

Robust Design: Uncertainty Representation and Propagation

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by

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Certificate

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Abstract

Robust design has been gaining wide attention, and its applications have been extended to making reliable decisions when designing complex engineering systems in a multidisciplinary design environment. Based on its fundamental principle, i.e., improving the quality of a product by minimizing the effects of variation without eliminating the causes, robust design has become one of the powerful tools to assist designers to make reliable decisions under uncertainty. The methods for robust design have progressed from the initial Taguchi's parameter design method to recent nonlinear programming methods that formulate robust design problems as nonlinear optimization problems with multiple objectives subject to feasibility robustness. Uncertainty identification and consistent modeling are prerequisites to performing robust design studies. Only very recently, the scientific and engineering community has begun to recognize the utility of defining multiple types of uncertainty. A brief introduction to the two different types of uncertainty, viz, aleatory and epistemic is given. A proper distinction between them has been made and the necessity of treating them separating has been discussed. Suggestions for use of interval analysis for propagation of ignorance (epistemic), and probability theory for propagation of variability (aleatory) has been made with representative examples. However, in most realistic problems, we must be able to deal with both types of uncertainties together. The Dempster-Shafer Evidence theory showed much promise to be considered as the unified theory to propagate both the aleatory and epistemic uncertainty satisfactorily.

Keywords:

- Robust Design
- Uncertainty
- Aleatory
- Epistemic
- Probability Theory
- Interval Analysis
- Dempster-Shafer Evidence Theory

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Nomenclature

ASDL	Aerospace System Design Laboratory
bpa	Basic probability assignment
CFD	Computational Fluid Dynamics
CDF	Cumulative Density Function
DOE	Design of Experiment
FEA	Finite Element Analysis
GaTech	Georgia Institute of Technology
GCDF	Generalized Cumulative Distribution Function
HAVOC	Hypersonic Aircraft Vehicle Optimization Code
MDO	Multi-Disciplinary Optimization
NASA	National Aeronautics and Space Administration
PDF	Probability Density Function
PhD	Doctorate of Philosophy
RBRD	Reliability-Based Robust Design
RSE	Response Surface Equation
RSM	Response Surface Methodology
SSTO	Single-Stage-To-Orbit
QED	Quantum Electro-Dynamics
A, B, \dots	Parameters or Evidences
\bar{A}	Complement of A
$Bel(.)$	Belief measure
$E(.)$	Expected value or Mean
\mathbf{e}	Error or Deviation
$F(.)$	Distribution function
$\mathbf{F}(\cdot)$	Simulation model
$m(.)$	Basic probability assignment measure

N_1, N_2, \dots	No of samples
$Nec(.)$	Necessity measure
n	Total no of sample
$P(.)$	Power Set
$prob(.)$	Probability measure
p	Probability of some event
$Pl(.)$	Plausibility measure
$Pos(.)$	Possibility measure
$var(.)$	Variance measure
Ω	Universal set
x	No of successes
$\mathbf{x, y, z}$	Design vector
$\Delta(.)$	Range of design vector
μ	Mean
σ	Standard Deviation

Chapter 1

Introduction

“If a man will begin with certainties, he shall end in doubts; but if he will be content to begin with doubts, he shall end in certainties.” - Sir Francis Bacon, English Renaissance writer.

Computational modeling and simulations have become essential elements in engineering. Recent advances in computational simulations have been supported by improvements in computational modeling and numerical algorithms, which were made affordable by growing computer power and software technology [1]. Much of engineering analysis and design today is performed employing computer simulation models of ranging fidelity. Effective utilization of computational simulation in place of expensive and time-consuming experimental tests enables engineers to achieve better designs with reduced cost and design cycle time. To exercise these models intelligently and eliminate the burden of manual iteration, manipulating inputs and reviewing outputs, optimization strategies are applied in a simulation-based design environment to search for designs that minimize or maximize design goals or objectives while satisfying all design requirements or constraints. Since most important decisions are made in the early phases of the design process when much of the key information is uncertain, there are increasing needs to use high-fidelity analyses in the very early stages [1].

In multidisciplinary design optimization (MDO), the design task is approached as an optimization problem by considering various disciplines simultaneously. It is a systematic approach to exploit the interactions between different disciplines in the early design phases. Efforts to use higher fidelity tools, such as computational fluid dynamics (CFD) and finite element analysis (FEA), in MDO are being actively pursued. Optimization algorithms, however, tend to push a design up to one or more constraint boundaries, leaving the designer with a design for which even slight uncertainties in the problem formulation or changes in the operating environment could produce failed, unsafe designs, and/or result in substantial performance degradation [2].

In reality, the knowledge that engineers have about the design problem is imperfect and incomplete. Very few real engineering problems are void of uncertainty; variation is inherent in material characteristics, loading conditions, simulation model accuracy, geometric properties, manufacturing precision, actual product usage, etc [2]. Uncertainty is ubiquitous in measurement data, simulation models, design parameters, and the operational environment of the product.

The traditional way of dealing with the uncertainties is to use conservative values of the uncertain quantities in the framework of deterministic design. Afterwards, safety factors are introduced to address the unavoidable uncertainties in the model and environments [1, 3, 4]. Also, in the traditional design optimization, engineers try to single out the best design point, i.e.,

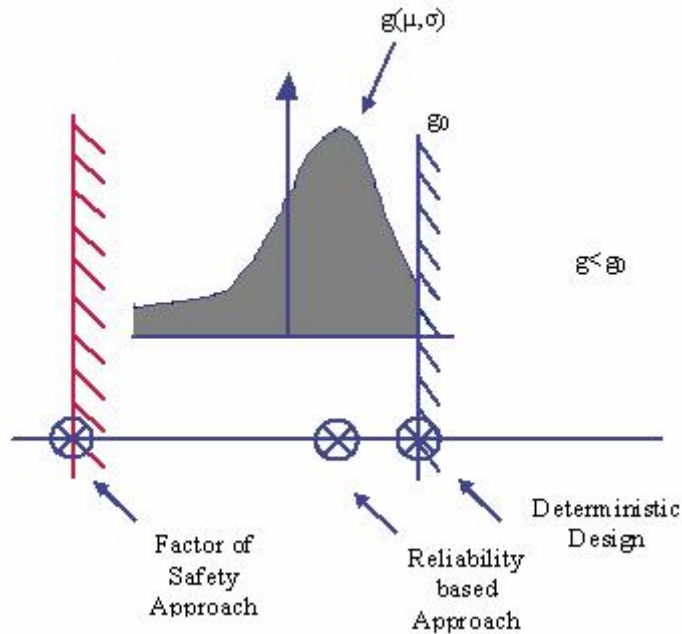


Figure 1.1: Pictorial representation of Deterministic design, Factor of Safety approach and Reliability approach

a search for the global optimum. Then, post optimality analysis is done to have sensitivity information with respect to design changes. A more rigorous treatment of the uncertainties can be found in reliability-based design philosophies that have been under development for the last half of a century and are gaining more and more momentum. More recently, the concept of “*robust design*” has become very popular. In recent years, probabilistic design analysis and optimization methods have been developed to address uncertainty and randomness through statistical modeling and probabilistic analysis, allowing the identification of designs that qualify as not only feasible, but as consistently feasible in the face of uncertainty.

1.1 Robust Design: a new design paradigm

Robust Design method is central to improving engineering productivity. Pioneered by Dr. Genichi Taguchi after the end of the Second World War, the method has evolved as a new design paradigm over the last five decades [5, 6]. There are many different opinions as to the meaning of robust design. A commonly used definition states that a robust design is a design that is insensitive (or less sensitive) to input variations. In other words, the best design is one which performs as expected in the face of both expected and unexpected variations; and it does so by virtue of the fact that the design is inherently insensitive to changes in the design parameters and service environment. Based on this definition, regions in the design space should be sought where the sensitivity of the important response quantities with respect to the key input variables is low (or ideally zero). While it is an attractive and powerful design concept, in practice it may be difficult (if not impossible) to achieve. For example, designs that are insensitive to all variations may be overly conservative and costly. Also, it should be remembered that designs which are not sensitive to key random variables cannot be “improved” by making changes to the mean values of those variables. This design characteristic may, in fact, be undesirable in many situations. In practice, what we are really concerned with is making sure that expected variations do not result in unacceptable performance; and among such designs the most desirable design is the one that is least sensitive to unexpected variation caused, for instance, by unintended use of the product or lack of knowledge of the uncertainties. Therefore, a more practical definition for robustness may be that a robust design is a design whose performance is not unacceptably compromised by expected variations in parameters which are known to affect its performance, and is more tolerant to unexpected variations [7].

Both robust design and reliability-based design try to deal with uncertainties. Their differences and similarities are difficult and probably unnecessary to describe, since there is no universal agreement on the definition and practice of the robust design philosophy, and the range of reliability theory is ever expanding [4]. In reliability-based design, all uncertain quantities are modeled as random variables (or processes if variation in time is important). If the statistical distributions of the input random variables are well established (i.e. when all uncertainties or variabilities are reducible (or expected)) then all of the uncertainties have been counted for in the design process and the result of reliability-based design would be robust by the more practical definition of robustness. When the distributions of the input random variables contain

uncertainty due, for example, to lack of data or unintended usage, the safety index or computed reliability will be subject to error. For example, if the mean value and standard deviations of the parameter was obtained with only few samples, we would suspect that they may not be the true means and standard deviations of the force. When more samples become available, we may find that the more refined mean value and standard deviation. The question is how to ensure the robustness of the reliability-based design when distributions of the random variables contain uncertainties. One method in reliability theory to address such problems is to model the input uncertainties using random variables with random means and/or standard deviations. The problem with this method is that another layer of uncertainties may be introduced when defining the distribution of the mean of a random variable. Instead of using this approach in this study, we will look into the different representation of uncertainties of the input random parameters by using the concept of robust design. In the context of reliability-based design, a definition of robustness that can be translated directly into a design criterion is that a robust design is one that is least sensitive to the change in the statistics of the input random variables within acceptable range of cost. We can call the design philosophy based on such a definition reliability-based robust design (RBRD) [4].

The structure of this report is as follows. The second chapter looks into some of the efforts made in the field of robust design starting from the initial Taguchi method to recent efforts in some leading universities across the world. The chapter 3 gives an introduction to the various types of uncertainties. A proper distinction between the different types has been made along with the necessity to treat them separately has been emphasized. In the fourth chapter, various methods to represent and propagate these uncertainties effectively have been discussed. The final chapter of this report covers conclusion and future works.

Chapter 2

Historical Perspective

Robust design has been gaining wide attention, and its applications have been extended to making reliable decisions when designing complex engineering systems in a multidisciplinary design environment. The methods for robust design have progressed from the initial Taguchi's parameter design method to recent nonlinear programming methods that formulate robust design problems as nonlinear optimization problems with multiple objectives subject to feasibility robustness.

2.1 Taguchi's method:

Dr. Genichi Taguchi, a Japanese Engineer is regarded as the pioneer of robust design methods [5, 6]. Parameter design by Taguchi method is off-line quality control method. Off line quality control methods are quality and cost control activities conducted at the product and process design stages to improve product manufacturability and reliability and to reduce product development and lifetime costs. Dr. Taguchi had a big effect on quality control and experimental design in the 1980s and 1990s. When Japan began its reconstruction efforts after World War II, it faced an acute shortage of good quality raw materials, high quality manufacturing equipment and skilled engineers. The challenge was to produce high quality products and continue to improve the quality under those circumstances. The task of developing a methodology to meet the challenge was assigned to Dr. Taguchi, who at that time was a manager in Nippon Telephone & Telegraph Company. Through his research in the 1950s and early 1960s, Dr. Taguchi developed the foundations of Robust Design and validated its basic philosophies by applying them in the development of many products [5].

Dr. Taguchi suggested that "quality" should be thought of, not as a product being inside or outside of specifications, but as the variation from the target. He proposed that quality should be designed into the product and not inspected into it. [6]. He recommended a three-stage design process, system design, parameter design or robust design followed by tolerance design.

- a. System design deals with the development of a basic functional prototype design, determination of materials, parts and assembly system and determination of the manufacturing process involved.
- b. Robust Design is a technique that reduces variation in a product by reducing the sensitivity of the design of the product to sources of variation rather than by controlling their sources. So, the purpose of the second stage is to find values of the “control parameters” for which performance is insensitive to variations in values of the uncontrollable “noise variables”. If we can design a process that has the robustness to noise factors that largely affects the variance of performance characteristics at a developing stage, it will very possible for the process to have robustness against other noise factors that could not be considered at the development stage. The aim of a parameter design experiment is so, to identify settings of the design parameters that maximize the chosen performance measure and are insensitive to noise factors.
- c. Tolerance Design is concerned with how much variation of the design and noise factors is permissible. It is a method for determining tolerances that minimizes the sum of product manufacturing and lifetime costs. The basic idea is to set tolerances around nominal settings identified by parameter design, not by convention. We have to first identify the control parameters. These are sometimes also called design parameters. They are the product design characteristics whose nominal settings can be specified by the product designers. Next, the problem and objectives are stated. It works best to have a session that includes all the interested parties, and not to work in isolation.

2.1.1 Taguchi's Parameter design method:

The aim of the second stage of Taguchi's method, the parameter design experiment or robust design is to identify settings of the design parameters that maximize the chosen performance measure. The estimation of a prediction equation that is valid over a wide region of the parameter space is clearly not the goal. Taguchi recognizes the presence of interactions among design parameters, but he downplays their importance relative to the main effects in constructing the design matrix. According to Taguchi, when there are limits on the number of test runs, it is better to include many design parameters in the design matrix even until no degrees of freedom are left for estimating the residual error than to include only a few design parameters and allow

for estimating interactions. With large number of design parameters, the number of test runs required to estimate all main effects and all pair interaction can be prohibitively large. A first order model involving only linear terms is often inadequate. The goal of a Taguchi's experimental design is to identify optimal settings for all the design parameter, not to build the model fitting of process. Taguchi has achieved substantial payoffs just by conducting many main effects only experiments and by checking the results by confirmation experiments. While such designs have been criticized due to no estimate of interactions, he is usually successful because main effects almost dominate interaction effects. If it can be proved that the system could be described well by even only main effects, the optimal condition determined by only main effect analysis can be very efficient and simple method for optimization. Orthogonal array has been used to minimize the number of test runs while keeping the pair wise balancing property in Taguchi's method for that purpose. These basic principles serve as a screening filter which allows the examination of the effects of many process variables, identifying those factors which have a major effect on process characteristics using a single trial with a few reactions [5, 6

Taguchi's method for identifying settings of design parameters that maximize performance characteristics (e.g. yield or productivity etc) is summarized below.

1. Identify initial and competing settings of design parameters, and identify important noise factors and their ranges.
2. Construct the design and noise matrices, and plan the parameter design experiments.
3. Conduct the parameter design experiments and evaluate the performance statistic for each test run of the design matrix.
4. Use the values of the performance statistic to predict new settings of the design matrix (if needed).
5. Confirm that the new settings indeed improve the performance statistic.

The design will be planned to determine the control factor's level that is less sensitive to noise factors. An orthogonal array containing the control factors will be arranged in the inner array, while an orthogonal array containing noise factors will be arranged in the outer array. Taguchi suggested that parameter design using noises that are deliberately created was more effective than not, if noises can be created purposely. The reason is that if noise is not induced

deliberately, many experiments must be performed to investigate the effects of noise factors diversely on process and it is very difficult to obtain reliable results under different noise conditions. If the experiments can be performed under various levels of noise i.e. with positive induction of noise to the design, we can obtain a realistic level of robustness. Therefore, a characteristic of Taguchi's parameter design is the deliberate creation of noise for the identification of control factor's level that is the least sensitive to the noises.

2.2 Recent Approaches:

Several new methods have been approached in recent time in the area of robust design. Recent developments in the Aerospace System Design Laboratory (ASDL) of Georgia Institute of Technology have proposed a new approach for design uncertainty modeling and propagation. DeLaurentis [7, 8] in his PhD thesis has provided a mechanism for consistent modeling of uncertainty from any conceivable source. His research has resulted in a design model which begins to address the challenges of uncertainty, high-fidelity analysis, and multidisciplinary synthesis for complex systems which emanate from simulation-based design. This has been found to be an important aspect of modern (and future) aerospace problems, where emphasis on life-cycle disciplines will introduce new uncertainties and require robust solutions.

In addition, a new technique for propagating this uncertainty through system synthesis is also developed to efficiently obtain robust design solutions [7]. Important elements of the new robust design technique include the use of a “probability of success” measure based on the cumulative distribution of the objective function. Also, the substantial cost of constructing these distribution functions at every point in the design space is mitigated by the use of the response surface method to approximate the functions at several discrete points. Once created, these efficient metamodels can be used again and again with different robustness objectives, avoiding the need to re-execute the expensive probabilistic analysis. The challenge problem that he dealt with is *What is the optimum level of stability relaxation corresponding to optimum wing and horizontal tail geometry, size, and position which maximizes robustness in system affordability while meeting stability, handling quality, and control authority constraints at critical points in the flight envelope?* [7]

The following five-step procedure has been developed by DeLaurentis [7]. The procedure utilizes the combined Design of Experiments (DOE)/Response Surface Methodology (RSM) to

relate the Cumulative Density Function (CDF) of a response random to the design variables under a selected uncertainty model. The procedure is also explained pictorially in Figure 3.

1. **Select the desired set of design variables and create a corresponding experimental design table.** Step 1 begins with selection of the design variables to be considered (X), ranges for those variables that define the extent of the design space, and the regression model equation. Next, a DOE table is selected with the resolution required by the model equation. Multiple center points are included in the DOE to quantify experimental error since the presence of uncertainty leads to non-repeatable results for simulation runs using the same design variable settings. The error term, e , in the regression now includes experimental error.

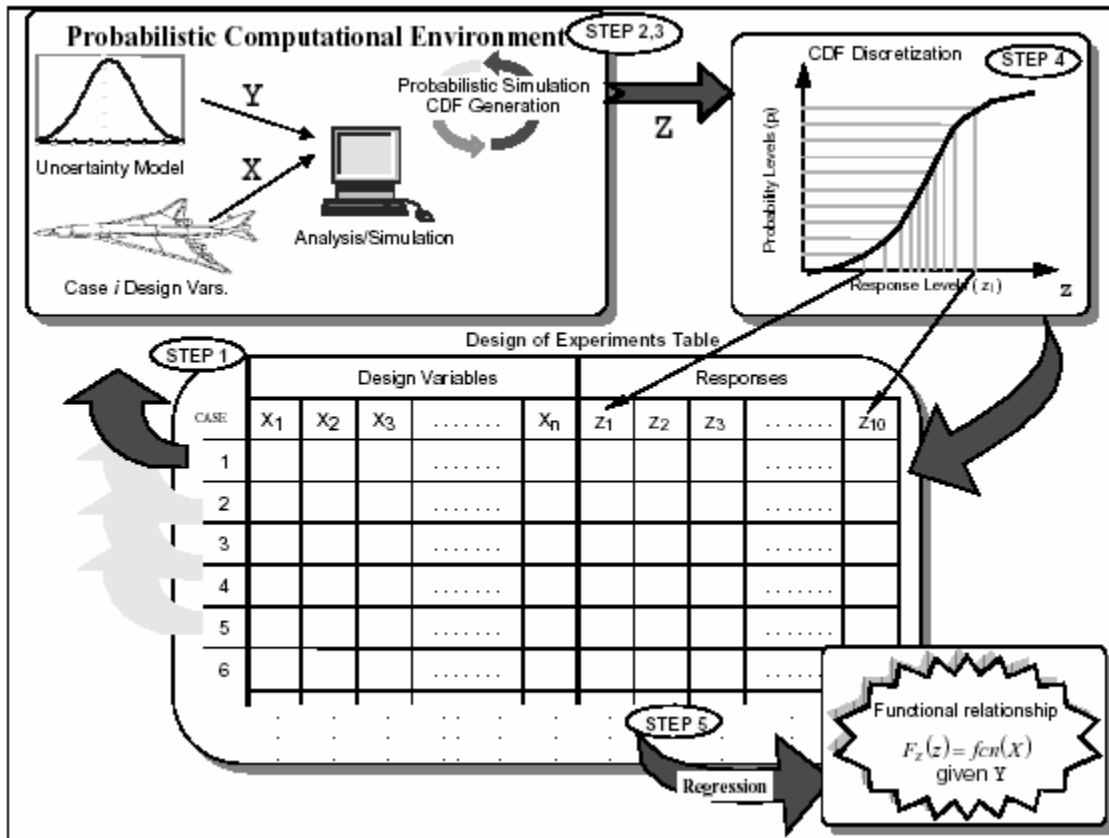


Figure 2.1: Summary of Steps for Generating Parametric CDFs [7]

2. **Construct an uncertainty model for the problem.** Using the generic model shown in Fig. 2.1, Step 2 calls for the establishment of a specific uncertainty model, accomplished by

assigning PDFs to the uncertain parameters represented by vector \mathbf{Y} . This assignment is based on such things as historical data, experimental results, expert opinion, and analysis.

3. **For each row in the experimental design table, generate a CDF for the objective and constraints.** Step 3 consists of the execution of an analysis code that produces response \mathbf{Z} given values for \mathbf{X} and \mathbf{Y} . For each row in the experiment table, the values of \mathbf{X} remain fixed as the PDFs for each random variable in \mathbf{Y} are sampled, culminating in a CDF for \mathbf{Z} .
4. **Discretize each CDF into n_p values corresponding to n_p probability levels for each objective and constraint; these n_p values for each objective and constraint are the responses in the experiment.** In Step 4, the constructed CDFs are discretized into n_p-1 intervals, resulting in n_p particular values of \mathbf{Z} (termed \mathbf{z}_o) and \mathbf{W} (\mathbf{w}_o), as shown in Eqn. (2.1) and Eqn. (2.2). These n_p values become the responses for which response surface equations are to be formed. An appropriate value for n_p depends on the accuracy desired in representing the CDF, though no less than $n_p=5$ is advised.

$$(\mathbf{z}_o)_i \text{ for } P(\mathbf{Z} \leq \mathbf{z}_o) = \frac{i}{n_p}, i=1, \dots, n_p \quad (2.1)$$

$$(\mathbf{w}_o)_i \text{ for } P(\mathbf{w} \leq \mathbf{w}_o) = \frac{i}{n_p}, i=1, \dots, n_p \quad (2.2)$$

5. **Form metamodel (response surface equation) for each response by multiple nonlinear regression.** The procedure culminates in Step 5 with the formation of the desired functional relationships between design variables and the CDFs via regression on the DOE data. For any set of design variable values, the Response Surface Equations (RSE) can be used to reconstruct the CDFs and perform any variety of robust design optimizations. The analysis does not have to be re-executed each time a new problem is to be solved, as long as problem falls within the original design space selected.

The approach developed is tested for validity on a multidisciplinary aircraft design problem involving the use of relaxed static stability technology for a supersonic commercial transport aircraft [8]. Wind tunnel data is used to examine the fidelity of stability and control analysis of the transport. Results of this examination are then used to form uncertainty models related to that analysis. Uncertainty is also modeled for the approximations of this analysis (via metamodels)

used during aircraft synthesis. The flexibility in shaping distributions based on knowledge of the uncertainty is a noted advantage of the approach. Finally, the probabilistic robust design method is exercised resulting in a comparison of robustness results with and without the technology. It is found that modest gains in robustness are obtained with RSS, certainly less substantial than the deterministic performance gains one would expect. The efficiency of the robust design approach employed, however, makes post-optimization sensitivity studies as well as the examination of different objectives feasible [8].

Another group from ASDL consisting of Mantis and Mavris [9, 10], have applied robust design techniques for a hypersonic vehicle system design. Mantis's PhD research encompasses the formulation and validation of a new design method, a systematic process for probabilistically assessing the impact of disciplinary uncertainty. The method implements Bayesian Statistics theory to quantify this source of uncertainty, and propagate its effects to the vehicle system level. Comparison of analytical and physical data for existing systems, modeled à priori in the given analysis tools, leads to quantification of uncertainty in those tools' calculation of discipline-level metrics. Then, after exploration of the new vehicle's design space, the quantified uncertainty is propagated probabilistically through the design space. This ultimately results in the assessment of the impact of disciplinary uncertainty on the confidence in the design solution: the final shape and variability of the probability functions defining the vehicle's system-level metrics. HAVOC, NASA Ames Research Center's hypersonic vehicle design code, provides the engineering analysis tool for the validation of this work. Application to a single-stage-to-orbit (SSTO) reusable launch vehicle concept, developed by the NASA Langley Research Center under the Space Launch Initiative, provides the proof of concept for this work. Although motivated by the hypersonic regime, Mantis [10] said that the proposed treatment of uncertainty applies to any class of aerospace vehicle.

2.3 Our perspective:

Based on its fundamental principle, i.e., improving the quality of a product by minimizing the effects of variation without eliminating the causes, robust design has become one of the powerful tools to assist designers to make reliable decisions under uncertainty. In designing complex engineering systems, Multidisciplinary Design Optimization (MDO) has become a systematic approach to optimization of complex, coupled engineering systems, where multidisciplinary

refers to the different aspects that must be included in designing a system that involves multiple interacting disciplines, such as those found in aircraft, spacecraft, automobiles, and industrial manufacturing applications. Though the usefulness of robust design is widely acknowledged for multidisciplinary design systems, its implementation is rare. One of the reasons is due to the complexity and computational burden associated with the evaluation of performance variations caused by the randomness (uncertainty) of a system.

Robust design applications in various other engineering fields are there from some times. We can find examples of robust design in manufacturing and structures community. Taguchi methods are a prime example of such efforts. Also Control & Guidance community has developed extensive methodologies for robust trajectory optimization. But the treatment of uncertainty is quite different there. It is clear that uncertainty identification and consistent modeling are prerequisites to performing robust design studies. Only very recently, the scientific and engineering community has begun to recognize the utility of defining multiple types of uncertainty [11]. In part the greater depth of study into the scope of uncertainty is made possible by the significant advancements in computational power we now enjoy. As systems become computationally better equipped to handle complex analyses, we encounter the limitations of applying only one mathematical framework (traditional probability theory) used to represent the full scope of uncertainty. Before we proceed further, let us have a clear idea about what this uncertainty is and how to handle it, in the next chapter.

Chapter 3

Uncertainty

A model can be described as an analyst's attempt to represent a system in a form that can be used as an explanatory and an exploratory tool [12]. But, it is impossible to capture all the subtleties of the system behavior in general. Therefore, at best any model is an approximation of the real system. A model in the physical sciences or engineering disciplines is usually a mathematical model, which is to say that it has a mathematical form which can produce numerical results that represent some observable aspects of system behavior. Such a mathematical model will generally have one or more parameters. Since any model is an approximate representation, it follows that there must be some uncertainty associated with the formulation, and predictions, of the model. For some models, however, this uncertainty is so small that it can essentially be ignored. For example, the mathematical formulation of many of the models created by physicists to explain natural phenomena are sufficiently well supported or verified that the models are very precise in their predictions, within a specified region of applicability. In addition, many of the parameters are so well known that they can be thought of as universal constants. An example of one such model is Newtonian mechanics and Newton's law of gravity, which is capable of making very accurate predictions of such things as planetary, motion, and can be used to define the trajectories of planets or space vehicles with great accuracy. Not only is the model rather simple but the parameter of the model, the gravitational constant, is known very accurately. Of course, under specific boundary conditions, and for particular problems, Newtonian mechanics breaks down and must be replaced with the General Theory of Relativity. Newtonian mechanics is an example of a deterministic model. However, a model need not necessarily be deterministic to be precise. Quantum Electrodynamics (QED) is a model which is capable of making very accurate predictions. But, because of the quantum mechanical nature of matter in the small scale, it does so only in a probabilistic sense, making *Characterization of uncertainty* predictions about the average behaviour of a population of events rather than about the outcome of a particular event. It is therefore a probabilistic model of the world.

Hence, it is generally recognized that there always exist uncertainties in any engineering system. For realistic representation of the actual systems, one must include uncertainties of various types in the mathematical model of the system. Uncertainty could occur in [13]

- parameters of the mathematical model
- in the accuracy of the mathematical model
- in the sequence of possible events that could occur in a discrete event system

3.1 Uncertainty Types:

The most widely recognized distinction in uncertainty types is between *aleatory* and *epistemic* uncertainty [11, 12, 13, 14, 15]. They are explained with examples in the next section.

3.1.1 Aleatory uncertainty:

Aleatory uncertainty arises because of natural, unpredictable variation in the performance of the system. The knowledge of experts cannot be expected to reduce aleatory uncertainty although their knowledge may be useful in getting a better estimate of the magnitude of the variability. So, this type of uncertainty is also referred to as *irreducible uncertainty*. Again, different researchers have termed it as *variability*, *objective uncertainty*, *inherent uncertainty* and *stochastic uncertainty*. Sources of aleatory uncertainty can commonly be singled out from other contributors to uncertainty by their representation as randomly distributed quantities that can take on values in an established or known range, but for which the exact value will vary by chance from unit to unit or from time to time. The mathematical representation most commonly used for aleatory uncertainty is a probability distribution. When substantial experimental data are available for estimating a distribution, there is no debate that the correct model for aleatory uncertainty is a probability distribution.

3.1.2 Epistemic uncertainty:

Epistemic uncertainty arises because of some level of ignorance or incomplete information of the system or the surrounding environment phase or activity of the modeling process. The key feature that this definition stresses is that the fundamental cause is incomplete information or incomplete knowledge of some characteristic of the system or the environment. As a result, an increase in knowledge or information can lead to a reduction in the predicted uncertainty of the

response of the system, all things being equal. This uncertainty is also termed as *ignorance*, *reducible uncertainty*, *subjective uncertainty* and *model form uncertainty*. Examples of sources of epistemic uncertainty are when there is little or no experimental data for a fixed (but unknown) physical parameter, limited understanding of complex physical processes, and the occurrence of fault sequences or environmental conditions not identified for inclusion in the analysis of the system. As opposed to aleatory uncertainty, the mathematical representation of epistemic uncertainty has proven to be much more of a challenge.

These both types of uncertainty are illustrated with examples in the next section.

3.2 Examples:

- a. *Toss of a coin:* Consider the experiment of tossing a coin. Suppose, the probability of getting heads is known to be p and hence the probability of tossing tails is $(1-p)$. Assuming the tosses are Bernoulli trials (independent with a constant probability of head), a binomial distribution could be used to codify the uncertainty in the outcome of the exercise. Here, the binomial distribution reflects the aleatory uncertainty in the process. Now suppose, the initial estimate of p is slightly inaccurate, for instance say the coin is slightly biased and p is not precisely known. Now, we will need to have an estimation of p . This uncertainty in p is now epistemic because the long-run frequency of head is not known. If we do sufficiently large no of coin tossing, we may get an accurate estimation of p . But this is not possible for all practical purposes and the best we can do is to fit a distribution to characterize p . Thus this confounding leads to an epistemic uncertainty about the aleatory uncertainty.
- b. *Estimating population of an endangered owl species:* Consider a specific type of endangered species present in a well defined region in a forest. We want to have an estimation of the future population of the species. For this we need the current population size and the mortality/birth rate. Now, the number of owls present at present is a particular number which is not varying. But that number is not known to us as it is extremely difficult to tally every single bird. So, this is the ignorance or epistemic uncertainty. On the other hand, the mortality/birth rate of the species has an inherent variability which depends on no of factors like weather condition etc. Additional effort may yield a better estimate of this aleatory uncertainty, but it will not tend to reduce it.

- c. *Measuring room temperature with thermometer*: The room temperature may vary because of different factors. So, it is aleatory in nature. Now, suppose there is a bias present in the thermometer itself, which we don't know precisely. This give rise to epistemic uncertainty.

3.3 Determining the nature of uncertainty and the necessity of separating them:

The incorporation of the appropriate uncertainty into the analysis of complex system is a topic of importance and widespread interest. Parry [12] in his paper has adopted the view that it is essential to maintain the distinction between these two types of uncertainty as they perform different functions in the model of the system created by the analyst. The aleatory uncertainty is a fundamental and integral part of the structure and form of the model, whereas the epistemic uncertainty is related to a characterization of how well we can represent the system by the model. In practice, however, many analysts have found that, for certain issues, especially those related to the modeling of the occurrence or the impact of particular physical phenomena, particularly in regimes that are outside our direct experience, it is difficult for them to distinguish between the two types. Parry [12] believes that the confusion has been exacerbated because the same mathematical tool, probability theory, is used to parameterize and quantify both types of uncertainty. It has not been uncommon for analysts to avoid addressing the issue by claiming that the distinction is irrelevant. However, as discussed in his paper, it is important to distinguish between the two, not only because it can impact the answer being given to a decision maker, and hence have an impact on the decision outcomes, but because it is essential to truly understand the nature of the model of the world that is being incorporated in the analysis. When it has been accepted that it is important to maintain the distinction between the two types of uncertainty, the most common question that analysts ask is whether the uncertainty associated with a particular model element is epistemic or aleatory, as if the type of uncertainty were a property of the issue being modeled. But, Parry insists that the question that should be asked by the analyst should be, 'how am I modeling this issue?' Understanding the modeling process is the key to an appropriate representation of uncertainty, and hence, ultimately, to making an appropriate use of the results.

Hora [14] argued that a sharp or natural distinction between these two types of uncertainty does not usually exist. The distinction arises because of the specific model to be quantified and the purposes to which the model is to be put. Which sources of uncertainty, variables, or

probabilities are labeled epistemic and which are labeled aleatory depends upon the mission of the study [14]. This is an important consideration in probability elicitation because once confounded, it may be very difficult to develop probability distributions that correctly reflect the various uncertainties. One cannot make the distinction between aleatory and epistemic uncertainties purely through physical properties or the experts' judgments. The same quantity in one study may be treated as having aleatory uncertainty while in another study the uncertainty maybe treated as epistemic. For example, a study of reactor safety may be designed to produce estimates of the long term risks averaged over a variety of weather conditions as discussed in the preceding section. In such a study the representation of uncertainty about the behaviour of the system averaged over very long period results in averaging over the aleatory uncertainty about short term weather fluctuations. Another study, in contrast, may be designed for emergency response and individual weather conditions may be meaningful if not critical. In the second study weather conditions may be dominate factors in determining risk and thus uncertainty in weather is the largest contributor to uncertainty in risk. Here uncertainty about the weather at the time of an emergency is resolvable and should be considered epistemic. Here, the same weather condition is treated as aleatory in the first study and as epistemic in the second study. So, the distinction between the uncertainties is then a matter of choice of scale and hence mutable [14]. Careful definition of the endpoint of the study is necessary to make these distinctions and to communicate them to the participants.

The normative specialist, the person leading and directing a probability elicitation, plays a key role in distinguishing between types of uncertainty [14]. The normative specialist must grasp both the nature of the models and the scientific bases that the experts employ in forming the judgments. This knowledge is needed to distinguish between the aleatory and epistemic sources of uncertainty. Without such an understanding, mistakes in probability elicitation are inevitable. The result is often a mixture of aleatory and epistemic uncertainties leaving the analyst with the difficult task of later unraveling the assessments, or even worse, having distributions inappropriate for the purpose they are put to.

The design of the probability elicitation process and the questions put to experts should acknowledge the need to separate types of uncertainty. Too often, the questions asked of experts do not clarify how these uncertainties are to be treated. Central to this issue is a clear understanding of the use that will be made of the judgments. To ensure that the expert is

responding in a manner suitable to the use, effective communication must exist between the experts and the parties seeking their knowledge. Achieving such effective communication is not easy. Often, parties are apparently agreeing while holding different but unarticulated views about the issues in question [14]. Hence, questions should be constructed so that these uncertainties retain distinct representations. Often, this is accomplished by using conditional probability distributions for the quantities having aleatory uncertainty and marginal probability distributions for quantities having epistemic uncertainty. The same strategy can be used for quantities embodying both types of uncertainty. For example, the measured concentration of a radionuclide at a given point that is traversed by a plume of radioactive materials as described earlier may embody both aleatory and epistemic uncertainty. The aleatory uncertainty reflects the random deviations in wind direction and heterogeneity in concentrations, perhaps as measured as a departure from the ideal Gaussian plume. The epistemic uncertainty reflects the variation due to lack of knowledge about how plumes will behave on average given the chosen set of weather conditions. To ensure that the elicitation process retrieves the desired information, the questions asked need to be carefully posed. In particular, questions about quantities that are integrated or averaged over time or space should be carefully constructed to decompose the elicitation quantities so that the two types of uncertainty are appropriately represented.

Often, the objective of using expert judgments is to obtain information about the parameters of a process. If one asks questions about the output of the process rather than the parameters of the process, the resulting distribution will embody both epistemic (here parametric) uncertainty and the aleatory uncertainty inherent in the process. One might then ask questions directly related to the parameters of the process and thus avoid the introduction of the confounding aleatory uncertainty. But, this approach may become problematic because the experts cannot reasonably be expected to respond to questions about parameters because parameters are most often not directly observable. For example, competing models may exist or there may be uncertainty about the correctness of the posited model. Various models will have different parameters. The expert could be put in the most uncomfortable position of having to respond to questions about something which is not believed. This potential conflict between the modelers' assumptions and the beliefs of the expert can be a major source of concern [14].

Again, the expert is not expected to know about all types of models and parameters involved. For example, consider the effort to quantify consequence models for nuclear reactor accidents. Suppose, the modeler proposed a Gaussian plume model given by,

$$C(x,y,z) = \frac{1}{2\pi\sigma_y\sigma_z} \exp\left[-\frac{1}{2}\left\{\left(\frac{y}{\sigma_y}\right)^2 + \left(\frac{z}{\sigma_z}\right)^2\right\}\right] \quad (3.1)$$

where $C(x,y,z)$ is the concentration in plume at a downwind distance x from a point source and y and z are displacements from the centre of the plume in the horizontal and vertical directions, respectively. Both σ_y and σ_z are functions of the downwind distance x where $\sigma_y = ax^b$ and $\sigma_z = cx^d$. Here the parameters of interest are a , b , c and d . These parameters are uncertain and the goal of the elicitation is to quantify uncertainty distributions for these parameters under a number of different classes of atmospheric conditions, i.e., stability classes. But, an expert who does not subscribe to the Gaussian model cannot be expected to provide meaningful judgments about its parameters. Questions about these parameters are meaningless to such an expert. As alternative models are available, an expert may subscribe to some other model, or to no model at all for that matter. So, a decision was made to ask for probability distributions only for observable or potentially observable quantities. This is a well established principle in probability elicitation [14]. Thus, questions about actual concentrations were asked from which distributions were obtained by solving an inverse problem. The inverse problem, in brief, is finding input distributions for the parameters of a model that cause the output of the model to follow some specified distribution. Usually the problem is under-determined so that some additional criteria such as maximum entropy is needed to select from the potential distributions.

In some cases, it is possible to model aleatory and epistemic uncertainty in terms of an additive or multiplicative model with independent components. For example, the uncertainty in rainfall during a given period, say a year, may be a function of the annual average rainfall, and a deviation from the annual average, say e . The rainfall, X , can then be expressed as $X = x + e$. Probability distributions on X and e or X and x can then be used to derive the distribution of the third quantity through de-convolution by LaPlace or Fourier transforms.

3.4 Aleatory and Epistemic uncertainty existing together:

As discussed in the previous section, because parameters that characterize aleatory uncertainty, or variability, and the probabilities that characterize epistemic uncertainty are dealing with

different issues, there is no option but to treat them separately. When we consider that both aleatory and epistemic uncertainties are present in a system certain other issues also come into the picture [14]. Reconsider the coin-tossing example discussed earlier. There a binomial distribution is used to represent the aleatory uncertainty in the outcome of the toss. If, in addition, there is uncertainty in p , probability of getting heads, then the probability distribution of getting x no of heads after n tosses, is given by,

$$Prob(x) = \int_0^1 f_{binomial}(x | p, n) dF(p) \quad (3.2)$$

where the probability measure $F(p)$ measures the epistemic uncertainty in p . The probability of getting a head in 2 tosses is then given by,

$$prob(1) = \int_0^1 f_{binomial}(1 | p, 1) dF(p) = E(p) \quad (3.3)$$

where $E(p)$ is the expected value or mean of the uncertainty distribution for p . Now if we calculate the probability of head in both the tosses, it is given as,

$$prob(2) = \int_0^1 f_{binomial}(2 | p, 2) dF(p) = [E(p)]^2 + var(p) \quad (3.4)$$

where $var(p)$ is the variance of the uncertainty distribution for p . Clearly, the outcomes of tossing the coin are no longer independent. By assumption, two tosses of the coin are independent events conditional on a given value of p . But, because of the epistemic uncertainty in the value p , there arises a dependency between the outcomes of the tosses. On the same note, the probability of a point up on the second toss given a point up on the first toss is $E(p) + [var(p)/E(p)]$ indicating a positive dependence, a dependence induced solely by the epistemic uncertainty in p .

Dependence introduced by epistemic uncertainty is sometimes ignored or not recognized in the modeling of complex systems [14]. Components of the same type, for example, may be assumed to fail independently in presence of only aleatory uncertainty. However, if epistemic uncertainty exists about the common failure probability for the components, the resulting failures will no longer be independent *a la* the coin example. Consider a system with several redundant components so that the system fails only if all of these components fail. Suppose that the failure probability for each of m components is known to be p . The system failure probability is then p^m .

If, however, the common component failure probability is uncertain and the uncertainty is described by a distribution $F(p)$ for p , then the failure probability for the system is

$$prob(failure) = \int_0^1 p^m dF(p) \quad (3.5)$$

The difference in the calculated values for system failure can be dramatic. If $m = 4$ and p is known to be $1/2$, the system failure probability is $1/16$. In contrast, if the common component failure probability is uncertain (epistemic uncertainty) and is assigned a uniform distribution on $[0,1]$ (note that the mean is still $1/2$), the resulting system failure probability rises to $1/5$, more than three times larger than when only the aleatory uncertainty is considered.

Chapter 4

Uncertainty Representation and Propagation

Methods to efficiently represent, aggregate, and propagate different types of uncertainty through computational models are clearly of vital importance. In literature, different uncertainty representation and propagation techniques have been developed and discussed.

Kim [1] described two approaches for estimation of uncertainty of a system: *the extreme condition approach* and *the statistical approach*. The extreme condition approach seeks the bounds of system output: the *interval analysis* is one example. The statistical approach finds the probability of failure/success of the system and frequently requires data sampling to construct a cumulative distribution function of the output through uncertainty propagation.

We will use the following simulation-based design model [16] as illustrated in Fig. 4.1 to explain the above two methodologies. The model consists of a chain of two simulation programs (imagining they are from two different disciplines) that are connected to each other through *linking variables* represented by the vector \mathbf{y} . The input to simulation model I is the vector of design variables \mathbf{x}_1 with aleatory uncertainty, described by a range $\Delta\mathbf{x}_1$ or certain distribution.

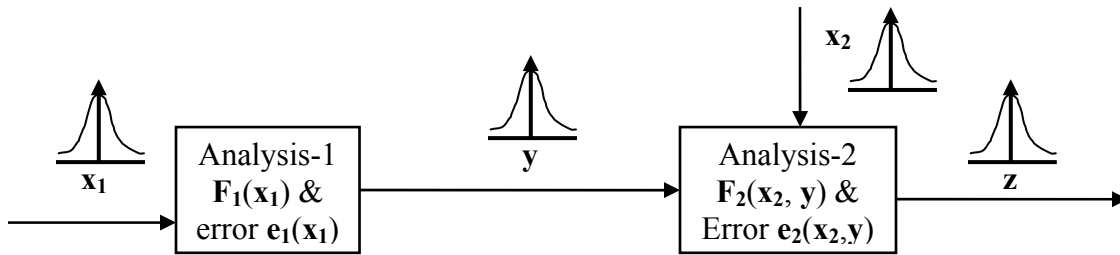


Figure 4.1: Simulation based Design Model of the proposed problem definition

For simulation model I, the output \mathbf{y} can be expressed as

$$\mathbf{y} = \mathbf{F}_1(\mathbf{x}_1) + \mathbf{e}_1(\mathbf{x}_1) \quad (4.1)$$

where $\mathbf{F}_1(\mathbf{x}_1)$ is the simulation model and $\mathbf{e}_1(\mathbf{x}_1)$ is the corresponding error model of the epistemic uncertainty. Additive error model is used to represent model structure uncertainty in this study, though its real form can be much more complicated.

For simulation model II, the inputs are the linking variable \mathbf{y} and the design variable \mathbf{x}_2 . The output vector \mathbf{z} can be expressed as

$$\mathbf{z} = \mathbf{F}_2(\mathbf{x}_2, \mathbf{y}) + \mathbf{e}_2(\mathbf{x}_2, \mathbf{y}) \quad (4.2)$$

where $\mathbf{F}_2(\mathbf{x}_2, \mathbf{y})$ is the simulation model and $\mathbf{e}_2(\mathbf{x}_2, \mathbf{y})$ is the corresponding error model. The output \mathbf{z} often represents system performance parameters that are used to model the design objectives and constraints. Because of the deviations (aleatory uncertainty) existing in \mathbf{x}_2 and \mathbf{y} , and the epistemic uncertainty $\mathbf{e}_2(\mathbf{x}_2, \mathbf{y})$, the final output \mathbf{z} will also have deviations.

4.1 Extreme Condition Approach:

The *extreme condition approach* is developed to obtain an interval or the *extremes* of the final output from a chain of simulation models [16]. The term *extreme* is defined as the minimum or the maximum value of the end performance (final output) corresponding to the given ranges of internal and external uncertainties. With this approach, the aleatory uncertainties are characterized by the intervals $[\bar{\mathbf{x}}_1 - \Delta\mathbf{x}_1, \bar{\mathbf{x}}_1 + \Delta\mathbf{x}_1]$ and $[\bar{\mathbf{x}}_2 - \Delta\mathbf{x}_2, \bar{\mathbf{x}}_2 + \Delta\mathbf{x}_2]$ ($\bar{\mathbf{x}}_1$ and $\bar{\mathbf{x}}_2$ denote the nominal values of \mathbf{x}_1 and \mathbf{x}_2 respectively). Correspondingly, the outputs of the two simulation models are described by the intervals $[\mathbf{y}^{\min}, \mathbf{y}^{\max}]$ and $[\mathbf{z}^{\min}, \mathbf{z}^{\max}]$, respectively.

Optimizations are used to find the maximum and minimum (extremes) of the outputs from simulation model I and simulation model II, respectively. The flow chart of the proposed procedure is illustrated in Fig. 4.2. The steps to obtain the range of output $[\mathbf{z}^{\min}, \mathbf{z}^{\max}]$, are presented as [16]:

- a) Given a set of nominal values $\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2$ and ranges $\Delta\mathbf{x}_1, \Delta\mathbf{x}_2$;
- b) For simulation model I, minimize and maximize $\mathbf{F}_1(\mathbf{x}_1)$ over the range of $[\bar{\mathbf{x}}_1 - \Delta\mathbf{x}_1, \bar{\mathbf{x}}_1 + \Delta\mathbf{x}_1]$ to obtain $\mathbf{F}_1^{\min}(\mathbf{x}_1)$ and $\mathbf{F}_1^{\max}(\mathbf{x}_1)$. The optimization model is

Given: The nominal value of $\bar{\mathbf{x}}_1$ and the range $\Delta\mathbf{x}_1$

Optimize: Minimize $\mathbf{F}_1(\mathbf{x}_1)$ to obtain $\mathbf{F}_1^{\min}(\mathbf{x}_1)$

Maximize $\mathbf{F}_1(\mathbf{x}_1)$ to obtain $\mathbf{F}_1^{\max}(\mathbf{x}_1)$

Subject to: $\bar{\mathbf{x}}_1 - \Delta\mathbf{x}_1 \leq \mathbf{x}_1 \leq \bar{\mathbf{x}}_1 + \Delta\mathbf{x}_1$

- c) Similar to step b), obtain the extreme values of epistemic uncertainty $e_1^{\min}(\mathbf{x}_1)$ and $e_1^{\max}(\mathbf{x}_1)$ over the range $[\bar{x}_1 - \Delta x_1, \bar{x}_1 + \Delta x_1]$;

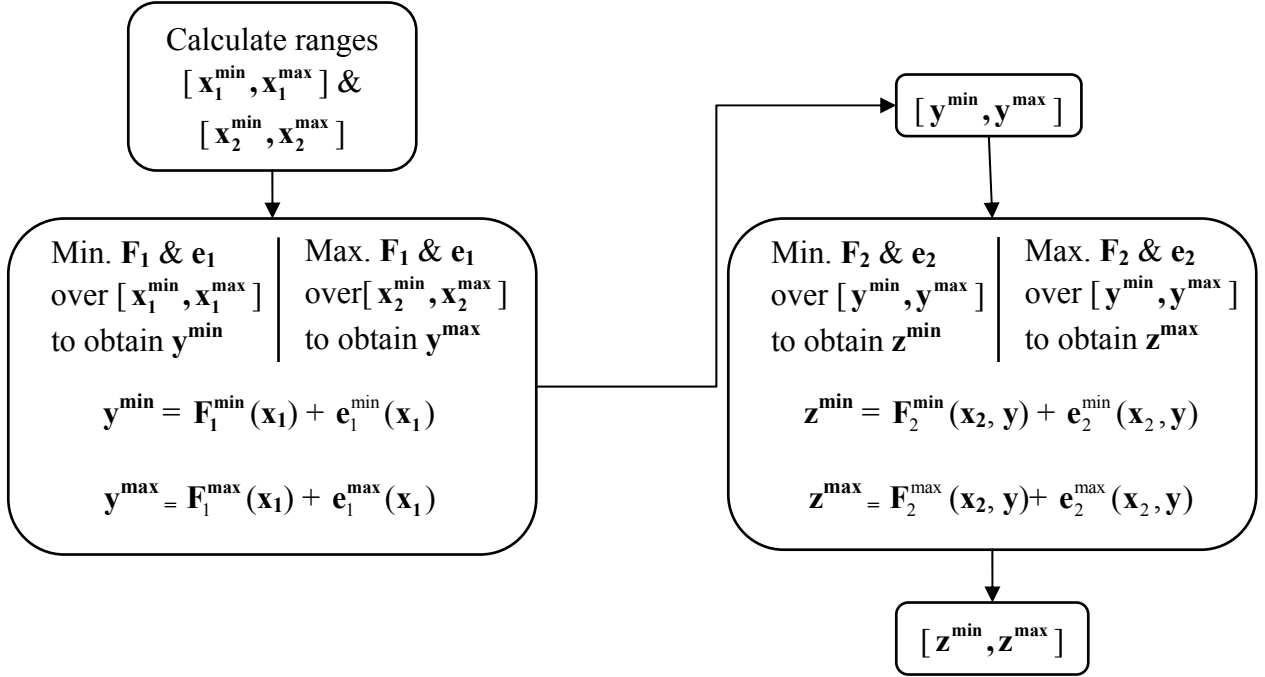


Figure 4.2: Flow-chart representation of Extreme Condition approach

- d) Obtain the interval [y^{\min}, y^{\max}] using

$$y^{\min} = F_1^{\min}(\mathbf{x}_1) + e_1^{\min}(\mathbf{x}_1) \quad (4.3)$$

$$y^{\max} = F_1^{\max}(\mathbf{x}_1) + e_1^{\max}(\mathbf{x}_1) \quad (4.4)$$

- e) For simulation model II, minimize and maximize $F_2(\mathbf{x}_2, \mathbf{y})$ over the range of $[\bar{x}_2 - \Delta x_2, \bar{x}_2 + \Delta x_2]$ and [y^{\min}, y^{\max}] to obtain $F_2^{\min}(\mathbf{x}_2, \mathbf{y})$ and $F_2^{\max}(\mathbf{x}_2, \mathbf{y})$, The optimization model is similar to the one in step b)

- f) Similar to step e), obtain the extreme values of epistemic uncertainty $e_2^{\min}(\mathbf{x}_2, \mathbf{y})$ and $e_2^{\max}(\mathbf{x}_2, \mathbf{y})$;

- g) Obtain the interval [z^{\min}, z^{\max}] using,

$$z^{\min} = F_2^{\min}(\mathbf{x}_2, \mathbf{y}) + e_2^{\min}(\mathbf{x}_2, \mathbf{y}) \quad (4.5)$$

$$\mathbf{z}^{\max} = \mathbf{F}_2^{\max}(\mathbf{x}_2, \mathbf{y}) + \mathbf{e}_2^{\max}(\mathbf{x}_2, \mathbf{y}) \quad (4.6)$$

Based on the computed interval $[\mathbf{z}^{\min}, \mathbf{z}^{\max}]$, the nominal value of \mathbf{z} is calculated as

$$\bar{\mathbf{z}} = \frac{1}{2}(\mathbf{z}^{\min} + \mathbf{z}^{\max}) \quad (4.7)$$

The deviation of \mathbf{z} can be calculated as

$$\Delta \mathbf{z} = \mathbf{z}^{\max} - \mathbf{z}^{\min} \quad (4.8)$$

The extreme condition approach identifies the interval of a system output based on the given intervals of the system inputs. It is applicable to the situation in which both the external uncertainties in \mathbf{x}_1 and \mathbf{x}_2 are expressed by ranges.

4.2 Statistical Approach:

The statistical approach is developed to estimate cumulative density function (*CDF*), probability density functions (*PDF*) or population parameters (for example, mean and variance) of the final outputs from a chain of simulation models [16]. Here we assume \mathbf{x}_1 , \mathbf{x}_2 , and the epistemic uncertainty $\mathbf{e}_1(\mathbf{x}_1)$ and $\mathbf{e}_2(\mathbf{x}_2, \mathbf{y})$ follow certain probabilistic distributions that may be obtained by field or experimental data, the information of similar existing products, and judgements by engineering experience. Note, since the distribution parameters (for example, mean and variance) of $\mathbf{e}_1(\mathbf{x}_1)$ and $\mathbf{e}_2(\mathbf{x}_2, \mathbf{y})$ are functions of \mathbf{x}_1 , \mathbf{x}_2 and \mathbf{y} , the final distributions of $\mathbf{e}_1(\mathbf{x}_1)$ and $\mathbf{e}_2(\mathbf{x}_2, \mathbf{y})$ are the accumulated effects of both the uncertainty in the error model and the aleatory uncertainty parameters such as \mathbf{x}_1 , \mathbf{x}_2 and \mathbf{y} . Monte Carlo simulation methods are used to propagate the effect of uncertainties through the simulation chain. The procedure is developed as [16]:

- a) Generate N_1 samples of \mathbf{x}_1 and N_2 samples of \mathbf{x}_2 as simulation inputs based on their distribution functions;
- b) For the given \mathbf{x}_1 , calculate the distribution parameters of the epistemic uncertainty $\mathbf{e}_1(\mathbf{x}_1)$ for simulation model I and generate N_3 samples of \mathbf{e}_1 based on the distribution function;
- c) Evaluate the corresponding output $\mathbf{y} = \mathbf{F}_1(\mathbf{x}_1) + \mathbf{e}_1$ for simulation model I;
- d) For each \mathbf{y} , calculate the distribution parameters of the epistemic uncertainty $\mathbf{e}_2(\mathbf{x}_2, \mathbf{y})$ of simulation model II and generate N_4 samples of \mathbf{e}_2 based on the distribution function;

- e) Evaluate the corresponding output $\mathbf{z} = \mathbf{F}_2(\mathbf{x}_2, \mathbf{y}) + \mathbf{e}_2$ for simulation model II;
- f) Calculate the mean value μ_z , the standard deviation s_z , or the *cdf* and *pdf* of \mathbf{z} based on $N \times M \times H$ samples of \mathbf{z} .

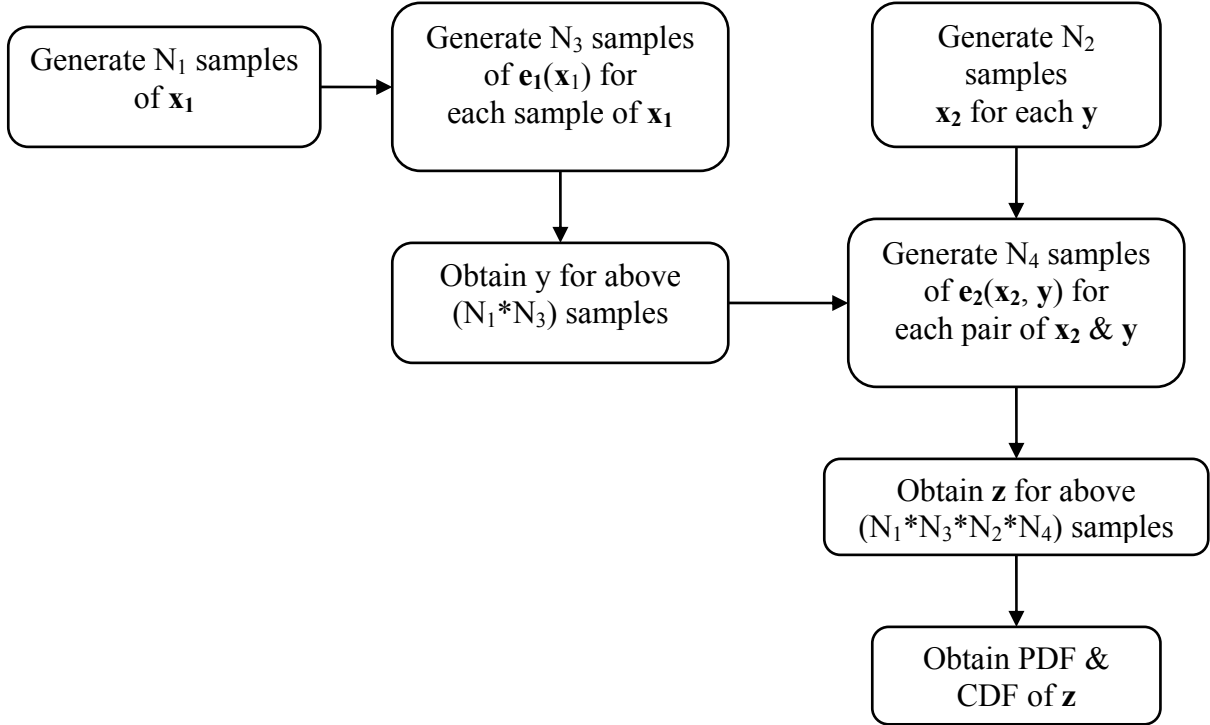


Figure 4.3: Flow-chart representation of Statistical approach

Fig. 4.3 depicts the process of this Monte Carlo simulation-based approach. This approach generates statistical estimates of the system output based on the given distributions of the inputs and error models. This gives us more information than the extreme condition approach by which only the best and worst performance are estimated. Because the statistical approach is based on the concept of Monte Carlo simulation, it often requires a large number of simulations. More effective sampling techniques such as Latin Hyper Cube and fractional factorial design can be used to reduce the amount of simulations [16].

4.3 Drawbacks:

So, these methods show how to propagate the aleatory uncertainty present in the parameter along with the epistemic model form uncertainty. But they don't take into consider the case when parameters with both aleatory and epistemic uncertainty present together. They don't distinguish

between the two types parameter uncertainty and treat them in the similar fashion. But, it is necessary to distinguish between the two types of uncertainty as they behave differently in calculations. Ferson et al [15] has discussed some empirical and computational issues that arise when both aleatory and epistemic uncertainties enter into the system using some simplified numerical examples. He argues that the two kinds of uncertainty should be propagated through mathematical expressions with different calculation methods. He suggests that interval analysis should be used to propagate ignorance, and probability theory should be used to propagate variability. He demonstrates how using an inappropriate method can yield erroneous results. He also shows how ignorance and variability can be represented simultaneously and manipulated in a coherent analysis that does not confound the two forms of uncertainty and distinguishes what is known from what is assumed.

He started with a simple example where the information about two parameters A and B are given as,

$$A \in [0.2, 0.4] \ \& \ B \in [0.3, 0.5] \quad (4.9)$$

Then what can we say about the possible range of the product AB . Ferson et al [15] argued that answer to such a question depends on what we believe about the parameters involved and what the nature of our uncertainty about them is. For instance, if we suppose both A and B are actually fixed quantities whose values we just don't happen to know, then we might use interval analysis [15] to arrive at an answer to the question. This approach asks about the possible range of the product given the stated ignorance about A and B . In this case, the smallest possible value would be obtained when A is 0.2 and B is 0.3, yielding the product 0.06. The largest possible value would be $0.4 \times 0.5 = 0.2$. No other pair of numbers from the respective intervals yields a product outside this range. Thus, the answer is that the product is a number somewhere between 0.06 and 0.2. Fig. 4.4(a) depicts this interval. On the other hand, if we think the parameters A and B are varying randomly, then we might use a very different approach to arrive at an answer. Since we don't have information on the probability of all events, we will use the uniform distribution function to model the uncertainty about each parameter, justified by Laplace's *Principle of Insufficient Reason* [11]. This can be interpreted that all simple events, for which a probability distribution is not known in a given sample space, are equally likely. Choosing any other distributions would constitute an assertion of additional knowledge about the parameters. These

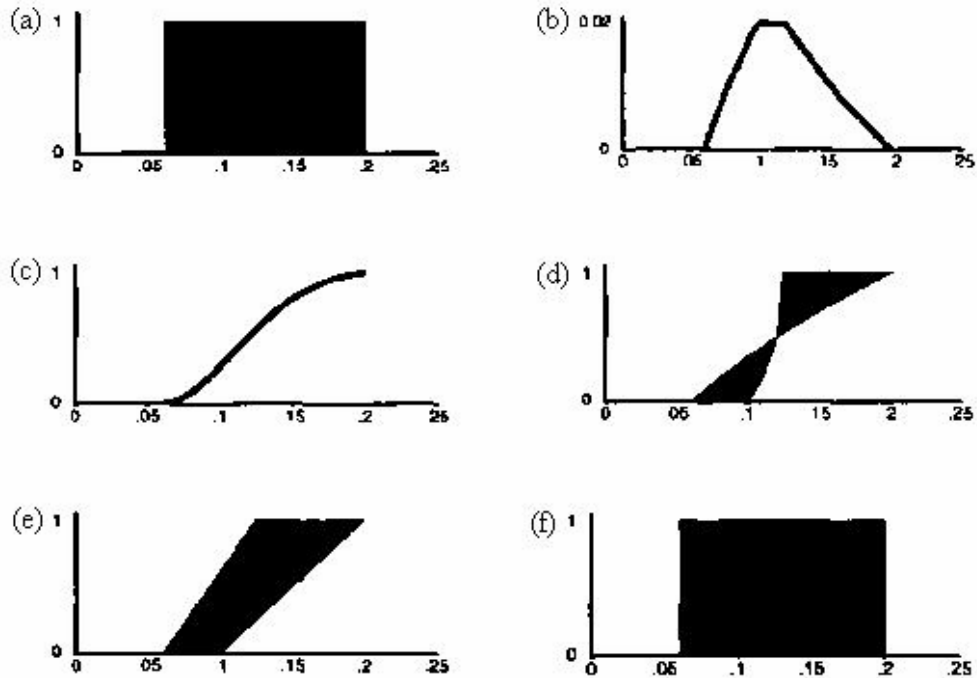


Figure 4.4: (a) Depiction of the interval $[0.06, 0.2]$ which is guaranteed to contain the product of two uncertain numbers $A = [0.2, 0.4]$ and $B = [0.3, 0.5]$. Note that this interval is not the same as a uniform probability distribution. (b) The probability distribution of the product of random variables $A = \text{uniform}(0.2, 0.4)$ and $B = \text{uniform}(0.3, 0.5)$ under an assumption of independence. (c) The cumulative form of the probability distribution of this product. (d) The smallest region guaranteed to contain the cumulative distribution of the product assuming the dependence between A and B is some linear correlation between -1 and $+1$. (e) The smallest region guaranteed to enclose the cumulative distribution of the product without making any assumption about the stochastic dependence between A and B . (f) The smallest region guaranteed to enclose the cumulative probability distribution of the product of A and B given no information other than the bounds on both parameters. Note that this region is very different from a uniform distribution. [15]

distributions are then combined by the ordinary rules of probability theory. In particular, if we assume they are independent, we can use probabilistic convolution or some Monte Carlo strategy to estimate the distribution of the product AB . The result is shown as a probability density function in Fig. 4.4(b). Fig. 4.4(c) is the cumulated form of the same probability distribution, which is slightly more convenient for the following discussion because the ordinate always ranges between zero and one.

The answers from interval analysis and probability theory agree in one sense. They both say the value of the product must lie between 0.06 and 0.2 . Yet the answers are clearly different. The

answer given by the probabilistic approach suggests that the product is much more likely to be a value near the central tendency than to be one of the extreme values. There is a clear concentration of probability mass in the centre of the distribution. But there is nothing in the original statement of the question that obviously justifies this concentration.

So, Ferson et al [15] pointed out that since we assumed independence between parameter A and parameter B , perhaps this assumption accounts for the observed central concentration of probability. So, we need to explore this idea by varying the correlation between A and B over the range of possible correlations. Fig. 4.4(d) shows the region circumscribed by the bounds on the cumulative distribution function that can result from the product of A and B , assuming correlations between A and B can be any value over the full theoretical range between $+1$ and -1 . This region is much narrower than the interval depicted in Fig. 4.4(a), so we see that ignoring correlation cannot have accounted for the discrepancy between the interval result and probability result.

The above described correlation only measures linear associations between the parameters. Fig. 4.4(e) gives the smallest region guaranteed to enclose the cumulative distribution of the product without making any assumption about the stochastic dependence between A and B . This result shows that, no matter what statistical dependency we assume between the two parameters, or even if we do not assume anything at all about their interdependence, we still get an answer that is narrower than suggested by interval analysis and which exhibits a persistent albeit weaker concentration of probability mass in the centre of the range.

So, finally Ferson et al [15] commented that the problem comes because we assumed particular shapes for them. One need not assume specific shapes for the distributions if there is no basis for doing so. Recently developed methods based on interval probabilities [15] allow us to express ignorance about probability itself. It is also possible to do calculations with these interval probabilities. Fig. 4.4(f) shows the region in which the probability distribution of the product AB must lie, given only the information about bounds on A and B . This result states that any value or distribution of values between 0.06 and 0.2 is possible and that we cannot say anything about which values are more likely than any others within this range. This interval-probabilistic approach thus agrees completely with the original result in Fig. 4.4(a) obtained from elementary interval analysis. They both represent an unknown quantity that may or may not be varying but

whose value(s) we know to be limited to a specific range. Although the results in Fig. 4.4(a) and (f) were obtained by completely different computational approaches, the interpretations of the results are identical.

So, assuming certain shape for the distribution using classical probability, if there is no basis for doing so, certainly leads us to erroneous results. Again, an additional assumption in classical probability is entailed by the axiom of additivity where all probabilities that satisfy specific properties must add to 1 [11]. This forces the conclusion that knowledge of an event necessarily entails knowledge of the complement of an event, i.e., knowledge of the probability of the likelihood of the occurrence of an event can be translated into the knowledge of the likelihood of that event not occurring. If an expert believes that a system may fail due to a particular component with a likelihood of 0.3, does that necessarily mean that the expert believes that the system will *not* fail due to that component with likelihood of 0.7? This articulates the challenge of modeling any uncertainty associated with an expert's subjective belief. Though the assumptions of additivity and the Principle of Insufficient Reason may be appropriate when modeling the random events associated with aleatoric uncertainty, these constraints are questionable when applied to an issue of knowledge or belief, i.e. the epistemic uncertainty. As a consequence of these concerns, applied mathematicians have investigated many more general representation of uncertainty to cope with particular situations involving epistemic uncertainty. Examples of these types of situations include [11]:

- a. When there is little information on which to evaluate a probability or
- b. When that information is nonspecific, ambiguous, or conflicting.

Analysis of these situations can be required, for an example in risk assessment, though probability theory lacks the ability to handle such information. Where it is not possible to characterize uncertainty with a precise measure such as a precise probability, it is reasonable to consider a measure of probability as an interval or a set. This characterization of a measure of probability as an interval or set has three important implications [11]:

1. It is not necessary to elicit a precise measurement from an expert or an experiment if it is not realistic or feasible to do so.

2. The Principle of Insufficient Reason is not imposed. Statements can be made about the likelihood of multiple events together without having to resort to assumptions about the probabilities of the individual events under ignorance.
3. The axiom of additivity is not imposed. The measures do not have to add to 1. When they do, it corresponds to a traditional probabilistic representation. When the sum is less than 1, called the subadditive case, this implies an incompatibility between multiple sources of information, e.g. multiple sensors providing conflicting information. When the sum is greater than 1, the superadditive case, this implies a cooperative effect between multiple sources of information, e.g. multiple sensors providing the same information.

Because there is more than one kind of uncertainty and probability theory may not apply to every situation involving uncertainty, many theories of generalized uncertainty based information have been developed. There are three major frameworks from which the problem of interval-based representation of uncertainty has been approached: imprecise probabilities (initial work by Walley, Fine; Kuznetsov); possibility theory (Zadeh; Dubois and Prade; Yager); and the Dempster-Shafer theory of evidence. (Dempster; Shafer; Yager; Smets) [11].

In this report we will discuss the Evidence theory proposed by Dempster and Shafer. The motivation for selecting Dempster-Shafer theory can be characterized by the following reasons:

1. The relatively high degree of theoretical development among the nontraditional theories for characterizing uncertainty.
2. The relation of Dempster-Shafer theory to traditional probability theory and set theory.
3. The large number of examples of applications of Dempster-Shafer theory in engineering in the past ten years.
4. The versatility of the Dempster-Shafer theory to represent and combine different types of evidence obtained from multiple sources.

4.4 Dempster-Shafer Evidence Theory:

Dempster-Shafer Theory (DST) is a mathematical theory of evidence. The seminal work on the subject is Shafer (1976) [17], which is an expansion of Dempster (1967) [11]. In a finite discrete space, Dempster-Shafer theory can be interpreted as a generalization of probability theory where probabilities are assigned to *sets* as opposed to mutually exclusive singletons. In traditional

probability theory, evidence is associated with only one possible event. In DST, evidence can be associated with multiple possible events, e.g., sets of events. As a result, evidence in DST can be meaningful at a higher level of abstraction without having to resort to assumptions about the events within the evidential set. Where the evidence is sufficient enough to permit the assignment of probabilities to single events, the Dempster-Shafer model collapses to the traditional probabilistic formulation. One of the most important features of Dempster-Shafer theory is that the model is designed to cope with varying levels of precision regarding the information and no further assumptions are needed to represent the information. It also allows for the direct representation of uncertainty of system responses where an imprecise input can be characterized by a set or an interval and the resulting output is a set or an interval.

4.4.1 Types of Evidence:

We consider four types of evidence from multiple sources: consonant evidence, consistent evidence, arbitrary evidence, and disjoint evidence [11]:

1. *Consonant evidence* (Fig. 4.5(a)) can be represented as a nested structure of subsets where the elements of the smallest set are included in the next larger set, all of whose elements are included in the next larger set and so on. This can correspond to the situation where information is obtained over time that increasingly narrows or refines the size of the evidentiary set.
2. *Consistent evidence* (Fig. 4.5(b)) means that there is at least one element that is common to *all* subsets.

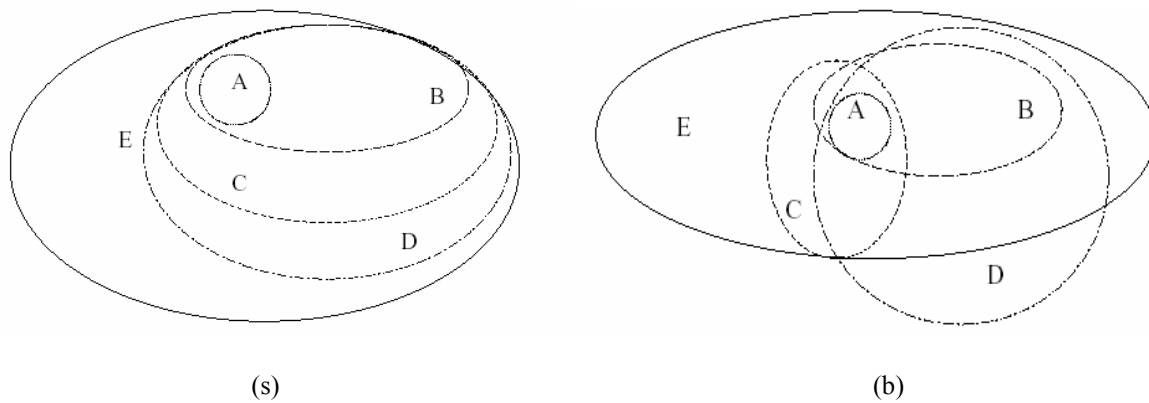


Figure 4.5: (a) Consonant Evidence, (b) Consistent Evidence

3. *Arbitrary evidence* (Fig. 4.6(a)) corresponds to the situation where there is no element common to *all* subsets, though some subsets may have elements in common.
4. *Disjoint evidence* (Fig. 4.6(b)) implies that any two subsets have no elements in common with any other subset.

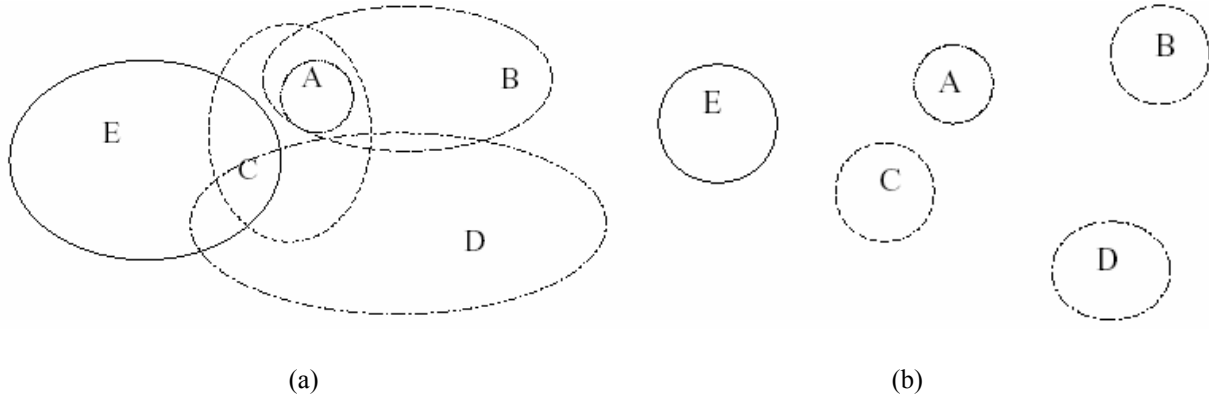


Figure 4.6: (a) Arbitrary Evidence, (b) Disjoint Evidence

4.4.2 The Evidence Theory:

There are three important functions in Dempster-Shafer theory: the *basic probability assignment* function (bpa or m) (analogue of probability), the *Belief* function (Bel), and the *Plausibility* function (Pl) [11, 18, 19, 20].

The basic probability assignment (bpa) is a primitive of evidence theory. It can be said as an analogue of probability, a weight associated to an elementary event. But, generally speaking, the term “basic probability assignment” does *not* refer to probability in the classical sense. The bpa, represented by m , defines a mapping of the power set to the interval between 0 and 1, where the bpa of the null set is 0 and the summation of the bpa’s of all the subsets of the power set is 1. The value of the bpa for a given set A (represented as $m(A)$), expresses the proportion of all relevant and available evidence that supports the claim that a particular element of Ω (the universal set) belongs to the set A but to no particular subset of A [Klir, 1998]. The value of $m(A)$ pertains only to the set A and makes no additional claims about any subsets of A . Any further evidence on the subsets of A would be represented by another bpa, i.e. $B \dot{\subset} A$, $m(B)$ would be the bpa for the subset B . Formally, this description of m can be represented with the following three equations:

$$m : P(\Omega) \rightarrow [0,1] \tag{4.10}$$

$$m(\phi) = 0 \quad (4.11)$$

$$\sum_{A \in P(\Omega)} m(A) = 1 \quad (4.12)$$

where $P(\Omega)$ represents the power set of Ω , ϕ is the null set, and A is a set in the power set. Every set $A \in P(\Omega)$ for which $m(A) > 0$ is called a *focal element*. The pair (F, m) , where F denotes the set of all focal elements of m , is called a *body of evidence* [11, 18, 19, 20].

From the basic probability assignment, the upper and lower bounds of an interval can be defined [11]. This interval contains the precise probability of a set of interest (in the classical sense) and is bounded by two nonadditive continuous measures called Belief and Plausibility. The lower bound *Belief* for a set A is defined as the sum of all the basic probability assignments of the proper subsets (B) of the set of interest (A) ($B \subseteq A$). The upper bound, *Plausibility*, is the sum of all the basic probability assignments of the sets (B) that intersect the set of interest (A) ($B \cap A \neq \phi$). Formally, for all sets A that are elements of the power set ($A \in P(\Omega)$),

$$Bel(A) = \sum_{B|B \subseteq A} m(B) \quad (4.13)$$

$$Pl(A) = \sum_{B|B \cap A \neq \phi} m(B) \quad (4.14)$$

For example, if $F = \{A_1, A_2, A_3, A_4\}$ with $m(A_1) = 0.1$, $m(A_2) = 0.2$, $m(A_3) = 0.3$, $m(A_4) = 0.4$, the Belief and Plausibility of the set B (Fig. 4.7) can be calculated as follows,

$$\begin{aligned} Bel(B) &= \sum_{A_j | A_j \subseteq B} m(A_j) \\ &= m(A_4) = 0.4 \end{aligned}$$

$$\begin{aligned} Pl(B) &= \sum_{A_j | A_j \cap B \neq \phi} m(A_j) \\ &= m(A_2) + m(A_3) + m(A_4) \\ &= 0.2 + 0.3 + 0.4 = 0.9 \end{aligned}$$

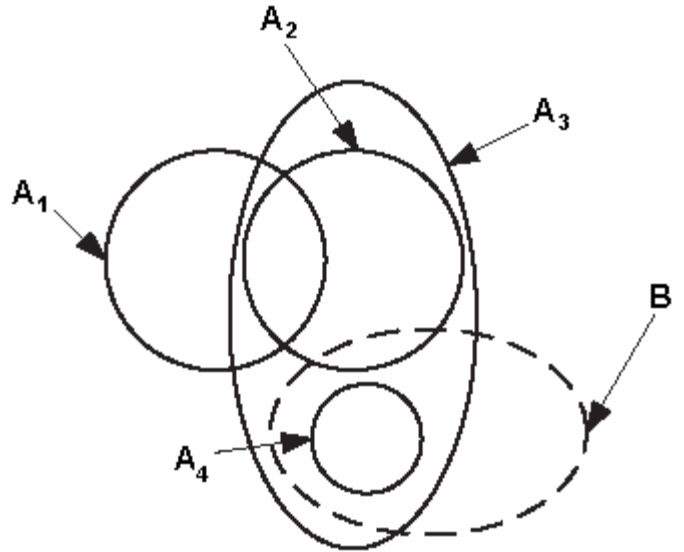


Figure 4.7 Example Problem

The two measures, *Belief* and *Plausibility* are nonadditive. It is not required for the sum of all the Belief measures to be 1 and similarly for the sum of the Plausibility measures. In fact,

$$Bel(A) + Bel(\bar{A}) \leq 1 \quad (4.15)$$

$$Pl(A) + Pl(\bar{A}) \geq 1 \quad (4.16)$$

It is possible to obtain the basic probability assignment from the *Belief* measure with the following inverse function:

$$m(A) = \sum_{B|B \subseteq A} (-1)^{|A-B|} Bel(B) \quad (4.17)$$

where $|A-B|$ is the difference of the cardinality of the two sets. In addition to deriving these measures from the basic probability assignment (m), these two measures can be derived from each other. For example, *Plausibility* can be derived from *Belief* in the following way:

$$Pl(A) = 1 - Bel(\bar{A}) \quad (4.18)$$

where \bar{A} is the classical complement of A . Furthermore,

$$Bel(A) \leq Pl(A) \quad (4.19)$$

for all $A \in P(\Omega)$. As a consequence of Eqn. 4.17 and 4.18, given any one of these measures ($m(A)$, $Bel(A)$, $Pl(A)$) it is possible to derive the values of the other two measures.

The precise probability of an event (in the classical sense) lies within the lower and upper bounds of *Belief* and *Plausibility*, respectively.

$$Bel(A) \leq prob(A) \leq Pl(A) \quad (4.20)$$

A belief measure (or a plausibility measure) becomes a *probability measure* when all focal elements are *singletons* or the evidences are disjoint. In this case, we have,

$$Bel(A) = prob(A) = Pl(A) \quad (4.20)$$

This corresponds to classical probability, where all the probabilities, $P(A)$ are uniquely determined for all subsets A of the universal set Ω [18, 20].

Again, when all the focal elements are nested, i.e. evidences are of consonant type, we obtain special plausibility measures, which are called *possibility measures* (or *consonant plausibility*

measures), and the corresponding special belief measures are called *necessity measures*. A possibility measure, Pos , is conveniently (and uniquely) determined by a *possibility distribution function*

$$r : \Omega \rightarrow [0,1] \quad (4.22)$$

via the formula

$$Pos(A) = \max_{x \in \Omega} r(x) \quad (4.23)$$

for all $A \in P(\Omega)$. The corresponding necessity measure, Nec , is then determined for all $A \in P(\Omega)$ by a formula equivalent to Eqn. (4.18),

$$Nec(A) = 1 - Pos(\bar{A}) \quad (4.24)$$

A theory that deals with consonant bodies of evidence in terms of possibility and necessity measures is usually called a *possibility theory* [18, 20]. Different researchers have suggested Possibility theory as an alternative for propagation of epistemic uncertainty.

So, we have seen that, the same evidence theory can also propagate the aleatory uncertainty part also along with the epistemic uncertainty. $Bel(A)$ and $Pl(A)$ may be viewed as lower and upper bounds on probabilities, respectively, where the actual probability is contained in the interval described by the bounds. This is termed as the bound on probabilities analysis described by Ferson et al [15], which they argued can handle both variability (aleatory) and ignorance (epistemic) in a single, comprehensive analysis which is faithful to both interval analysis and probability theory. He showed a simple example by taking two parameters, where the uncertainty associated with one is aleatory and with the other it is epistemic. The parameter with aleatory uncertainty is represented by a normal distribution $normal(5,1)$ (Fig. 4.8(a)) and the one with epistemic is represented by an interval $[0.5,1]$ (Fig. 4.8(b)). Then the product, sum, quotient and difference are represented in Fig. 4.8(c), (d), (e) and (f), respectively. The rules of combination are discussed in the later section of this report.

The answer that probability bounds analysis gives us is not a single probability distribution. Rather, it is a region within which the probability distribution of the result must lie. This is to say that, whatever the true value(s) of the uncertain quantity we have represented with the interval, the distribution of the product lies somewhere within the black region of the graphs. This answer

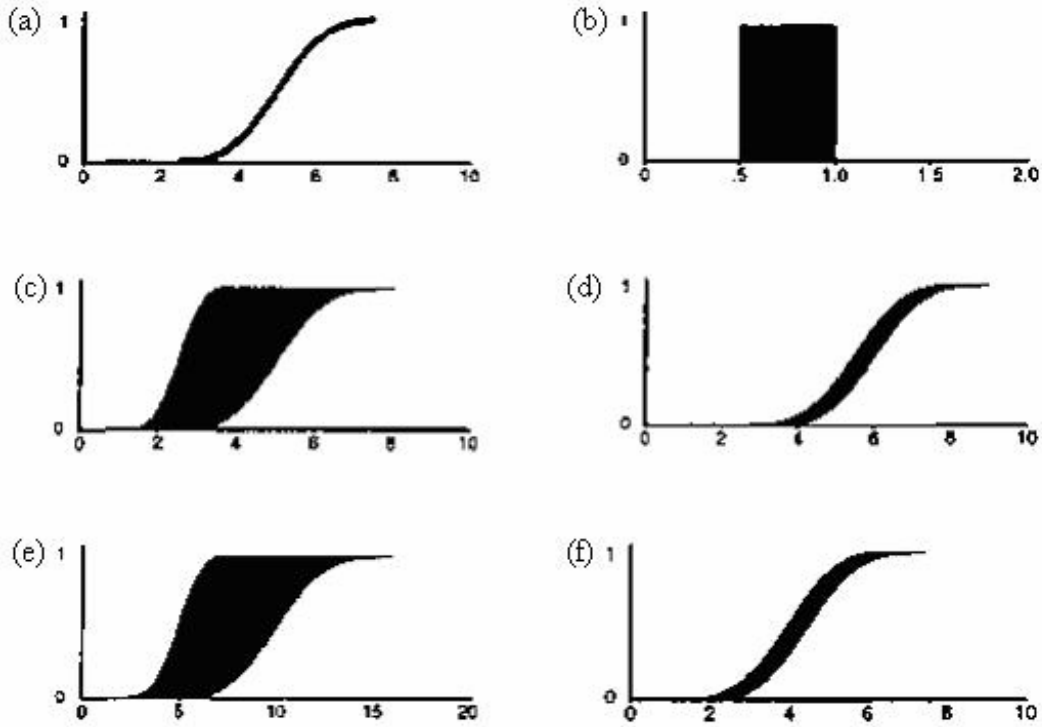


Figure 4.8 (a) A cumulative normal probability distribution with mean 5.0 and standard deviation 1.0, symbolized by the expression $\text{normal}(5,1)$. (b) An interval between one half and one, symbolized by the expression $[0.5,1]$. (c) The product of the distribution and interval, $\text{normal}(5,1) \times [0.5,1]$. The black region envelopes all the cumulative distributions that could arise as this product. (d) The sum $\text{normal}(5,1) + [0.5,1]$. (e) The quotient $\text{normal}(5,1)/[0.5,1]$. (f) The difference $\text{normal}(5,1) - [0.5,1]$. [15]

fully expresses the uncertainty induced by the two factors. Any more precise an answer would simply be underestimating the degree of uncertainty present in the calculation.

The horizontal span of the probability bounds is a function of the variability (aleatory) in the result. The vertical breadth of the bounds is a function of our ignorance (epistemic). A pure risk analysis problem with perfectly characterized probability distributions as inputs will yield a pure probability distribution as the result. Values, distributions and dependencies that are imperfectly known contribute to a widening of the bounds. The greater the ignorance, the wider the vertical distance between bounds, and the more difficult to make precise probabilistic statements about the expected frequencies of extreme events. But this is what one wants; after all, ignorance should muddle the answer to some extent. Something is obviously amiss information-theoretically if we can combine ignorance and gain more precision than we started with.

The discussion above can be effectively summarized by the representative figure below.

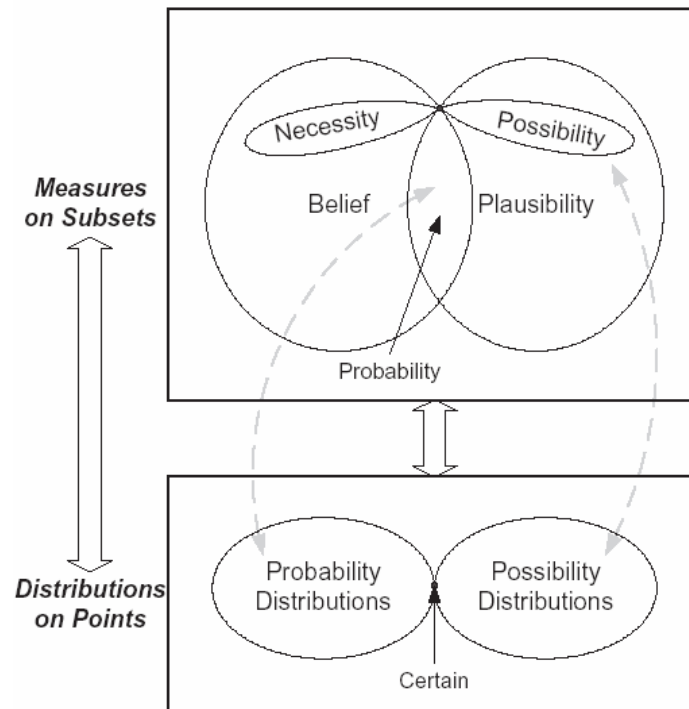


Figure 4.9: Summary of Dempster-Shafer Evidence theory

Hence, Evidence theory can be used as the *unified theory* to handle both types of uncertainty satisfactorily.

4.3.3 Combination of Evidences: The Dempster Rule

There are multiple possible ways in which evidence can be combined in Dempster-Shafer theory. The original combination rule of multiple basic probability assignments known as the Dempster rule is critical to the original conception of Dempster-Shafer theory. It is a generalization of Bayes' rule [11]. This rule strongly emphasizes the agreement between multiple sources and ignores *all* the conflicting evidence through a normalization factor. This can be considered a strict AND-operation.

Here, the measures of *Belief* and *Plausibility* are derived from the combined basic assignments. Dempster's rule combines multiple belief functions through their basic probability assignments (m). These belief functions are defined on the same frame of discernment, but are based on *independent* arguments or bodies of evidence. The issue of independence is a critical factor when combining evidence and is an important research subject in Dempster-Shafer theory [11]. The

combination rule results in a belief function based on conjunctive (AND) pooled evidence [11]. Specifically, the combination (called the joint m_{12}) is calculated from the aggregation of two bpa's m_1 and m_2 in the following manner:

$$m_{12}(A) = \frac{\sum_{B \cap C = A} m_1(B)m_2(C)}{1 - K} \quad \text{when } A \neq \phi \quad (4.25)$$

$$m_{12}(\phi) = 0 \quad (4.26)$$

$$\text{where } K = \sum_{B \cap C = \phi} m_1(B)m_2(C) \quad (4.27)$$

K represents basic probability mass associated with conflict. This is determined by the summing the products of the bpa's of all sets where the intersection is null. The denominator in Dempster's rule, $1-K$, is a normalization factor [11].

Using the operations discussed above, now we will consider the aggregation of three sources of information where the information is given as intervals. Interval-based data is common to problems involving parametric uncertainty for various practical problems. Suppose there is an experiment that provides multiple intervals for an uncertain parameter from three sources A, B, and C that must be combined. The intervals associated with sources A, B, and C are summarized in the Tables 4.1, 4.2, and 4.3, respectively. Figure 4.10 depict the intervals and the basic probability assignments graphically with a "generalized cumulative distribution function" (GCDF). This is the probabilistic concept of cumulative distribution function generalized to Dempster-Shafer structures where the focal elements (intervals) are represented on the x-axis and the cumulative basic probability assignments on the y-axis.

Interval	M_1
[1, 4]	0.5
[3, 5]	0.5

Table 4.1: source A data

Interval	M_2
[1, 4]	0.3333
[2, 5]	0.3333
[3, 6]	0.3333

Table 4.2: source B data

Interval	m_3
[6, 10]	0.3333
[9, 11]	0.3333
[12, 14]	0.3333

Table 4.3: source C data

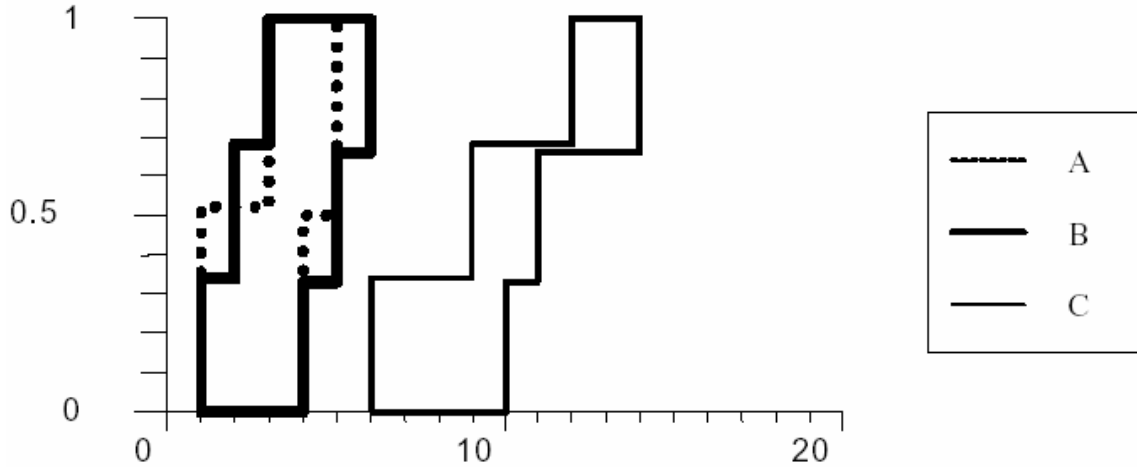


Figure 4.10: The GCDF's of A, B, and C without any combination operation

As is evident in Figure 4.10 and Table 4.1, 4.2 and 4.3, the data for A and B is consistent with each other. However the data for A and C are disjoint. First, we will consider the combination of consistent data (A and B) and then the combination of the disjoint data (A and C) with the combination rules discussed above. The calculation using Dempster rule is summarized in Table 4.4.

		A				
		Interval	m	Interval	M	
		[1, 4]	0.5	[3, 5]	0.5	
B	Interval	M				
	[1, 4]	0.3333	[1, 4]	0.16665	[3, 4]	0.16665
	[2, 5]	0.3333	[2, 4]	0.16665	[3, 5]	0.16665
	[3, 6]	0.3333	[3, 4]	0.16665	[3, 5]	0.16665

Table 4.4: Combination of A and B using Dempster's rule

The resulting structure of the combination of A and B using Dempster's rule is depicted in Figure 4.11. The bpa's for like intervals are summed, i.e. [1,4] has a value for m of 0.16665, [2,4] has an m value of 0.16665, [3,4] has a value of 0.3333; and [3,5] has an m value of 0.3333.

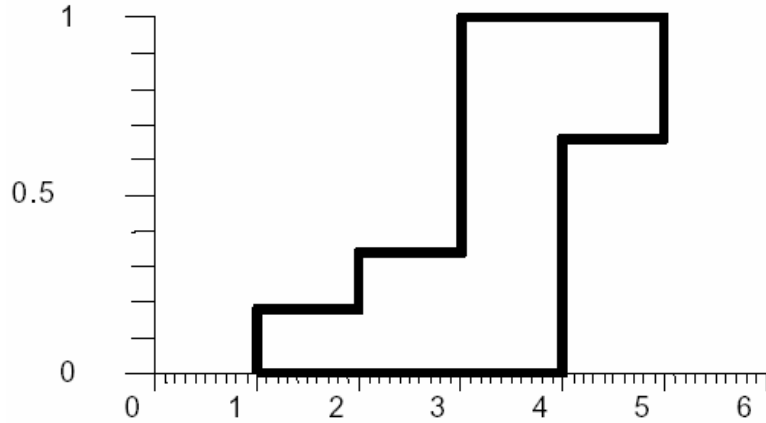


Figure 4.11: The GCDF of the combination of A and B using Dempster's rule

The combination of A and C using Dempster's rule is not possible due to the normalization factor. Since the intervals of A and C are completely disjoint, we will have $K=1$ and so we cannot use Eq. 4.25. This is a serious drawback of the Dempster rule of combination. The normalizing factor $(1-K)$ has the effect of *completely* ignoring conflict and attributing any probability mass associated with conflict to the null set [11]. In the case above we cannot simply proceed with the Dempster rule. In case partial conflict, this rule will yield counterintuitive results. The problem with conflicting evidence and Dempster's rule was originally pointed out by Lotfi Zadeh in his review of Shafer's book, *A Mathematical Theory of Evidence* [11]. Zadeh provides a compelling example of erroneous results. Suppose that a patient is seen by two physicians regarding the patient's neurological symptoms. The first doctor believes that the patient has either meningitis with a probability of 0.99 or a brain tumor, with a probability of 0.01. The second physician believes the patient actually suffers from a concussion with a probability of 0.99 but admits the possibility of a brain tumor with a probability of 0.01. The distributions can be represented by the following:

Physician 1:

$m_1(A) = 0.99$ (probability of meningitis)

$m_1(B) = 0.01$ (probability of brain tumor)

Physician 2:

$m_2(B) = 0.01$ (probability of brain tumor)

$m_2(C) = 0.99$ (probability of concussion)

So, we can see that both physicians has given brain tumor the least probability assignment. Now, let us calculate the combined probability of the patient having brain tumor based on the Dempster rule. The combination of the m's based on the physicians is summarized in Table 4.5.

			Physician 1			
			A	B	C	Disease
			0.99	0.01	0	m_1
Physician 2	Disease	M_2				
	A	0	$m_1(A)m_2(A)=0$	$m_1(A)m_2(A)=0$	$m_1(A)m_2(A)=0$	
	B	0.01	$m_1(A)m_2(A)=0.0099$	$m_1(A)m_2(A)=0.0001$	$m_1(A)m_2(A)=0$	
	C	0.99	$m_1(A)m_2(A)=0.9801$	$m_1(A)m_2(A)=0.0099$	$m_1(A)m_2(A)=0$	

Table 4.5: Dempster Combination of Physician 1 and Physician 2

The only nonzero value is for the combination of B ,

$$m_1(B)m_2(B) = (0.01) \times (0.01) = 0.0001 \quad (4.28)$$

For K , there are three cells that contribute to conflict represented by empty intersections. Using Equation 4.27,

$$K = (0.99)(0.01) + (0.99)(0.01) + (0.99)(0.99) = 0.9999 \quad (4.29)$$

Now using Eq. 4.25, $m(\text{brain tumor})$ based on both the physicians judgments is given by,

$$m_{12}(B) = \frac{0.01 \times 0.01}{1 - 0.9999} = 1 \quad (4.30)$$

Also $m_{12}(A) = m_{12}(C) = 0$. So, Dempster rule applied as it given that $\text{Bel}(\text{brain tumor}) = 1$. Clearly, this rule of combination yields a result that implies complete support for a diagnosis that both physicians considered to be very unlikely.

In light of this simple but dramatic example of the counterintuitive results of normalization factor in Dempster's rule, a number of methods and combination operations have been developed to address this problem posed by strongly conflicting evidence.

4.5 Alternate Approaches:

Dempster-Shafer theory essentially combines the Bayesian notion of probabilities with the classical idea of sets where a numerical value signifying confidence can be assigned to sets of

simple events rather than to just mutually exclusive simple events. But, the classical Dempster rule fails to combine information in case of conflicting evidences. Much of the recent research in the combination rules in Dempster-Shafer theory is devoted to advancing a more accurate mathematical representation of conflict. In literature we can find four modified Dempster rules [11]:

1. Yager's rule
2. Inagaki's unified combination rule
3. Zhang's center combination rule
4. Dubois and Prade's disjunctive pooling rule.

Again three different types of combination rules, other than Dempster's rule for the aggregation of evidence from multiple sources, have been suggested by different researchers, which are [11]:

1. discount and combine
2. convolutive averaging
3. mixing

We won't discuss these alternatives in this report. Further studies are needed in this context to employ any of these combination rules in an application.

Chapter 5

Conclusion and Future Works

The concept of robust design has been discussed in this report. Starting from the initial Taguchi method to the latest developed by Aerospace System Design Lab of GaTech are briefly described. The necessity of defining multiple type uncertainty has been raised and proper distinction between them has been made. The various tools for representing and propagating the two kinds of uncertainty are briefly introduced. The Dempster-Shafer Evidence theory showed much promise to be considered as the unified theory to propagate both the aleatory and epistemic uncertainty satisfactorily. One of the future works will be to develop a formal procedure by which one could effectively propagate the different types of uncertainty through a system of analysis.

Then we will look into some of the problems in aerospace engineering field under study in some leading research institute (like ASDL, GaTech) in the light of this new approach. The already solved problems will help us in gaining more experience. They can also be treated as verifying tool for this new methodology or suggesting any improvement, if needed.

The ultimate goal of this project is to build up of the necessary framework and understanding to routinely perform robust design. We want to build up the adequate computer resources for uncertainty quantification and then integration of these methods with various analysis modules, especially high fidelity analysis modules like CFD, FEM etc, which is scheduled for the third stage.

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