Area-Efficient Drawings of Outerplanar Graphs*

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Abstract. It is well-known that a planar graph with \( n \) nodes admits a planar straight-line grid drawing with \( O(n^2) \) area [3, 8], and in the worst case it requires \( \Omega(n^2) \) area. It is also known that a binary tree with \( n \) nodes admits a planar straight-line grid drawing with \( O(n) \) area [6]. Thus, there is wide gap between the \( \Theta(n^2) \) area-requirement of general planar graphs and the \( \Theta(n) \) area-requirement of binary trees. It is therefore important to investigate special categories of planar graphs to determine if they can be drawn in \( o(n^2) \) area.

Outerplanar graphs form an important category of planar graphs. We investigate the area-requirement of planar straight-line grid drawings of outerplanar graphs. Currently the best known bound on the area-requirement of such a drawing of an outerplanar graph with \( n \) vertices is \( O(n^2) \), which is that same as for general planar graphs. Hence, a fundamental question arises that can be draw an outerplanar graph in this fashion in \( o(n^2) \) area?

In this paper, we provide a partial answer to this question by proving that an outerplanar graph with \( n \) vertices and degree \( d \) can be drawn in this fashion in area \( O(dn^{1.48}) \) in \( O(n \log n) \) time. This implies that an outerplanar graph with \( n \) vertices and degree \( d \), where \( d = o(n^{0.52}) \), can be drawn in this fashion in \( o(n^2) \) area.

From a broader perspective, our contribution is in showing a sufficiently large natural category of planar graphs that can be drawn in \( o(n^2) \) area.

1 Introduction

A drawing \( \Gamma \) of a graph \( G \) maps each vertex of \( G \) to a distinct point in the plane, and each edge \( (u,v) \) of \( G \) to a simple Jordan curve with endpoints \( u \) and \( v \). \( \Gamma \) is a straight-line drawing, if each edge is drawn as a single line-segment. \( \Gamma \) is a polyline drawing, if each edge is drawn as a connected sequence of one or more line-segments, where the meeting point of consecutive line-segments is called a bend. \( \Gamma \) is a grid drawing if all the nodes have integer coordinates. \( \Gamma \) is a planar drawing, if edges do not intersect each other in the drawing. In this paper, we concentrate on grid drawings. So, we will assume that the plane is covered by a

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rectangular grid. Let \( \Gamma \) be a grid drawing. Let \( R \) be the smallest rectangle with sides parallel to the \( X \)-and \( Y \)-axes, respectively, that covers \( \Gamma \) completely. The width (height) of \( \Gamma \) is equal to \( 1 + \) width of \( R \) (1 + height of \( R \)). The area of \( \Gamma \) is equal to \( (1 + \text{width of } R) \cdot (1 + \text{height of } R) \), which is equal to the number of grid points contained within \( R \). The degree of a graph is equal to the maximum number of edges incident on a vertex.

It is well-known that a planar graph with \( n \) vertices admits a planar straight-line grid drawing with \( O(n^2) \) area \([3, 8]\), and in the worst case it requires \( \Omega(n^2) \) area. It is also known that a binary tree with \( n \) nodes admits a planar straight-line grid drawing with \( O(n) \) area \([6]\). Thus, there is wide gap between the \( \Theta(n^2) \) area-requirement of general planar graphs and the \( \Theta(n) \) area-requirement of binary trees. It is therefore important to investigate special categories of planar graphs to determine if they can be drawn in \( o(n^2) \) area.

Outerplanar graphs form an important category of planar graphs. We investigate the area-requirement of planar straight-line grid drawings of outerplanar graphs. Currently the best known bound on the area-requirement of such a drawing of an outerplanar graph with \( n \) vertices is \( O(n^2) \), which is that same as for general planar graphs. Hence, a fundamental question arises: can we draw an outerplanar graph in this fashion in \( o(n^2) \) area?

In this paper, we provide a partial answer to this question by proving that an outerplanar graph with \( n \) vertices and degree \( d \) can be drawn in this fashion in area \( O(dn^{1+\delta}) = O(dn^{1.48}) \) in \( O(n) \) time. This implies that an outerplanar graph with \( n \) vertices and degree \( O(n^\delta) \), where \( 0 \leq \delta < 0.52 \) is a constant, can be drawn in this fashion in \( o(n^2) \) area.

From a broader perspective, our contribution is in showing a sufficiently large natural category of planar graphs that can be drawn in \( o(n^2) \) area.

In Section 4, we present our drawing algorithm. This algorithm is based on a tree-drawing algorithm of \([2]\). The connection between the two algorithms comes from the fact that the dual of a maximal outerplanar graph is a tree.

## 2 Previous Results

There has been little work done on planar straight-line grid drawings of outerplanar graphs. Let \( G \) be an outerplanar graph with \( n \) vertices. Currently the best known bound on the area-requirement of such a drawing of an outerplanar graph with \( n \) vertices is \( O(n^2) \), which is that same as for general planar graphs. However, in 3D, we can construct a crossings-free straight-line grid drawing of \( G \) with \( O(n) \) volume \([4, 5]\).

\([1]\) shows that \( G \) admits a planar polyline drawing as well as a visibility representation with \( O(n \log n) \) area. \([7]\) shows that \( G \) admits a planar polyline drawing with \( O(n) \) area, if \( G \) has degree 4. The technique of \([7]\) can be easily extended to construct a planar polyline drawing of \( G \) with \( O(d^2n) \) area, if \( G \) has degree \( d \) \([1]\).
3 Preliminaries

We assume a 2-dimensional Cartesian space. We assume that this space is covered by an infinite rectangular grid, consisting of horizontal and vertical channels.

We denote by $|G|$, the number of vertices (nodes) in a graph (tree) $G$. A rooted tree is one with a pre-specified root. An ordered tree is a rooted tree with a pre-specified left-to-right order of the children for each node. Let $T$ be an ordered binary tree with $n$ nodes. Let $p$ and $\delta$ be two constants such that $p = 0.48$ and $0 < \delta \leq 0.0004$. A spine $S$ of $T$ is a path $v_0, v_1, v_2, \ldots, v_m$, where $v_0, v_1, v_2, \ldots, v_m$ are nodes of $T$, that is defined recursively as follows (as defined in the proof of Lemma A.1 in [2]):

- $v_0$ is the same as the root of $T$, and $v_m$ is a leaf of $T$;
- let $\alpha_i$ and $\beta_i$ be the the left and right subtrees with the maximum number of nodes among the subtrees that are rooted at any of the nodes in the path $v_0, v_1, \ldots, v_i$; let $L_i$ and $R_i$ be the subtrees rooted at the left and right children of $v_i$, respectively. Then,
  - if $|\alpha_i|^p + |R_i|^p \leq (1-\delta)n^p$ and $|L_i|^p + |\beta_i|^p > (1-\delta)n^p$, set $v_{i+1}$ to be the left child of $v_i$,
  - if $|\alpha_i|^p + |R_i|^p > (1-\delta)n^p$ and $|L_i|^p + |\beta_i|^p \leq (1-\delta)n^p$, set $v_{i+1}$ to be the right child of $v_i$,
  - if $|\alpha_i|^p + |R_i|^p \leq (1-\delta)n^p$ and $|L_i|^p + |\beta_i|^p \leq (1-\delta)n^p$, we terminate the construction as follows:
    - if $|L_i| \leq |R_i|$, set the spine to be the concatenation of path $v_0, v_1, \ldots, v_i$ and the leftmost path from $v_i$ to a leaf $v_m$,
    - otherwise (i.e. $|L_i| > |R_i|$), set the spine to be the concatenation of the path $v_0, v_1, \ldots, v_i$ and the rightmost path from $v_i$ to a leaf $v_m$.
- in [2] it is shown that the case $|\alpha_i|^p + |R_i|^p > (1-\delta)n^p$ and $|L_i|^p + |\beta_i|^p > (1-\delta)n^p$ is not possible.

$v_0, v_1, \ldots, v_m$ are called spine nodes. A subtree $T'$ of $T$ is a subtree of $S$, if it is rooted at the non-spine child $c$ of a spine node $v_i$; $T'$ is a left (right) subtree of $S$, if $c$ is the left (right) child of $v_i$.

We will use Lemma A.1 of [2], which is given below:

**Lemma 1 (Lemma A.1 of [2]).** Let $p = 0.48$. For any left subtree $\alpha$ and right subtree $\beta$ of a spine, $|\alpha|^p + |\beta|^p \leq (1-\delta)n^p$, for any constant $\delta$, $0 < \delta \leq 0.0004$.

An outerplanar graph is a planar graph for which there exists an embedding with all vertices on the exterior face. Throughout this paper, by the term outerplanar graph we will mean a maximal outerplanar graph, i.e., an outerplanar graph to which no edge can be added without destroying its outerplanarity. It is easy to see that each internal face of a maximal outerplanar graph is a triangle. Two vertices of a graph are neighbors, if they are connected by an edge. The dual tree $T_G$ of an outerplanar graph $G$ is defined as follows:

- there is a one-to-one correspondence between the nodes of $T_G$ and the internal faces of $G$, and
there is an edge $e = (u, v)$ in $T_G$ if and only if the faces of $G$ corresponding to $u$ and $v$ share an edge $e'$ on their boundaries. $e$ and $e'$ are duals of each other.

For example, Figure 1(b), shows the dual tree of the outerplanar graph of Figure 1(a).

Let $P = v_0 v_1 \ldots v_q$ be a path of $T_G$. Let $H$ be the subgraph of $G$ corresponding to $P$. A beam drawing of $H$ is shown in Figure 2, where the vertices of $H$ are placed on two horizontal channels, and the faces of $H$ are drawn as triangles.

A line-segment with end-points $a$ and $b$ is a flat line-segment if $a$ and $b$ are grid points, and either belong to the same horizontal channel, or belong to adjacent horizontal channels.

Let $B$ be a flat line-segment with end-points $a$ and $b$, such that $b$ is at least one unit to the right of $a$. Let $G$ be an outerplanar graph with two distinguished adjacent vertices $u$ and $v$, such that the edge $(u, v)$ is on the external face of $G$; $u$ and $v$ are called the poles of $G$. Let $D$ be a planar straight-line drawing of $G$. $D$ is a feasible drawing of $G$ with base $B$ if:

- the two poles of $G$ are mapped to $a$ and $b$ each,
- each non-pole vertex of $G$ is placed at least one unit above the lower of $a$ and $b$, and is placed at least one unit to the right of $a$ and at least one unit to the left of $b$. 

![Diagram](image)
4 Outerplanar Graph Drawing Algorithm

The drawing algorithm, which we call *Algorithm OpDraw*, is recursive in nature. In each recursive step, it takes as input an outerplanar graph $G$ with pre-specified poles, and a long-enough flat line-segment $B$, and constructs a feasible drawing $D$ of $G$ with base $B$ by constructing a drawing $M$ of the subgraph $Z$ corresponding to a spine of $T_G$, splitting $G$ into several smaller outerplanar graphs after removing $Z$ and some other vertices from it, constructing feasible drawings of each smaller outerplanar graph, and then combining their drawings with $M$ to obtain $D$.

We now give the details of the actions performed by *Algorithm OpDraw* in each recursive step (see Figure 3) (a):

- Let $u$ and $v$ be the poles of $G$. Let $T_G$ be the dual tree of $G$. Let $r$ be the node of $T_G$ that corresponds to the internal face $F$ of $G$ that contains both $u$ and $v$. Convert $T_G$ into an ordered tree as follows:
  - make $T_G$ a rooted tree by making $r$ its root,
  - and for each node $w$, let $w'$ be the parent of $w$ in $T_G$ (which now is a rooted tree). Let $c$ ($d$) be the children of $w$ such that the face corresponding to $c$ immediately follows (precedes) the face corresponding to $w'$ in the counter-clockwise order of internal faces incident on the face corresponding to $w$. Make $c$ the leftmost child of $w$, and $d$ the rightmost child of $w$. Assign the children of $w$ the same left-to-right order as the counter-clockwise order in which the faces that correspond to them are incident on the face corresponding to $w$.

Note that $T_G$ is a binary tree because each internal face of $G$ is a triangle.
- Draw $F$ as a triangle such that $u$ and $v$ coincide with the end-points of $B$, and the third vertex $w$ of $F$ is placed one unit above the lower of $u$ and $v$.
Fig. 3. The drawing of the outerplanar graph of Figure 1(a) constructed by Algorithm OpDraw: (a) When $v$ is one unit above $u$, (b) when $u$ and $v$ are in the same horizontal channel, and (c) when $u$ is one unit above $v$. 
(We will determine later on the horizontal distances of \( w \) from \( u \) and \( v \), when we analyze the area-requirement of the drawing.) In the rest of this section, we will assume that \( v \) is placed one unit above \( u \). (The cases, where \( u \) and \( v \) are in the same horizontal channel, and where \( u \) is placed one unit above \( v \) are similar, and are shown in Figures 3(b) and 3(c), respectively).

- Let \( P = v_0 v_1 \ldots v_q \) be the spine of \( G \), where \( v_0 = r \). Assume that the edge \( (v_0, v_1) \) is the dual of edge \((u, w)\) (the case where \( (v_0, v_1) \) is the dual of edge \((u, w)\) is symmetrical). Let \((v_0, v')\) be the dual of edge \((u, w)\). Let \( H \) be the subgraph of \( G \) corresponding to the subtree of \( T_G \) rooted at \( v' \). Recursively construct a feasible drawing \( D_H \) of \( H \) with \( uv \) as the base.

- Let \( c_0 = w, c_1, \ldots, c_m = (c_0, c_1, c_2, \ldots, c_s \) be the clockwise order of the neighbors of \( v \) different from \( u \), where, for each \( i \) \((1 \leq i \leq m) \), the face \( c_{i-1} c_{i} c'_{i} \) corresponds to the spine node \( v_i \), and for each \( i \) \((1 \leq i \leq s) \), the face \( c_{i-1} c_{i} c'_{i} \) corresponds to a non-spine node \( v_i' \) of \( T_G \). (In Figure 3(a), \( m = 3 \), and \( s = 2 \).) Place the vertices \( c_1, \ldots, c_m = (c_0, c_1, c_2, \ldots, c_s \) in the same horizontal channel one unit above \( w \). (We will determine later on the horizontal distances between these vertices.)

- Let \((v_i, x_i)\) be the dual of edge \((c_{i-1}, c_i)\). Let \( K_i \) be the subgraph of \( G \) corresponding to the subtree of \( T_G \) rooted at \( x_i \). For each \( i \), where \( 1 \leq i \leq m - 1 \), recursively construct a feasible drawing of \( K_i \) with \( c_{i-1} c_i \) as the base.

- Let \((v_i', x_i')\) be the dual of edge \((c_{i-1}', c_i')\). Let \( K_i' \) be the subgraph of \( G \) corresponding to the subtree of \( T_G \) rooted at \( x_i \). For each \( i \), where \( 1 \leq i \leq s \), recursively construct a feasible drawing \( D_i' \) of \( K_i' \) with \( c_{i-1}' c_i' \) as the base.

- Let \( \alpha_0, \alpha_1, \ldots, \alpha_t \) be the vertices of \( K_m \), such that \( \alpha_0, \alpha_1, \ldots, \alpha_h \) \((0 \leq h \leq t) \) is the clockwise order of the neighbors of \( c_{m-1} \) in \( K_m \), and \( \alpha_h, \alpha_{h+1}, \ldots, \alpha_t \) is the clockwise order of the neighbors of \( c_m \) in \( K_m \). For example, in Figure 3(a), \( h = 4 \), and \( t = 5 \). Let \( j \) be the index such that the dual of edge \( (\alpha_{j-1}, \alpha_j) \) belongs to \( P \) (if no such \( j \) exists, then we can do the following: if \( K_m \) consists of only one internal face, namely, \( c_{m-1} c_m \), then set \( j = 0 \). Otherwise, the leaf \( v_0 \) of \( P \) will correspond to either the face \( c_0 \alpha_1 c_{m-1} \) or the face \( c_{t-1} c_t c_m \); in the first case, set \( j = 1 \), and in the second case, set \( j = t \).) For example, in Figure 3(a), \( j = 3 \). Place \( \alpha_0, \alpha_1, \ldots, \alpha_{j-1} \) in the same horizontal channel, and \( \alpha_j, \alpha_{j+1}, \ldots, \alpha_t \) along a line making \( 45^\circ \) angle with the horizontal channels, such that

  - \( \alpha_t \) is in the same vertical channel as \( c_m \), and at least one unit above the horizontal channel \( X \) occupied by \( c_s' \) (we will give the exact value of the vertical distance between \( \alpha_t \) and \( X \) a little while later),
  - for each \( k \), where \( j - 1 \leq k \leq t - 1 \), \( \alpha_k \) is one unit above and one unit to the left of \( \alpha_{k+1} \), and
  - \( \alpha_0 \) is in the same vertical channel as \( c_{m-1} \).

(We will determine later on the horizontal distances between \( \alpha_0, \alpha_1, \ldots, \alpha_{j-1} \).)

- For each \( i \), where \( 0 \leq i \leq t \), removing \( \alpha_{i-1} \) and \( \alpha_i \), splits \( K_m \) into two subgraphs, one containing \( c_{m-1} \) and \( c_m \), and another subgraph \( L_i \). Let \( L_i \) be the subgraph of \( K_m \) consisting of the vertices of \( L_i' \), \( \alpha_{i-1} \), and \( \alpha_i \), and the edges between them. Recursively construct a feasible drawing of each \( L_i \), where \( 0 \leq i \leq j - 1 \), with \( \alpha_{i-1} \alpha_i \) as the base.
Let $S = \beta_0, \beta_1, \ldots, \beta_\mu$ be the clockwise order of the neighbors of $\alpha_{j-1}, \alpha_j, \ldots, \alpha_t$ in the subgraphs $L_j, L_{j+1}, \ldots, L_t$, where each $\beta_k$ is different from $\alpha_{j-1}, \alpha_j, \ldots, \alpha_t$. In $S$, we first place the neighbors of $\alpha_{j-1}$, then of $\alpha_j$, and so on, finally placing the neighbors of $\alpha_t$. For each $k$, where $j-1 \leq k \leq t$, we place the neighbors of $\alpha_k$ into $S$ in the same order as their clockwise order around $\alpha_k$.

For example, in Figure 3(a), $\mu = 8$. Let $\epsilon$ be the index such that the dual of the edge $(\beta_{j-1}, \beta_\epsilon)$ belongs to $P$ (if no such $\epsilon$ exists, then we can do the following: if $L_j$ consists of only one internal face, namely, $\alpha_{j-1} \alpha_j \beta_0$, then set $\epsilon = 0$. Otherwise, the leaf $v_\epsilon$ of $P$ will correspond to either the face $\beta_0 \beta_1 \alpha_{j-1}$ or the face $\beta_{\mu-1} \beta_\mu \alpha_j$; in the first case, set $\epsilon = 1$, and in the second case, set $\epsilon = \mu$). For example, in Figure 3(a), $\epsilon = 2$.

Place $\beta_0, \beta_1, \ldots, \beta_{\epsilon-1}$ in the same horizontal channel from left-to-right, and place $\beta_\epsilon, \beta_{\epsilon+1}, \ldots, \beta_\mu$ in another horizontal channel from right-to-left, such that:

- $\beta_0, \beta_1, \ldots, \beta_{\epsilon-1}$ are placed one unit above $\alpha_{j-1}$,
- $\beta_\epsilon, \beta_{\epsilon+1}, \ldots, \beta_\mu$ are placed one unit below $\alpha_t$,
- $\beta_0$ and $\beta_\mu$ are at either to the right of, or on the same vertical channel as $\alpha_t$,
- $\beta_{\epsilon-1}$ and $\beta_\epsilon$ are on the same vertical channel, and
- the distance between $\beta_{\epsilon-1}$ and $\beta_\epsilon$ is equal to 2 plus the vertical distance between $\alpha_{j-1}$ and $\alpha_t$.

For each $i$, where $0 \leq i \leq \epsilon - 1$, if there is an edge $e = (\beta_{i-1}, \beta_i)$ in $G$, then do the following: Notice that removing $e$ from $G$, split it into two subgraphs, one that contains $\alpha_{j-1}, \alpha_j, \ldots, \alpha_t$, and another subgraph $M_1'$ that does not contain any of them. Let $M_i$ be the subgraph of $G$ consisting of $\beta_{i-1}, \beta_i$, the vertices of $M_1'$, and the edges between them. Recursively construct a feasible drawing of $M_i$ with $\overline{\beta_{i-1}\beta_i}$ as its base.

For each $i$, where $\epsilon \leq i \leq \mu$, if there is an edge $e = (\beta_{i-1}, \beta_i)$ in $G$, then do the following: Notice that removing $e$ from $G$, splits it into two subgraphs, one that contains $\alpha_{j-1}, \alpha_j, \ldots, \alpha_t$, and another subgraph $N_i'$ that does not contain any of them. Let $N_i$ be the subgraph of $G$ consisting of $\beta_{i-1}, \beta_i$, the vertices of $N_1'$, and the edges between them. Recursively construct a feasible drawing $D_i''$ of $N_i$ with $\overline{\beta_{i-1}\beta_i}$ as its base, and then flip $D_i''$ upside-down.

Let $(v_{i-1}, v_\rho)$ be the edge of $P$ that is the dual of the edge $(\beta_{i-1}, \beta_i)$. For example, in Figure 1(b), $\rho = 9$. Let $R$ be the subgraph of $G$ that corresponds to the subpath $v_{i-1}v_\rho v_{\rho+1} \ldots v_\epsilon$. Construct a beam drawing $E$ of $R$. For each edge $e$ on the external face of $R$, do the following: Let $e = (\gamma_1, \gamma_2)$. Removing $\gamma_1$ and $\gamma_2$ from $G$ splits it into two subgraphs, one containing $\beta_0, \beta_1, \ldots, \beta_\mu$, and the other subgraph $Q_e$ not containing them. Let $Q_e$ be the subgraph of $G$ containing $\gamma_1, \gamma_2$, and the vertices of $Q_e$, and the edges between them. If $e$ is on the top or bottom boundary of $E$, then recursively construct a feasible drawing $D_e$ of $Q_e$ with $\overline{\gamma_1\gamma_2}$ as its base. If $e$ is on the bottom boundary of $E$, then flip $Q_e$ upside down. (Note that if $e$ is on the right boundary of $E$, then $Q_e$ will contain just the edge $e$ because $v_\rho$ is a leaf of $T_G$.)

We are now ready to give the vertical distance between $\alpha_t$ and $X$: it is equal to $1 + \theta$, where $\theta$ is maximum height of any of $D_i', D_i''$, and $D_e$, where $e$ is on
the bottom boundary of $E$. Note that this will guarantee that the vertices of each $D'_e$ and $D_e$ will occupy horizontal channels that are either above or the same as the horizontal channel that contains $c_i = w, c_1, \ldots, c_m (= c'_0, c'_1, c'_2, \ldots, c'_{\mu})$. This ensures that there are no crossings between the edges of any $D'_e$ or $D_e$, and any edge of the form $(v, c'_j)$.

Let $h(n)$ and $w(n)$ be the height and width, respectively, of a feasible drawing $D$ of $G$ with base $B$, constructed by the Algorithm $OPDraw$. Here, $n$ is the number of vertices in $G$. Let $d$ be the degree of $G$. Note that, by the definition of feasible drawings, $w(n)$ will be equal to one plus the horizontal separation between the end-points of $B$.

It is easy to prove using induction that $w(n) = n$ is sufficient. As for the horizontal distances between $u$ and $v$, between $c_{i-1}$ and $c_i$ (for $1 \leq i \leq m - 1$, between $c'_{i-1}$ and $c'_i$ (for $1 \leq i \leq s$), between $c_{\epsilon - 1}$ and $c_{\epsilon}$ (for $1 \leq i \leq \epsilon - 1$), between $\beta_{\epsilon - 1}$ and $\beta_{\epsilon}$ (for $1 \leq i \leq \epsilon - 1$), and between $\beta_{\epsilon - 1}$ and $\beta_{\epsilon}$ (for $\epsilon + 1 \leq i \leq \mu$), it is sufficient to set them to be equal to $|H| - 1$, $|K_{\epsilon}| - 1$, $|K'_\epsilon| - 1$, $|L_{\epsilon}| - 1$, $|M_{\epsilon}| - 1$, and $|N_{\epsilon}| - 1$, respectively. It is also sufficient to set the distance between the end-points of each edge $e$ on the top or bottom boundary of $E$, to be equal to $|Q_e| - 1$.

As for $h(n)$, first notice that, because $G$ has degree $d$, $t - (j - 1)$ is less than $2d$, and hence, the distance between $\beta_{\epsilon - 1}$ and $\beta_{\epsilon}$ is less than $2d + 2$.

Let $h'$ be a function, such that $h'(f) = h(n)$, where $f$ is the number of internal faces in $G$, i.e., the number of nodes in the dual tree $T_G$ of $G$.

From the construction of $D$, we have that:

$$
h'(f) \leq \max\left\{ \max_{1 \leq i \leq s} \{h'(|T_{K_i}|)\}, \max_{1 \leq i \leq \mu - 1} \{h'(|T_{K_{\epsilon}}|)\}, \max_{1 \leq i \leq \mu} \{h'(|T_{K'_\epsilon}|)\}, \max_{1 \leq i \leq \mu - 1} \{h'(|T_{L_{\epsilon}}|)\}, \max_{1 \leq i \leq \mu - 1} \{h'(|T_{M_{\epsilon}}|)\}, \max_{1 \leq i \leq \mu - 1} \{h'(|T_{N_{\epsilon}}|)\}\right\} + O(d),
$$

Since $P$ is a spine of $T_G$, and

- the dual trees of $H$, $K_i$, $L_i$, $M_i$, and $Q_e$ (in the case when edge $e$ is on top boundary of $E$), are either right subtrees of $P$, or belong to the right subtrees of $P$, and
- the dual trees of $K'_\epsilon$, $N_i$, and $Q_e$ (in the case when edge $e$ is on bottom boundary of $E$), are either left subtrees of $P$, or belong to the left subtrees of $P$,

from Lemma 1, it follows that:

$$
h'(f) \leq \max_{f_1 + f_2 \leq 1 - \delta} \{h'(f_1) + h'(f_2) + O(d)\}.
$$

Using induction, we can show that $h'(f) = O(df^{0.48})$ (see also [2]). Since $f = O(n)$, $h(n) = h'(f) = O(df^{0.48}) = O(dn^{0.48})$. 

Theorem 1. Let $G$ be an outerplanar graph with degree $d$ and $n$ vertices. We can construct a planar straight-line grid drawing of $G$ with area $O(dn^{1.48})$ in $O(n)$ time.

Proof. Arbitrarily select any edge $e = (u, v)$ on the external face of $G$, and designate $u$ and $v$ as the poles of $G$. Let $B$ be any horizontal line-segment with length $n - 1$, such that the end-points of $B$ are grid points. Let $\delta$ be any user-defined constant in the range $(0, 0.0004]$. Construct a feasible drawing of $G$ with base $B$ using Algorithm OpDraw. From the discussion given above, it follows immediately that the area of the drawing is $O(dn^{1+0.48}) = O(dn^{1.48})$. It is easy to see the algorithm runs in $O(n)$ time.

Corollary 1. Let $G$ be an outerplanar graph with $n$ vertices and degree $d$, where $d = o(n^{0.52})$. We can construct a planar straight-line grid drawing of $G$ with $o(n^2)$ area in $O(n)$ time.

References