

CENSORED DISCRETE LINEAR l_1 APPROXIMATION*

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Abstract. The censored linear l_1 approximation problem is to minimize the nonconvex piecewise linear function $F(x) = \sum_{i=1}^m |y_i - \max(z_i, x^T a_i)|$. The problem arises in regression models where the range of the dependent variable is restricted. Unlike the maximum likelihood and least squares estimators the censored l_1 estimator provides a consistent estimator without an assumption that the errors are normally distributed.

This paper presents a compact characterization of the generalized gradient of F , and necessary and sufficient conditions for a (strict) local minimizer of F . A reduced gradient algorithm for linear programming and l_1 approximation is extended to provide a stable finite direct descent method for calculating a local minimizer of F . This provides an efficient method of calculating the censored l_1 estimator.

Key words. censored l_1 approximation, censored LAD estimation, generalized gradient, reduced gradient algorithm

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1. Introduction. The censored discrete linear l_1 approximation problem is to minimize over $x \in \mathbb{R}^n$ the nonconvex piecewise linear function $F: \mathbb{R}^n \rightarrow \mathbb{R}$ defined by

$$(1.1) \quad F(x) = \sum_{i=1}^m |y_i - \max(z_i, x^T a_i)|,$$

where $y_i \in \mathbb{R}$, $z_i \in \mathbb{R}$ and $a_i \in \mathbb{R}^n$ for $i = 1, \dots, m$.

The interest in this problem arises from the censored and truncated least absolute deviation (LAD) estimators proposed by Powell [8] and [10] for regression coefficients in two models with limited dependent variables—that is, regression models where the range of the dependent variable is restricted to some subset of the real line. Consider a linear process, where the dependent variable is restricted to be nonnegative, namely

$$(1.2) \quad y_i = \max(0, \tilde{x}^T a_i + \varepsilon_i).$$

Here \tilde{x} is the underlying vector of parameters to be estimated, ε_i are unobservable errors and $y_i \in \mathbb{R}$ and $a_i \in \mathbb{R}^n$ are observable data. The consistency of the maximum likelihood and least squares estimators depends critically upon the assumption that the errors are normally distributed. However the censored LAD estimator, that is the global minimizer of (1.1) with $z_i = 0 \forall i = 1, \dots, m$, provides a consistent estimator of \tilde{x} which does not depend upon the functional form of the distribution of the errors.

An important feature of (1.1) is the generality obtained by taking z_i arbitrary. Thus the results of this paper apply to censored l_1 estimation ($z_i = 0 \forall i$), truncated l_1 estimation ($z_i = y_i/2 \forall i$), general lower and upper bounds on the observed dependent variable, as well as to standard l_1 estimation ($z_i = -\infty \forall i$). The results could also be extended to regression quantile estimation [9].

One aim of this paper is to provide an efficient algorithm for computing the censored LAD estimator, and thus remove a major obstacle to its use. For instance Paarsch [7] published a simulation study of the censored l_1 estimator which uses a grid-search method to minimize (1.1)! Although Powell's proof [10] of the consistency and asymptotic normality of the censored LAD estimator assumes the unique global

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minimizer of F is calculated, in practice only a local minimizer can be calculated as F is not convex. However on some randomly generated test problems this seemed to be a major difficulty only when m is of the same order of magnitude as n . When $m \gg n$, as in most practical applications, the algorithm usually found the global minimizer of F (see § 5 for further comments).

The following example [11] of temperature accelerated life tests on electrical insulation in 40 motorettes illustrates the type of problem on which the algorithm can be used. Ten motorettes were tested at each of four temperatures. Testing was terminated at different times at each temperature giving the data in Table 1. The model fitted in [11] is

$$\log_{10} H = \tilde{x}_1 + 1000\tilde{x}_2 / (T + 273.2) + \varepsilon,$$

where H is the failure time and T is the temperature. At each temperature there is an upper bound H_t (the time at which testing was stopped) on the observed failure times, so the logarithms of the observed failure times are given by

$$\min(\log_{10} H_i, x_1 + 1000\tilde{x}_2 / (T + 273.2) + \varepsilon).$$

A problem with upper bounds on the observed variable can be converted into one with lower bounds by changing the sign of the data y_i, z_i and a_i , as

$$|y_i - \max(z_i, x^T a_i)| = |-y_i - \min(-z_i, -x^T a_i)|.$$

TABLE 1
Data for motorette example.

	Test temperature T °C			
	150	170	190	200
Failure times H in hours		1764	408	408
		2722	408	408
		3444	1344	504
		3542	1344	504
		3780	1440	504
		4860		
		5196		
Termination time H_t	8064	5448	1680	528
	10 units	3 units	5 units	5 units

The next section considers the differential properties of F and establishes a concise characterization of the generalized gradient $\partial F(x)$ of $F(x)$. Section 3 strengthens the usual necessary conditions $0 \in \partial F(x)$ of nonsmooth optimization (Clarke [5]) to provide necessary and sufficient conditions for a (strict) local minimizer of F . The interpolation result for l_1 approximation (see for example Watson [12, p. 119]) extends to censored l_1 approximation. That is a global minimizer of F is characterized by n (when the vectors $a_i, i = 1, \dots, m$ have rank n) linear equations $r_i(x) \equiv \max(y_i, z_i) - a_i^T x = 0$. This reduces the search for a minimizer of F to a finite number of points.

A finite direct descent algorithm is developed in § 4. This algorithm is a generalization of a reduced gradient algorithm for l_1 approximation, which in turn is simple an extension of a reduced gradient algorithm for linear programming (see Osborne [6]). The algorithm which is numerically stable, can be implemented in a convenient tableau form for relatively small dense problems. An efficient algorithm, where for

large m the number of operations per iteration is dominated by $nm \log m$, can be developed. For fixed n the number of iterations needed to find a solution appears to grow slowly with m .

2. Differential properties. The function F defined by (1.1) can be written as the sum of a nonconvex function and a convex function, namely

$$(2.1) \quad F(x) = \sum_{i: y_i > z_i} |y_i - \max(z_i, x^T a_i)| + \sum_{i: y_i \leq z_i} -y_i + \max(z_i, x^T a_i).$$

The character of the component functions is illustrated by the following one-dimensional examples. The situation where $y_i > z_i$ is typified by the function $f_1(x) = |1 - \max(0, x)|$, where $y_1 = 1, z_1 = 0$ and $a_1 = 1$, which is sketched in Fig. 1. The situation where $y_i \leq z_i$ is typified by the function $f_2(x) = \max(0, x)$ where $y_1 = 0, z_1 = 0$ and $a_1 = 1$, which is sketched in Fig. 2.

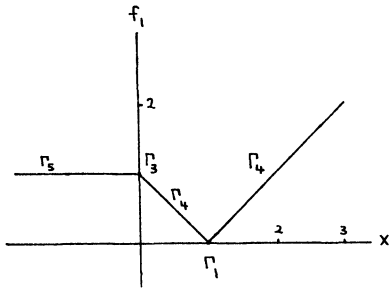


FIG. 1. A function with $y_i > z_i$.

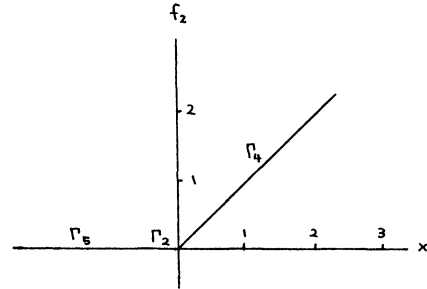


FIG. 2. A function with $y_i \leq z_i$.

It should be noted that for censored estimation problems with a lower bound z_i on the dependent variable

$$y_i = \max(z_i, \tilde{x}^T a_i + \varepsilon_i) \geq z_i.$$

Thus the inequalities $y_i \leq z_i$ could be replaced by $y_i = z_i$ in the definition of the sets $\Gamma_i(x)$ below. However this does not produce any further simplification, so is not used in the rest of the paper. Define the functions $r_i(x)$ by

$$(2.2) \quad r_i(x) = \max(y_i, z_i) - x^T a_i \quad \text{for } i = 1, \dots, m,$$

and the index sets $\Gamma_j(x), j = 1, \dots, 5$ by

$$(2.3) \quad \begin{aligned} \Gamma_1(x) &= \{i \in 1, \dots, m: y_i > z_i \text{ and } x^T a_i = y_i\}, \\ \Gamma_2(x) &= \{i \in 1, \dots, m: y_i \leq z_i \text{ and } x^T a_i = z_i\}, \\ \Gamma_3(x) &= \{i \in 1, \dots, m: y_i > z_i \text{ and } x^T a_i = z_i\}, \\ \Gamma_4(x) &= \{i \in 1, \dots, m: x^T a_i > z_i \text{ and } x^T a_i \neq y_i\}, \\ \Gamma_5(x) &= \{i \in 1, \dots, m: x^T a_i < z_i\}. \end{aligned}$$

As an example the nonempty sets $\Gamma_i(x)$ are marked on Figs. 1 and 2. Also define the index set $\mathcal{A}(x)$ by

$$(2.4) \quad \mathcal{A}(x) = \Gamma_1(x) \cup \Gamma_2(x) = \{i \in 1, \dots, m: r_i(x) = 0\}.$$

The sets $\Gamma_j(x)$, $j = 1, \dots, 5$ form a disjoint partition of $\{1, \dots, m\}$, that is

$$\bigcup_{j=1}^5 \Gamma_j(x) = \{1, \dots, m\} \quad \forall x \in \mathbb{R}^n,$$

$$\Gamma_i(x) \cap \Gamma_j(x) = \emptyset \quad \forall i \neq j, \quad \forall x \in \mathbb{R}^n.$$

A definition of the generalized gradient $\partial F(x)$ (Clarke [5]) of a piecewise smooth function F at the point x is

$$(2.5) \quad \partial F(x) = \text{conv} \{u \in \mathbb{R}^n : \exists \text{ a sequence } \{x^{(k)}\} \text{ such that } x^{(k)} \rightarrow x, \nabla F(x^{(k)}) \text{ exists } \forall k, \text{ and } \nabla F(x^{(k)}) \rightarrow u \text{ as } k \rightarrow \infty\},$$

where $\text{conv } G$ denotes the convex hull of G . For functions $F: \mathbb{R}^n \rightarrow \mathbb{R}$, $\partial F(x)$ is a nonempty compact convex set in \mathbb{R}^n .

For $i \in \Gamma_4(x)$ the component functions $|y_i - \max(z_i, x^T a_i)|$ are smooth (continuously differentiable) in a neighbourhood of x , with gradient $-\theta_i a_i$ where $\theta_i = \text{sign}(r_i(x))$. For $i \in \Gamma_5(x)$ the component functions $|y_i - \max(z_i, x^T a_i)| = |y_i - z_i|$ are also smooth in a neighbourhood of x , but with zero gradient. The gradient $g \equiv g(x)$ of the smooth component functions is thus

$$(2.6) \quad g(x) = - \sum_{i \in \Gamma_4(x)} \theta_i a_i.$$

For $i \in \mathcal{A}(x)$ the component functions $|y_i - \max(z_i, x^T a_i)|$ are nonsmooth, but convex in a neighbourhood of x , with generalized gradients (subdifferentials) given by, for $i \in \Gamma_1(x)$

$$\partial |y_i - \max(z_i, x^T a_i)| = \text{conv} \{-a_i, a_i\} = \{u \in \mathbb{R}^n : u = -\lambda_i a_i, -1 \leq \lambda_i \leq 1\},$$

and for $i \in \Gamma_2(x)$

$$\partial(-y_i + \max(z_i, x^T a_i)) = \text{conv} \{0, a_i\} = \{u \in \mathbb{R}^n : u = -\lambda_i a_i, -1 \leq \lambda_i \leq 0\}.$$

For $i \in \Gamma_3(x)$ the component functions are nonsmooth and nonconvex at x with generalized gradient

$$\partial |y_i - \max(z_i, x^T a_i)| = \text{conv} \{-a_i, 0\} = \{u \in \mathbb{R}^n : u = -\lambda_i a_i, 0 \leq \lambda_i \leq 1\} \quad \text{for } i \in \Gamma_3(x).$$

As generalized gradients satisfy $\partial(F_1(x) + F_2(x)) \subseteq \partial F_1(x) + \partial F_2(x)$, one has $\partial F(x) \subseteq G(x)$ where $G(x)$ is the nonempty compact convex polytope defined by

$$(2.7) \quad G(x) = \left\{ v \in \mathbb{R}^n : v = g(x) - \sum_{i \in \mathcal{A}(x) \cup \Gamma_3(x)} \lambda_i a_i \text{ where } \right.$$

$$\left. |\lambda_i| \leq 1, i \in \Gamma_1(x), -1 \leq \lambda_i \leq 0, i \in \Gamma_2(x), 0 \leq \lambda_i \leq 1, i \in \Gamma_3(x) \right\}.$$

F may not be convex, so the inclusion $\partial F(x) \subseteq G(x)$ can be strict as the following example illustrates. Let $n = 2$, $m = 3$, $z^T = (0, 0, 0)$, $y^T = (1, 2, 2)$, $a_1^T = (1, -1)$, $a_2^T = (1, 1)$ and $a_3^T = (-1, 3)$. The contours of F are sketched in Fig. 3, whilst the sets $\partial F(x)$ and $G(x)$ at the point $\bar{x}^T = (3/2, 1/2)$ are sketched in Fig. 4. At the point $\bar{x}^T = (3/2, 1/2)$ $\Gamma_1 = \{1, 2\}$, $\Gamma_3 = \{3\}$ and $\Gamma_2 = \Gamma_4 = \Gamma_5 = \emptyset$. Thus the possible extreme values of λ in (2.7) are

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix},$$

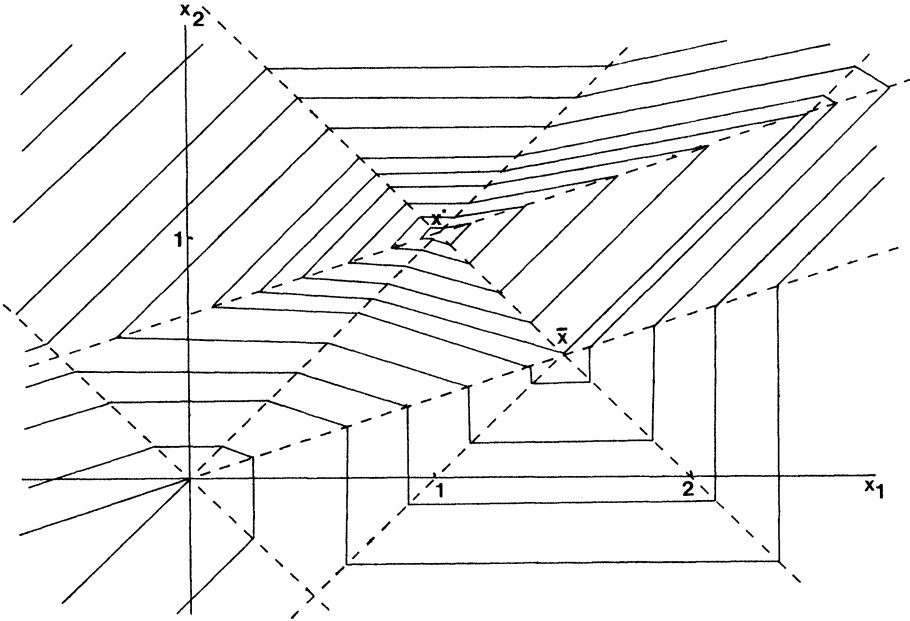


FIG. 3. Contours of Example 1.

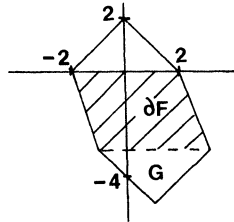


FIG. 4. $G(\bar{x})$ and $\partial F(\bar{x})$.

giving

$$G(\bar{x}) = \text{conv} \left\{ \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}.$$

However from (2.5) and Fig. 3 one can see that

$$\partial F(\bar{x}) = \text{conv} \left\{ \begin{bmatrix} -1 \\ -3 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\},$$

which is a strict subset of $G(\bar{x})$.

The following result gives conditions which ensure the equality of the sets $\partial F(x)$ and $G(x)$.

LEMMA 1. Let $\hat{\lambda}_i = \pm 1$ for $i \in \Gamma_1(x)$, $\hat{\lambda}_i = -1$ or 0 for $i \in \Gamma_2(x)$ and $\hat{\lambda}_i = 0$ or 1 for $i \in \Gamma_3(x)$. Define the set $\mathcal{F}(\hat{\lambda})$ by

$$\begin{aligned} \mathcal{F}(\hat{\lambda}) = \{ \omega \in \mathbb{R}^n : & z_i \leq \omega^T a_i \leq y_i \text{ for } i \in \Gamma_1(x) \text{ with } \hat{\lambda}_i = 1 \\ & y_i \leq \omega^T a_i \text{ for } i \in \Gamma_1(x) \text{ with } \hat{\lambda}_i = -1 \\ & z_i \leq \omega^T a_i \text{ for } i \in \Gamma_2(x) \text{ with } \hat{\lambda}_i = -1 \text{ and } i \in \Gamma_3(x) \text{ with } \hat{\lambda}_i = 1 \\ & \omega^T a_i \leq z_i \text{ for } i \in \Gamma_2(x) \cup \Gamma_3(x) \text{ with } \hat{\lambda}_i = 0 \}. \end{aligned}$$

Then $\partial F(x) = G(x)$ if and only if $\mathcal{F}(\hat{\lambda})$ has nonempty interior for every $\hat{\lambda}$ which corresponds to an extreme point of $G(x)$.

Proof. (a) Assume $\mathcal{F}(\hat{\lambda})$ has nonempty interior for every $\hat{\lambda}$ corresponding to an extreme point of $G(x)$. As $\partial F(x) \subseteq G(x)$ one only has to show $G(x) \subseteq \partial F(x)$. Note that $\mathcal{F}(\hat{\lambda})$ is the subset of \mathbb{R}^n in which the linear function specified by $\hat{\lambda}$ is active, so the gradient of this linear function is

$$\hat{v} = g - \sum_{i \in \mathcal{A} \cup \Gamma_3} \hat{\lambda}_i a_i.$$

Let $\bar{x} \in \text{int } \mathcal{F}(\hat{\lambda})$, and set $s = \bar{x} - x$. Then $F(x + \alpha s) = F(x) + \alpha \hat{v} \forall \alpha \in [0, 1]$; moreover $\nabla F(x + \alpha s)$ exists and $\nabla F(x + \alpha s) = \hat{v}$ for all α in $(0, 1]$. Thus from (2.5) $\hat{v} \in \partial F(x)$. This holds for all extreme points \hat{v} of $G(x)$ giving $G(x) \subseteq \partial F(x)$.

(b) Assume $\partial F(x) = G(x)$. Let $\hat{\lambda}$ correspond to an extreme point \hat{v} of $G(x)$, and hence $\partial F(x)$. Then from (2.5) there exists a sequence $\{x^{(k)}\}$ such that $x^{(k)} \rightarrow x$, $\nabla F(x^{(k)})$ exists for all k and $\nabla F(x^{(k)}) \rightarrow \hat{v}$ as $k \rightarrow \infty$. As F is piecewise linear, this implies $\hat{v} = \nabla F(x^{(k)})$ for all k sufficiently large ($x^{(k)} \neq x$). As $\nabla F(x^{(k)})$ exists, one has $x^{(k)} \in \text{int } \mathcal{F}(\hat{\lambda})$ for these k . \square

If $\text{rank } \{a_i : i \in \mathcal{A}(x) \cup \Gamma_3(x)\} = n$, then there are vectors $\hat{\lambda}$ for which $\mathcal{F}(\hat{\lambda})$ has an empty interior. The sets $\partial F(x)$ and $G(x)$ are equal if and only if these $\hat{\lambda}$ correspond to points \hat{v} which are not extreme points of $G(x)$. If $\Gamma_3(x)$ is empty, then F is convex in a neighbourhood of x , so $\partial F(x) = G(x)$. Thus difficulties only arise at points where $\Gamma_3(x)$ is nonempty.

A key tool in the development of optimality conditions in nonsmooth optimization is the one-sided directional derivative $F'(x; s)$ defined by

$$F'(x; s) = \lim_{\alpha \rightarrow 0^+} \frac{F(x + \alpha s) - F(x)}{\alpha},$$

which exists for all $x, s \in \mathbb{R}^n$ as F is a continuous piecewise linear function. For the function (1.1) one has

$$(2.8) \quad F'(x; s) = s^T g + \sum_{i \in \Gamma_1(x)} |s^T a_i| + \sum_{i \in \Gamma_2(x)} \max(0, s^T a_i) - \sum_{i \in \Gamma_3(x)} \max(0, s^T a_i).$$

As F may be nonconvex at points x where $\Gamma_3(x)$ is nonempty, one only has

$$(2.9) \quad F'(x; s) \leq \max_{u \in \partial F(x)} u^T s.$$

If $\Gamma_3(x)$ is empty, then (2.9) holds with equality.

3. Characterization of minimizers. The inclusion $\partial F(x) \subseteq G(x)$ means that the well-known necessary conditions $0 \in \partial F(x^*)$ [5] for x^* to be a local minimizer of F carry through to $0 \in G(x^*)$. However sufficient conditions are only immediately available when $\Gamma_3(x^*)$ is empty, as then F is convex in a neighbourhood of x^* . In that case $\partial F(x^*) = G(x^*)$ and $0 \in G(x^*)$ is both necessary and sufficient, whilst $0 \in \text{int } G(x^*)$ ensures that x^* is a strict local minimizer of F . Two one-dimensional examples; F_1 with $y = (1, 1/2)^T, z = (0, 0)^T, a_1 = 1, a_2 = -1/2$ and F_2 with $y = (1, 1, 2)^T, z = (0, 0, 0)^T, a_1 = 2, a_2 = 1, a_3 = 1$, are sketched in Figs. 5 and 6. F_1 has strict local minima at $x = -1$ with $F_1 = 1$ and at $x = 1$ with $F_1 = 1/2$, whilst any $x \in [1/2, 1]$ is a local minimizer of F_2 with $F_2 = 1$.

When $\Gamma_3(x)$ is nonempty, the situation is more complicated. However one can obtain an interpolation result similar to the discrete l_1 case where a solution is

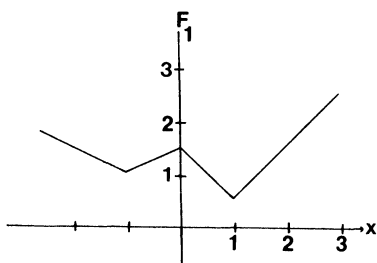


FIG. 5. Distinct local minimizers.

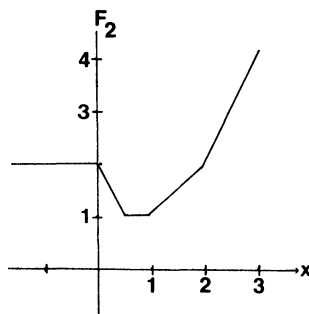


FIG. 6. Multiple minimizers.

characterized in terms of the residual functions $r_i(x)$ which are zero. Define

$$A = [a_i : i \in \mathcal{A}(x)].$$

LEMMA 2. Let x be a point with $0 \in G(x)$ and $\text{rank}(A) < n$. Then $F'(x; s) \leq 0$ for all directions s satisfying $s^T A = 0$. Moreover if there exists an index $k \in \Gamma_3(x)$ with $a_k \notin \mathcal{R}(A)$, the range of space of A , then there exists a descent direction s with $s^T A = 0$.

Proof. As $0 \in G(x)$ there exists a vector λ satisfying $|\lambda_i| \leq 1$, $i \in \Gamma_1(x)$, $-1 \leq \lambda_i \leq 0$, $i \in \Gamma_2(x)$, $0 \leq \lambda_i \leq 1$, $i \in \Gamma_3(x)$ and

$$g = \sum_{i \in \mathcal{A}(x) \cup \Gamma_3(x)} \lambda_i a_i.$$

Let s satisfy $s^T A = 0$. If $\Gamma_3(x)$ is nonempty, then

$$s^T g = \sum_{i \in \Gamma_3(x)} \lambda_i s^T a_i.$$

From (2.8)

$$\begin{aligned} F'(x; s) &= s^T g - \sum_{i \in \Gamma_3(x)} \max(0, s^T a_i) \\ &= \sum_{i \in \Gamma_3(x)} [\lambda_i s^T a_i - \max(0, s^T a_i)] \\ &= \sum_{\substack{i \in \Gamma_3(x) \\ s^T a_i \leq 0}} \lambda_i s^T a_i + \sum_{\substack{i \in \Gamma_3(x) \\ s^T a_i > 0}} (\lambda_i - 1) s^T a_i \\ &\leq 0, \end{aligned}$$

as $0 \leq \lambda_i \leq 1$ for all $i \in \Gamma_3(x)$. If $\Gamma_3(x)$ is empty, then $s^T A = 0$ implies $s^T g = 0$ and hence $F'(x; s) = 0$. Now let $k \in \Gamma_3(x)$ be such that $a_k \notin \mathcal{R}(A)$. If $0 < \lambda_k \leq 1$ and s is chosen so that $s^T a_k < 0$, then $F'(x; s) < 0$. Alternatively if $0 \leq \lambda_k < 1$ and s is chosen so that $s^T a_k > 0$, then $F'(x; s) < 0$. \square

Note that if $k \in \Gamma_3(x)$, $0 < \lambda_k < 1$, and $a_k \notin \mathcal{R}(A)$ then any direction s such that $s^T A = 0$ is a descent direction. It is only if λ_k is at one of its bounds that an additional restriction needs to be placed on s to obtain a descent direction.

The significance of the above result is that in searching for a minimizer one need only consider points where $a_i \in \mathcal{R}(A)$ for all $i \in \Gamma_3(x)$. Typically methods also involve a line search, and the following result shows that only points where a new residual $r_i(x)$ becomes zero need be considered in minimizing F along a line. Define

$$(3.1) \quad I(x; s) = \{i \in 1, \dots, m : r_i(x)(s^T a_i) > 0\}.$$

LEMMA 3. *Let s be a direction such that $F'(x; s) \leq 0$. If $I(x; s)$ is empty, then $F(x + \alpha s) = F(x) \forall \alpha \geq 0$. Otherwise any local minimum of $F(x + \alpha s)$ over $\alpha \geq 0$ is attained at a point $\alpha_i > 0$ satisfying $r_i(x + \alpha_i s) = 0$ for some $i \in I(x; s)$.*

Proof. Define

$$F'(x + \alpha^- s; s) \equiv \lim_{\zeta \rightarrow \alpha^-} F'(x + \zeta s; s) = -F'(x + \alpha s; -s).$$

The only points α where $F'(x + \alpha s; s)$ changes are those that satisfy either $r_i(x + \alpha s) = 0$ or $(x + \alpha s)^T a_i = z_i$ for some $i \in \{1, \dots, m\}$. Let α be a point at which only the second of these conditions is satisfied, then from (2.8)

$$\begin{aligned} F'(x + \alpha s; s) &= F'(x + \alpha^- s; s) - \sum_{\substack{i \in \Gamma_2(x + \alpha s) \\ s^T a_i > 0}} s^T a_i + \sum_{\substack{i \in \Gamma_2(x + \alpha s) \\ s^T a_i < 0}} s^T a_i \\ &< F'(x + \alpha^- s; s). \end{aligned}$$

Thus points α satisfying just $(x + \alpha s)^T a_i = z_i$ need not be considered in looking for minimizers of $F(x + \alpha s)$. It is easily verified that there exists an $\alpha > 0$ with $r_i(x + \alpha s) = 0$ if and only if $i \in I(x; s)$. If $I(x; s) = \emptyset$ then $F'(x + \alpha s; s)$ is a nonincreasing function of $\alpha \geq 0$. As F is continuous and bounded below by zero the only possibility is that $F'(x + \alpha s; s) = 0$ for all $\alpha \geq 0$. \square

Solving $r_i(x + \alpha_i s) = 0$ gives

$$\alpha_i = \frac{\max(y_i, z_i) - x^T a_i}{s^T a_i}, \quad i \in I(x; s),$$

as the points to be considered in a line search. These may include separate local minima as well as points which do not correspond to minima of $F(x + \alpha s)$ (see Figs. 5 and 6). The global minimum may also be attained at other points.

Lemmas 2 and 3 enable one to obtain the following result, which states that when the vectors $a_i, i = 1, \dots, m$ have rank n (which is usually true as $m \gg n$ in practice) the global minimum of F is attained at a point characterized by n linear equations $r_i(x) = 0$. This reduces the search for a global minimizer of F to a finite number of points, which may also include nonglobal local minimizers of F (for example $x = -1$ in Fig. 5.). However certain nonglobal local minima are excluded (for example any $x < 0$ in Fig. 6). This result corresponds to the interpolation result for discrete l_1 approximation (see for example Watson [12, p. 119]), remembering that $r_i(x) = 0$ if and only if $i \in \mathcal{A}(x)$.

THEOREM 1. *If the vectors $a_i, i = 1, \dots, m$ have rank m_r , then there exists a global minimizer x^* of $F(x)$ with $\text{rank}(A^*) = m_r$, and hence $\mathcal{A}(x^*) \equiv \Gamma_1(x^*) \cup \Gamma_2(x^*)$ contains at least m_r indices.*

Proof. Suppose \bar{x} is a global minimizer of F with $\text{rank}(\bar{A}) < m_r$. Then $0 \in G(\bar{x})$. Let $s \neq 0$ be a direction satisfying $s^T \bar{A} = 0$. As $\text{rank}(\bar{A}) < m_r$ there exists an index k with $k \notin \mathcal{A}(\bar{x})$ and $s^T a_k \neq 0$. The sign of s can then be chosen so that $r_k(\bar{x})(s^T a_k) > 0$, and hence $k \in I(\bar{x}; s)$. Then $F'(\bar{x}; s) \leq 0$ by Lemma 2, and by Lemma 3 the minimum of $F(\bar{x} + \alpha s)$ over $\alpha \geq 0$ is attained at a point $\alpha_i > 0$ where $r_i(\bar{x} + \alpha_i s) = 0$ for some $i \in I(\bar{x}; s)$. This process can be repeated until $\text{rank}(A) = m_r$. \square

The following theorem strengthens the condition $0 \in G(x)$ to provide necessary and sufficient conditions for local minimizers and strict local minimizers of F . The proof also illustrates how descent directions may be calculated. Let \mathcal{A}^* denote $\mathcal{A}(x^*)$, $g^* = g(x^*)$ and so on.

THEOREM 2. *The point x^* is a local minimizer of F if and only if there exist multipliers λ and $\mu^{(j)}$, $j \in \Gamma_3^*$ satisfying*

(1)

$$(3.2) \quad g^* = \sum_{i \in \mathcal{A}^*} \lambda_i a_i, \quad a_j = \sum_{i \in \mathcal{A}^*} \mu_i^{(j)} a_i \quad \forall j \in \Gamma_3^*;$$

(2)

$$(3.3) \quad \begin{aligned} -\lambda_i + \sum_{j \in \Gamma_3^*} \max(0, \mu_i^{(j)}) &\leq 1 \quad \forall i \in \mathcal{A}^*, \\ \lambda_i + \sum_{j \in \Gamma_3^*} \max(0, -\mu_i^{(j)}) &\leq 1 \quad \forall i \in \Gamma_1^*, \\ \lambda_i + \sum_{j \in \Gamma_3^*} \max(0, -\mu_i^{(j)}) &\leq 0 \quad \forall i \in \Gamma_2^*. \end{aligned}$$

Moreover the point x^* is a strict local minimizer of F if and only if the following condition also holds.

(3) *The set of vectors a_i , $i \in \mathcal{A}^*$ such that all the inequalities (3.3) are strict has rank n .*

Proof. (a) Suppose there exist multipliers λ and $\mu^{(j)}$, $j \in \Gamma_3(x^*)$ satisfying conditions (1) and (2). Now x^* is a local minimizer of F if and only if $F'(x^*; s) \geq 0 \forall s \in \mathbb{R}^n$. Moreover x^* is a strict local minimizer of F if and only if $F'(x^*, s) > 0 \forall s \neq 0, s \in \mathbb{R}^n$. Now from (2.8).

$$(3.4) \quad \begin{aligned} F'(x^*; s) &= \sum_{i \in \Gamma_1^*} [\lambda_i s^T a_i + |s^T a_i|] + \sum_{i \in \Gamma_2^*} [\lambda_i s^T a_i + \max(0, s^T a_i)] \\ &\quad - \sum_{j \in \Gamma_3^*} \max\left(0, \sum_{i \in \mathcal{A}^*} \mu_i^{(j)} s^T a_i\right) \\ &\geq \sum_{i \in \mathcal{A}^*} \left[\lambda_i \operatorname{sign}(s^T a_i) + \eta_i - \sum_{j \in \Gamma_3^*} \max(0, \mu_i^{(j)} \operatorname{sign}(s^T a_i)) \right] |s^T a_i| \\ &= \sum_{i \in \mathcal{A}^*} \zeta_i |s^T a_i| \geq 0, \end{aligned}$$

where $\eta_i = 1$ for $i \in \Gamma_1^*$ and $\eta_i = \max(0, \operatorname{sign}(s^T a_i))$ for $i \in \Gamma_2^*$. The last inequality follows as $\zeta_i \geq 0 \forall i \in \mathcal{A}^*$ from condition (2). Moreover if condition (3) holds, then $\zeta_i > 0$ for a set of indices i such that the corresponding vectors a_i have rank n , and one obtains $F'(x^*; s) > 0 \forall s \neq 0$.

(b) Suppose x^* is a local minimizer of F . Then $0 \in G^*$, so there exist multipliers $\tilde{\lambda}$ such that

$$g^* = \sum_{i \in \mathcal{A}^* \cup \Gamma_3^*} \tilde{\lambda}_i a_i.$$

As a descent direction cannot exist at x^* , Lemma 2 implies there exist $\mu^{(j)}$ satisfying

$$a_j = \sum_{i \in \mathcal{A}^*} \mu_i^{(j)} a_i \quad \forall j \in \Gamma_3^*.$$

Hence there exist multipliers λ and $\mu^{(j)}$, $j \in \Gamma_3^*$ satisfying (3.2).

Now suppose there exists an index $k \in \mathcal{A}^*$ such that

$$-\sigma \lambda_k + \sum_{j \in \Gamma_3^*} \max(0, \sigma \mu_k^{(j)}) > \begin{cases} 1, & k \in \Gamma_1^*, \\ \max(0, \sigma), & k \in \Gamma_2^*, \end{cases}$$

where $\sigma = +1$ or -1 . Let Λ be a subset of \mathcal{A}^* such that $k \in \Lambda$ and the vectors a_i , $i \in \Lambda$ form a basis for $\mathcal{R}(A)$. For $i \in \mathcal{A}^*$ such that $i \notin \Lambda$ the multipliers λ_i and $\mu_i^{(j)}$, $j \in \Gamma_3^*$

can be chosen arbitrarily whilst still satisfying (3.2). Let s be a direction satisfying $s^T a_k = \sigma$ and $s^T a_i = 0$ for $i \neq k$, $i \in \Lambda$. Also for all $j \in \Gamma_3^*$ and $i \in \mathcal{A}^*/\Lambda$ let $\mu_i^{(j)} = 0$ and $\lambda_i = -\text{sign}(s^T a_i)$. Then from (3.4)

$$F'(x^*; s) = \sigma \lambda_k + \left\{ \begin{array}{ll} 1, & k \in \Gamma_1^* \\ \max(0, \sigma), & k \in \Gamma_2^* \end{array} \right\} - \sum_{j \in \Gamma_3^*} \max(0, \sigma \mu_k^{(j)}) < 0.$$

This contradicts the fact that x^* is a local minimizer of F , so establishing the inequalities (3.3).

Finally suppose x^* is a strict local minimizer of F . Then exactly as above one establishes conditions (1) and (2). Suppose that condition (3) does not hold. Then there exists an index $k \in \mathcal{A}^*$ which satisfies (3.5) with equality, and a direction s such that $s^T a_k = \sigma$ and $s^T a_i = 0$ for all indices i for which the inequalities in (3.3) are strict. One can then choose a subset Λ of \mathcal{A}^* such that $k \in \Lambda$, the vectors a_i , $i \in \Lambda$ form a basis for $\mathcal{R}(A)$, and $s^T a_i = 0$ for $i \in \Lambda/\{k\}$ implies $s^T a_i = 0$ for all the indices i for which the inequalities (3.3) are strict. Then following the proof above one has a direction s such that $F'(x^*; s) = 0$, contradicting the fact that x^* is a strict local minimizer of F . \square

An immediate consequence of the inequalities (3.3) is that

$$\begin{aligned} |\lambda_i| &\leq 1, & i \in \Gamma_1^*, \\ -1 &\leq \lambda_i \leq 0, & i \in \Gamma_2^*, \\ \sum_{j \in \Gamma_3^*} |\mu_i^{(j)}| &\leq \begin{cases} 2, & i \in \Gamma_1^*, \\ 1, & i \in \Gamma_2^*. \end{cases} \end{aligned}$$

This theorem includes the degenerate situation when the vectors a_i , $i \in \mathcal{A}^*$ are linearly dependent, in which case the multipliers λ and $\mu^{(j)}$ are not uniquely determined.

4. A reduced gradient algorithm. As F is nonconvex, the censored l_1 approximation problem cannot be posed as a linear programming problem, as the discrete linear l_1 approximation problem can. Thus this section presents a direct descent method based on the reduced gradient method for l_1 approximation. A detailed discussion of the reduced gradient method, and its equivalence to the modified simplex algorithm of Barrodale and Roberts [1], can be found in Osborne [6]. An equally viable alternative, which is not considered here, is to develop a projected gradient algorithm based on the work of Bartels, Conn and Sinclair [2] for the l_1 problem. Osborne [6] also gives a comparison of the reduced and projected gradient methods for linear programming and discrete linear l_1 approximation. Note that the distinction between the reduced and projected gradient algorithms is only of importance whilst a complete active set (one with n elements) is being built up.

Let vectors a_i , $i = 1, \dots, m$ have rank m , (in most practical situations $m_r = n$ as $m \gg n$). The basic idea is to generate a sequence of points satisfying sets of equations $r_i(x) = 0$ for $i \in \mathcal{M}$ (an approximation to \mathcal{A}^*). The number of elements t in \mathcal{M} increases until $t = m_r$, and thereafter one element at a time is changed until $\mathcal{M} = \mathcal{A}^*$ and the optimality conditions are satisfied. At each point x a step is made in a direction s satisfying $F'(x; s) \leq 0$ (usually $F'(x; s) < 0$) and $r_i(x + as) = 0$ for all or all but one $i \in \mathcal{M}$.

Let \mathcal{M} be an index set with t elements such that $\mathcal{M} \subseteq \mathcal{A}$, and let

$$A = [a_i : i \in \mathcal{M}] = [a_{\mathcal{M}(1)} a_{\mathcal{M}(2)} \dots a_{\mathcal{M}(t)}].$$

It will be shown later that, because of the way elements are added to \mathcal{M} , A always has full rank. Define the $n \times n$ matrix B by

$$B = [A|E],$$

where E comprises the columns of the $n \times n$ identity matrix chosen to make B nonsingular. Let

$$H = \text{span} \{a_i: i \in \mathcal{M}\}.$$

The reduced gradient algorithm is based on the fact that the vectors $B^{-T}e_j$ for $j = t+1, \dots, n$, where $e_j \in \mathbb{R}^n$ is the j th unit vector, form a basis for the orthogonal complement of H , and so provide a suitable basis from which to choose a search direction.

Using the above notation a typical iteration of the basic algorithm can now be given. Each step will then be discussed along with possible improvements. The implementation of the algorithm along with its numerical properties is considered in § 5. The sets Γ_i , $i = 1, \dots, 5$ are defined by (2.3).

Step 1. Calculate

$$(4.1) \quad g = \sum_{i \in \Gamma_4} \theta_i a_i,$$

where $\theta_i = \text{sign}(r_i(x))$ and $r_i(x) = \max(y_i, z_i) - x^T a_i$.

Step 2. Calculate multipliers: solve

$$(4.2) \quad Bu = g \quad \text{for } u,$$

and

$$(4.3) \quad B_{v^{(j)}} = a_j \quad \text{for } v^{(j)}, j \in \Gamma_3.$$

Step 3. Check optimality

a) Calculate

$$(4.4) \quad \delta_i(\sigma) = -\sigma u_i + \sum_{j \in \Gamma_3} \max(0, \sigma v_i^{(j)}) - \eta_i(\sigma), \quad i = 1, \dots, n,$$

where $\sigma = +1$ and -1 , and

$$(4.5) \quad \eta_i(\sigma) = \begin{cases} 1 & \text{if } \mathcal{M}(i) \in \Gamma_1 \text{ and } 1 \leq i \leq t, \\ \max(0, \sigma) & \text{if } \mathcal{M}(i) \in \Gamma_2 \text{ and } 1 \leq i \leq t, \\ 0 & \text{if } t+1 \leq i \leq n. \end{cases}$$

$$(4.6) \quad \gamma_i = \max(\delta_i(1), \delta_i(-1)), \quad i = 1, \dots, n,$$

$$(4.7) \quad p = \text{argmax} \{ \gamma_i, i = 1, \dots, t, \tau \gamma_i, i = t+1, \dots, n \},$$

$$(4.8) \quad \sigma_p = \begin{cases} 1 & \text{if } \gamma_i = \eta_i(1), \\ -1 & \text{if } \gamma_i = \eta_i(-1) \end{cases}$$

If $\gamma_p > 0$ go to step 4.

If $t = n$ STOP: Rank n termination.

b) If $t < n$ and $J = \{i \in 1, \dots, m: y_i > z_i > x^T a_i\} \neq \emptyset$ set $j = J(1)$ and

$$(4.9) \quad \text{(i) Solve } B\bar{v}^{(j)} = a_j.$$

(ii) Let

$$(4.10) \quad p = \text{argmax} \{ |\bar{v}_i^{(j)}|, i = t+1, \dots, n \},$$

$$(4.11) \quad \sigma_p = \text{sign}(\bar{v}_p^{(j)}).$$

(iii) If $|\bar{v}_p^{(j)}| > 0$ go to step 4

Otherwise select next $j \in J$ and go to (i).

If $\bar{v}_i^{(j)} = 0 \forall i = t+1, \dots, n$ and $\forall j \in J$ STOP: Rank deficient termination.

Step 4. Compute search direction

$$(4.12) \quad s = \sigma_p B^{-T} e_p.$$

Step 5. Line search

$$(4.13) \quad I = \{i \in 1, \dots, m: i \notin \mathcal{M}, r_i(x)(a_i^T s) > 0\},$$

$$(4.14) \quad \alpha_i = \frac{r_i(x)}{a_i^T s} \quad \text{for } i \in I.$$

$$(4.15) \quad q = \operatorname{argmin} \{F(x + \alpha_i s), i \in I\}.$$

Step 6. Update $x_{\text{new}} = x + \alpha_q s$.

If $p \leq t$ then $\mathcal{M}_{\text{new}} = \mathcal{M} + \{q\} - \{p\}$
 otherwise $\mathcal{M}_{\text{new}} = \mathcal{M} + \{q\}$.

Update B and go to Step 1.

Given a starting point x the algorithm is initialized by setting $\mathcal{M} = \emptyset$, so $t = 0$ and $B = I_n$ (the $n \times n$ identity matrix). Step 1 calculates the gradient of the component functions $|y_i - \max(z_i, x^T a_i)|$ which are smooth at x , whilst step 2 calculates the multipliers u and $v^{(j)}$ for $j \in \Gamma_3$. The usual situation is that Γ_3 is empty, in which case steps 2 and 3 simplify considerably. Step 3 then checks the optimality conditions (3.2) and (3.3), and if they are not satisfied calculates the index p by an unnormalized steepest edge test so that the direction s in step 4 is a descent direction. Step 5 then calculates the next point by a line search along s , where from Lemma 3 the minimum is known to lie at one of a finite number of points.

Step 3 needs further discussion. Consider the nondegenerate case where $\mathcal{M} = \mathcal{A}$, that is $\mathcal{A}(x^{(0)}) = \emptyset$ where $x^{(0)}$ is the starting point and the index q calculated in step 5 is uniquely determined. For the search directions $s^{(i)}$ defined by

$$(4.16) \quad s^{(i)} = \sigma B^{-T} e_i, \quad i = 1, \dots, n,$$

where $\sigma = \pm 1$, (2.8), (4.2)-(4.5) yield

$$F'(x; s^{(i)}) = -\delta_i(\sigma) \quad \text{for } i = 1, \dots, n.$$

If the index p is chosen by

$$p = \operatorname{argmax} \{\gamma_i, i = 1, \dots, n\},$$

with σ defined by (4.11), then the search direction $s^{(p)}$ minimizes $F'(x; s^{(i)})$ over $i = 1, \dots, n$ and $\sigma = \pm 1$. If $\gamma_p > 0$, then $F'(x; s^{(p)}) < 0$ so $s^{(p)}$ corresponds to the steepest descent edge direction.

The first part of the optimality conditions is that there exist multipliers satisfying (3.2), that is $g \in H$ and $a_j \in H$ for all $j \in \Gamma_3$. This is true if and only if

$$u_i = 0 \quad \text{and} \quad v_i^{(j)} = 0 \quad \forall j \in \Gamma_3, \quad \text{for } i = t+1, \dots, n,$$

or equivalently if and only if

$$\gamma_i = 0, \quad i = t+1, \dots, n.$$

Note that $\gamma_i \geq |u_i|$ for $i = t+1, \dots, n$ with equality if and only if $\Gamma_3 = \emptyset$. The factor τ in (4.7) is a positive weight (typically $\tau = 100$) to ensure that the columns of E are favoured for deletion until $\gamma_i, i = t+1, \dots, n$ are all small or $t = n$ (the usual situation). This permits relaxing off active equations $r_i(x) = 0$ when a very large negative directional derivative would result, before (3.3) is satisfied.

If $t = n$ and $\gamma_i < 0$, $i = 1, \dots, n$ then the sufficient conditions are satisfied and x is a strict local minimizer of F . However if $t < n$ then x can be a local minimizer, but there exists a nearby point with the same function value and at which there is a descent direction. This is only likely to occur with relatively small values of m or poor choices of starting point, a typical example being any point $x < 0$ in Fig. 4.

Step 3b) thus checks to see if there are any directions along which a line search will result in an increase in t without increasing F . This ensures that $t = m_r = \text{rank}(a_i, i = 1, \dots, m)$, so excluding certain types of nonglobal local minima. Let

$$J = \{j \in 1, \dots, m: y_i > z_j > x^T a_j\},$$

and for every $j \in J$ let

$$K^{(j)} = \{i \in t+1, \dots, n: a_j^T s^{(i)} \neq 0\}.$$

For any $i \in K^{(j)}$ a line search in the direction $s^{(i)}$ defined in (4.16) with the sign σ chosen so that $a_j^T s^{(i)} > 0$ will increase t without increasing F . For $i \in K^{(j)}$ let

$$(4.17) \quad \beta^{(i)} = \min \left\{ \frac{z_j - x^T a_j}{a_j^T s^{(i)}}, j \in J \right\},$$

where σ is always chosen so that $a_j^T s^{(i)} > 0$. If

$$\beta^{(i)} < \min \{ \alpha_i: i \in I(x; s^{(i)}) \},$$

where I is defined by (3.1), then as $t+1 \leq i \leq n$ $F'(x; s^{(i)}) = 0$, moreover

$$F(x + \alpha s^{(i)}) = F(x) \quad \forall \alpha \in [0, \beta^{(i)})$$

and

$$(4.18) \quad F'(x + \beta^{(i)} s^{(i)}; s^{(i)}) = a_k^T s^{(i)} < 0,$$

where k is the index which achieves the minimum in (4.17) (if k is not uniquely defined, there will simply be more negative contributions to (4.18)). As the set J can be large, step 3b) simply finds the first $j \in J$ for which $K^{(j)}$ is nonempty. The search direction chosen is that which would give the most negative directional derivative (4.18).

Finally the updating of B must be considered. Let

$$\bar{B} = B + [a_q - B e_p] e_p^T.$$

In order that the structure of B is preserved

$$B_{\text{new}} = \bar{B} P,$$

where if $t+1 \leq p \leq n$ then P is the permutation matrix which corresponds to swapping the p th column of \bar{B} with the $(t+1)$ st. Otherwise P is the identity matrix.

LEMMA 4. *If B is nonsingular, then B_{new} is nonsingular.*

Proof. B_{new} is nonsingular if and only if \bar{B} is nonsingular. Now

$$\bar{B} = B(I + [B^{-1} a_q - e_p] e_p^T).$$

The result follows as

$$\det(I + [B^{-1} a_q - e_p] e_p^T) = 1 - e_p^T (B^{-1} a_q - e_p) = e_p^T B^{-1} a_q = \text{sign}(u_p) s^T a_q \neq 0. \quad \square$$

As B is initially the identity matrix, all matrices B are nonsingular.

In the nondegenerate case the above algorithm is finite, as on every iteration either F decreases or if F does not decrease t (the number of elements in \mathcal{M}) increases. Thus

as t never decreases, the sets \mathcal{M} never repeat. As there are only a finite number of possible sets \mathcal{M} the algorithm must terminate in a finite number of iterations.

The degenerate case arises if \mathcal{M} is a strict subset of \mathcal{A} , in which case the direction s generated in steps 3a) and 4 may not be a descent direction. Some allowance for this can be made by changing the definition of I in (4.13) to

$$(4.19) \quad I = \{i \in 1, \dots, m: i \notin \mathcal{M}, s^T a_i \neq 0 \text{ and } r_i(x)(s^T a_i) \geq 0\}.$$

It is then possible for (4.14) and (4.15) to produce $\alpha_q = 0$, so that an element in \mathcal{M} is exchanged without F decreasing. In this case it is theoretically possible for the sets \mathcal{M} to cycle. However the various techniques available for resolving degeneracy in linear programming can be extended to remove this difficulty.

Some remarks should be made on the use of the unnormalized steepest edge tests (4.4)–(4.7) used to calculate a descent direction. The use of (4.7) assumes that γ_i , $i = 1, \dots, n$ are of comparable magnitude, and calculates p by computing a minimum over unnormalized directional derivatives $F'(x; s)$. If these are normalized so that $\|s\| = 1$ one obtains

$$(4.20) \quad p = \operatorname{argmax} \left\{ \frac{\gamma_i}{\|B^{-T} e_i\|} \quad i = 1, \dots, t, \frac{\tau \gamma_i}{\|B^{-T} e_i\|} \quad i = t+1, \dots, n \right\}.$$

This test can be computed economically by setting up a recursion for the quantities

$$(4.21) \quad \chi_i = \|B^{-T} e_i\|^2, \quad i = 1, \dots, n.$$

One obtains

$$(4.22) \quad \chi_i^{\text{new}} = \chi_i - 2\zeta_{ip}\phi_i + \zeta_{ip}^2 \chi_p$$

where

$$(4.23) \quad \phi = B^{-1}(B^{-T} e_p), \quad \psi = B^{-1} a_q \quad \text{and} \quad \zeta_{ip} = (\psi_i - \delta_{ip})/\psi_p$$

where δ_{ip} is the Kronecker delta. A complete discussion of the relative merits of unnormalized and normalized steepest edge tests in the context of linear programming and l_1 approximation can be found in Osborne [6].

Although conceptually one of the simplest, step 5 of the algorithm, the line search, is one of the most time-consuming steps for large m , as the number of elements in I in (4.13) is often a significant fraction of m (see § 5 for more comments). A more efficient line search algorithm can be developed as one knows the points at which the slope of $F(x + \alpha s)$ changes and the amount by which it changes. These are the points α_i , $i \in I$ given by (4.14) and (4.19) where a minimum may occur, and at which the slope changes by

$$(4.24) \quad \Delta g(\alpha_i) = \begin{cases} 2|s^T a_i| & \text{if } y_i > z_i \\ |s^T a_i| & \text{if } y_i \leq z_i \end{cases}$$

The only other points where the slope changes are given by

$$(4.25) \quad \beta_i = \frac{z_i - x^T a_i}{s^T a_i} \quad \text{for } i \in \tilde{I},$$

where

$$(4.26) \quad \tilde{I} = \{i \in 1, \dots, m: y_i > z_i \text{ and } (z_i - x^T a_i)s^T a_i > 0\}.$$

At these points the change in slope is given by

$$(4.27) \quad \Delta g(\beta_i) = -|s^T a_i|.$$

A local minimizer along the line is characterized by the slope changing from nonpositive to positive upon passing through that point. Let n_I denote the number of elements in the set I . The partition sort techniques of Clark and Osborne [4] can be used to find an α_i corresponding to a local minimum of $F(x + \alpha s)$ in $O(n_I)$ operations. This has the disadvantage that the algorithm is then slightly more likely to converge to a local minimum rather than a global minimum of F . Alternatively if one requires the global minimum along the line, as in step 5 of the algorithm, the $\alpha_i, i \in I$ must be completely sorted. This can be done by a QUICKSORT algorithm taking $O(n_I \log n_I)$ operations. In either case one only needs a partition sort of the $\beta_i, i \in I$, as for any $k \in I$

$$(4.28) \quad F'(x + \alpha_k s; s) = -\gamma_p + \sum_{i: \alpha_i \leq \alpha_k} \Delta g(\alpha_i) + \sum_{i: \beta_i \leq \alpha_k} \Delta g(\beta_i).$$

5. Implementation and numerical experience. Osborne [6] gives a convenient tableau representation for relatively small dense l_1 problems, which can be extended to censored l_1 problems in the following way.

$$(5.1) \quad W = [I_n | A_F],$$

where

$$A_F = [a_1, a_2, \dots, a_m],$$

and let

$$w^{(k)T} = [x^{(k)T} | r^{(k)T}],$$

where $x^{(k)}$ is the point on the k th iteration and $r^{(k)}$ is the corresponding vector of residuals $r_i^{(k)} = \max(y_i, z_i) - a_i^T x^{(k)}$. Also let

$$B^{(k)} = L^{(k)} U^{(k)},$$

where $L^{(k)}$ is unit lower triangular and $U^{(k)}$ is upper triangular. Define

$$W^{(k)} = L^{(k)-1} W.$$

The key point is that at each stage one works with the tableau $W^{(k)}$. The multipliers $u^{(k)}$ satisfying (4.2) are calculated by solving

$$U^{(k)} u^{(k)} = g^{(k)}$$

by backsubstitution, where

$$g^{(k)} = \sum_{i \in \Gamma_4^{(k)}} \theta_i^{(k)} L^{(k)-1} a_i.$$

Note that the vectors $L^{(k)-1} a_i$ are just certain columns of $W^{(k)}$. Also as $U^{(k)} = L^{(k)-1} B^{(k)}$, the columns of $U^{(k)}$ are simply those columns of $W^{(k)}$ which correspond to the columns of W forming $B^{(k)}$. The vectors $v^{(j,k)}$ satisfying (4.3) are similarly given by a backsubstitution to solve

$$U^{(k)} v^{(j,k)} = L^{(k)-1} a_j.$$

To calculate the search direction, one computes

$$\hat{s}^{(k)} \equiv \sigma_p U^{(k)-T} e_p = L^{(k)T} s^{(k)},$$

which involves a forward substitution with special structure from the unit vector e_p . Then

$$\Delta w^{(k)T} \equiv \hat{s}^{(k)T} W^{(k)} = s^{(k)T} W = [s^{(k)T} | s^{(k)T} A_F].$$

The line search (4.13)–(4.15) requiring $r_i^{(k)} \equiv w_{n+i}^{(k)}$ and $a_i^T s^{(k)} \equiv \Delta w_{n+i}^{(k)}$ provides the steplength $\alpha_q^{(k)}$ and the column a_q to be pivoted into $B^{(k)}$. The vector $w^{(k)}$ is updated by

$$w^{(k+1)} = w^{(k)} + \alpha_q^{(k)} \Delta w^{(k)}.$$

The factors of $B^{(k)}$ are updated by the Bartels–Golub scheme [3]. $W^{(k)}$ must be updated when the p th column of $U^{(k)}$ is deleted and $L^{(k)-1} a_q$ is added. After the appropriate column shifts one has

$$\bar{U}^{(k)} = [U_1^{(k)} \cdots U_{p-1}^{(k)} U_{p+1}^{(k)} \cdots U_n^{(k)} L^{(k)-1} a_q].$$

The subdiagonal elements $\bar{U}_{i+1,i}^{(k)}$ for $i = p+1, \dots, n$ are zeroed by row operations to produce $U^{(k+1)}$. If necessary rows are interchanged to ensure that the multipliers in the elimination do not exceed one in magnitude. This ensures there is no unnecessary growth of rounding error. It also means that $L^{(k)-1}$ can lose its lower triangular structure, which causes no difficulties as $L^{(k)}$ is never explicitly used.

The quantities (4.22) and (4.23) required to implement the normalized steepest edge test (4.20) can also be efficiently calculated. The details are omitted as the following numerical results refer to the basic algorithm with the unnormalized steepest edge test (4.4)–(4.8) and the basic line search on function values given by (4.13)–(4.15).

To test the numerical performance of the algorithm, some pseudo-random censored estimation problems are generated by the following procedure. A vector \tilde{x} is produced by taking $\tilde{x}_i \in \mathcal{R}[a, b]$, $i = 1, \dots, n$, where $\mathcal{R}[a, b]$ generates a sequence of numbers uniformly distributed on $[a, b]$. The input vectors a_i , $i = 1, \dots, m$ are generated in a similar manner. With error terms $\varepsilon_i \in \mathcal{R}[a_e, b_e]$, the observed data values y_i are then generated by (1.2). The algorithm described in § 4 is then used to calculate a local minimizer of (1.1) with $z_i = 0$, $i = 1, \dots, m$.

For each value of n and m 10 different problems were generated, and for each problem the algorithm was applied from 10 different starting points generated by $x_i^{(0)} \in \mathcal{R}[a, b]$. The results with $a = -10$, $b = 10$, $a_e = -5$ and $b_e = 5$ are collected in Table 1. The reported figures are the minimum number of iterations, the median number of iterations and the maximum number of iterations taken to converge to a local minimum. A figure in brackets indicates the number of nonglobal local minima found and the number of times the algorithm converged to a nonglobal local minimum out of the total of 100 runs for each value of n and m . Only problems with $m > n$ were solved, whilst time restrictions limited the results available for the larger values of n and m .

If the problem has a nonglobal local minimum, then it was usual for several of the starting points to produce convergence to it. Unfortunately there seems to be no way of verifying that a point is a global minimizer, even for the very special case where $a_i \geq 0$ and $y_i \geq z_i$ for $i = 1, \dots, m$.

It was previously remarked that the line search (4.13)–(4.15) based on function values is expensive as the number of points α_i to be checked is usually a significant fraction of m . For the problems in Table 2 the average number of points in the line search is around $m/2$. Thus the line search requires $O(m^2)$ operations. For large m the rest of the algorithm requires $O(nm)$ operations, which is dominated by the line search. Hence a more efficient line search based on a sorting algorithm and the changes

TABLE 2
Results for some random censored estimation problems.

$m \backslash n$	2	5	10	15	20	25
10	2 3 5	5 6 9 (7, 14)				
20	2 4 7	5 8 13 (1, 2)	10 13 18 (14, 24)	15 15 20 (39, 43)		
40	2 4 7	6 11 21 (1, 4)	13 20 28	19 25 29 (2, 4)	20 28 46 (32, 38)	25 27 39 (62, 67)
60	2 4 7	6 13 21 (4, 18)	18 24 34 (2, 4)	21 33 44 (2, 4)	25 39 54 (5, 18)	31 43 71 (12, 21)
80	2 4 8 (1, 2)	8 13 20 (3, 12)	18 27 37 (1, 8)	27 38 49 (1, 2)	32 46 63 (4, 13)	41 50 69 (3, 10)
100	2 4 9 (1, 1)	8 14 25 (3, 15)	20 29 41	29 41 58 (2, 7)	38 52 71 (3, 6)	45 59 79 (1, 1)
200	2 5 10	11 18 27 (2, 6)	22 36 51 (2, 12)	37 52 66 (3, 8)	51 69 90 (1, 3)	72 90 113 (2, 8)
400	3 7 12	11 20 36	31 43 59 (1, 3)	48 65 89 (1, 3)	68 87 113 (2, 14)	80 103 126 (4, 14)
600	3 6 11	9 22 30	32 45 63 (1, 3)	55 70 93 (1, 2)	74 95 116 (1, 1)	
800	2 6 12	14 23 32 (1, 2)	33 49 70 (9, 23)			
1000	3 7 10	13 23 33 (1, 4)				

(4.24) to (4.28) in the directional derivative, which would require $O(m)$ or $O(m \log m)$ operations, should be used.

Finally to check consistency ($x_m^* \rightarrow \tilde{x}$ as $m \rightarrow \infty$, where x_m^* is the global minimizer of (1.1)) and \sqrt{m} -consistency ($\sqrt{m} \|x_m^* - \tilde{x}\|$ is bounded in probability) the average values of $\|x_m^* - \tilde{x}\|$ for $n = 5$ (which is typical of the other values of n) for the 10 different problems of Table 2 are listed in Table 3.

TABLE 3
Average values of $\|x_m^* - \tilde{x}\|$ for $n = 5$.

m	10	20	40	60	80	100
$\ x_m^* - \tilde{x}\ $	4.324	.679	.389	.277	.290	.255
$\sqrt{m} \ x_m^* - \tilde{x}\ $	13.7	3.04	2.46	2.15	2.59	2.55
m	200	400	600	800	1000	
$\ x_m^* - \tilde{x}\ $.179	.147	.130	.099	.095	
$\sqrt{m} \ x_m^* - \tilde{x}\ $	2.53	2.94	3.18	2.80	3.00	

Finally Table 4 gives the results for the motorette example given in § 1. For this problem the function (1.1) has a nonunique global minimizer, so the points listed in Table 4 are the global minimizers x^* characterized by $r_i(x^*) = 0$ for $i \in \mathcal{A}(x^*)$. From all starting points tried the algorithm converged to one of these points in 2 or 3 iterations. Note that observations with identical data values z_i , y_i and a_i were grouped together to minimize the effects of degeneracy. The iterated least squares solution

reported in [11] is $x^* = (-5.818, 4.204)^T$, whilst the maximum likelihood solution is $x^* = (-6.027, 4.314)$. Both these points are very close to the convex hull of the points in Table 4.

TABLE 4
Global minima for motorette example.

\mathcal{A}^*	x_1^*	x_2^*
1, 6	-3.386	3.086
1, 7	-.967	2.062
6, 13	-4.578	3.615
6, 15	-5.054	2.826
6, 16	-4.855	3.737
7, 15	-6.022	4.303
7, 16	-5.822	4.214
13, 15	-5.371	3.982
13, 16	-5.039	3.828

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