interesting to see if the bound can be further reduced.

We obtained a canonical ordering tree for a 4-connected plane triangulation with at most $\lceil \frac{n+1}{2} \rceil$ leaves. An interesting problem deserves further consideration is whether there is a 4-connected plane triangulation G such that every canonical ordering tree of G has at least this many leaves.

We also showed that every 4-connected plane graph has a VR with height at most $\lceil \frac{3n}{4} \rceil$. It deserves further study to improve this bound.

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