10

Evidence Theory for Engineering Applications

William L. Oberkampf
Sandia National Laboratories

Jon C. Helton
Sandia National Laboratories

10.1 Introduction

10.1.1 Background

Computational analysis of the performance, reliability, and safety of engineered systems is spreading rapidly in industry and government. To many managers, decision makers, and politicians not trained in computational simulation, computer simulations can appear most convincing. Terminology such as "virtual prototyping," "virtual testing," "full physics simulation," and "modeling and simulation-based acquisition" are extremely appealing when budgets are highly constrained; competitors are taking market share; or when political constraints do not allow testing of certain systems. To assess the accuracy and usefulness of computational simulations, three key aspects are needed in the analysis and experimental process: computer code and solution verification; experimental validation of most, if not all, of the mathematical models of the engineered system being simulated; and estimation of the uncertainty associated with analysis inputs, physics models, possible scenarios experienced by the system, and the outputs of interest in the simulation. The topics of verification and validation are not addressed here, but these are covered at length in the literature (see, for example, [1–6]). A number of fields have contributed to the development of uncertainty estimation techniques and procedures, such as nuclear reactor safety, underground storage of radioactive and toxic wastes, and structural dynamics (see, for example, [7–18]).

Uncertainty estimation for engineered systems is sometimes referred to as the simulation of nondeterministic systems. The mathematical model of the system, which includes the influence of the environment on the system, is considered nondeterministic in the sense that: (i) the model can produce nonunique system responses because of the existence of uncertainty in the input data for the model, or (ii) there are multiple alternative mathematical models for the system. The mathematical models, however, are assumed to be deterministic in the sense that when all necessary input data for a designated model is specified, the model produces only one value for every output quantity. To predict the nondeterministic response of the system, it is necessary to evaluate the mathematical model, or alternative mathematical models, of the system multiple times using different input data. This presentation does not consider chaotic systems or systems with hysteresis, that is, mathematical models that map a unique input state to multiple output states.

Many investigators in the risk assessment community segregate uncertainty into *aleatory* uncertainty and *epistemic* uncertainty. Aleatory uncertainty is also referred to as variability, irreducible uncertainty, inherent uncertainty, stochastic uncertainty, and uncertainty due to chance. Epistemic uncertainty is also referred to as reducible uncertainty, subjective uncertainty, and uncertainty due to lack of knowledge. Some of the investigators who have argued for the importance of distinguishing between aleatory uncertainty and epistemic uncertainty are noted in [19–32]. We believe the benefits of distinguishing between aleatory and epistemic uncertainty include improved interpretation of simulation results by decision makers and improved ability to allocate resources to decrease system response uncertainty or risk. Note that in the present work we use the term "risk" to mean a measure of the likelihood and severity of an adverse event occurring [13, 33].

Sources of aleatory uncertainty can commonly be singled out from other contributors to uncertainty by their representation as randomly distributed quantities that take values in an established or known range, but for which the exact value will vary by chance from unit to unit or from time to time. The mathematical representation most commonly used for aleatory uncertainty is a probability distribution. When substantial experimental data is available for estimating a distribution, there is no debate that the correct mathematical model for aleatory uncertainty is a probability distribution. Propagation of these distributions through a modeling and simulation process is well developed and is described in many texts (see, for example, [31, 34–38]).

Epistemic uncertainty derives from some level of ignorance about the system or the environment. For this presentation, *epistemic uncertainty* is defined as any lack of knowledge or information in any phase or activity of the modeling process [39]. The key feature stressed in this definition is that the fundamental source of epistemic uncertainty is incomplete information or incomplete knowledge of some characteristic of the system or the environment. As a result, an increase in knowledge or information can lead to a reduction in the predicted uncertainty of the response of the system, all things being equal. Examples of sources of epistemic uncertainty are: little or no experimental data for a fixed (but unknown) physical parameter, a range of possible values of a physical quantity provided by expert opinion, limited understanding of complex physical processes, and the existence of fault sequences or environmental conditions not identified for inclusion in the analysis of a system. For further discussion of the sources of epistemic uncertainty in engineering systems see, for example, [40, 41].

Epistemic uncertainty has traditionally been represented with a random variable using subjective probability distributions. However, a major concern is that when there is little or no closely related experimental data, a common practice is to simply pick some familiar probability distribution and its associated parameters to represent one's belief in the likelihood of possible values that could occur. Two important weaknesses with this common approach are of critical interest when the assessment of epistemic uncertainty is the focus. First, even small epistemic uncertainty in parameters for continuous probability distributions, such as a normal or Weibull, can cause very large changes in the tails of the distributions. For example, there can be orders-of-magnitude change in the likelihood of rare events when certain distribution parameters are changed by small amounts. Second, when epistemic uncertainty is represented as a probability distribution and when there are multiple parameters treated in this fashion, one can obtain misleading results. For example, suppose there are ten parameters in an analysis that are only *thought* to be within specified intervals; for example, the parameters are estimated from expert opinion, not measurements. Assume each of these parameters is treated as a random variable and assigned the least informative distribution (i.e., a uniform distribution). If extreme system responses correspond to extreme values of these parameters (i.e., values near the ends of the uniform distribution), then their probabilistic combination could predict a very low probability for such extreme system responses. Given that the parameters are only known to occur within intervals, however, this conclusion is grossly inappropriate.

10.1.2 Improved Models for Epistemic Uncertainty

During the past two decades, the information theory and expert systems communities have made significant progress in developing a number of new theories that can be pursued for modeling epistemic uncertainty. Examples of the newer theories include fuzzy set theory [17, 42–46], interval analysis [47, 48], evidence (Dempster-Shafer) theory [49–55], possibility theory [56, 57], and theory of upper and lower previsions [58]. Some of these theories only deal with epistemic uncertainty; most deal with both epistemic and aleatory uncertainty; and some deal with other varieties of uncertainty (e.g., nonclassical logics appropriate for artificial intelligence and data fusion systems [59]).

A recent article summarizes how these theories of uncertainty are related to one another from a hierarchical viewpoint [60]. The article shows that evidence theory is a generalization of classical probability theory. From the perspective of bodies of evidence and their measures, evidence theory can also be considered a generalization of possibility theory. However, in evidence theory and in possibility theory, the mechanics of operations applied to bodies of evidence are completely different [54, 57]. The mathematical foundations of evidence theory are well established and explained in several texts and key journal articles [49–55, 61–64]. However, essentially all of the published applications of the theory are for simple model problems — not actual engineering problems [41, 65–74]. Note that in some of the literature, evidence theory is referred to as the theory of random sets.

In evidence theory there are two complementary measures of uncertainty: belief and plausibility. Together, belief and plausibility can be thought of as defining lower and upper limits of probabilities, respectively, or interval-valued probabilities. That is, given the information or evidence available, a precise (i.e., single) probability distribution cannot be specified. Rather, a range of possible probabilities exists, all of which are consistent with the evidence. Belief and plausibility measures can be based on many types of information or evidence (e.g., experimental data for long-run frequencies of occurrence, scarce experimental data, theoretical evidence, or individual expert opinion or consensus among experts concerning the range of possible values of a parameter or possibility of the occurrence of an event). We believe that evidence theory could be an effective path forward in engineering applications because it can deal with both thoroughly characterized situations (e.g., precisely known probability distributions) and situations of near-total ignorance (e.g., only an interval containing the true value is known).

There are two fundamental differences between the approach of evidence theory and the traditional application of probability theory. First, evidence theory uses two measures — belief and plausibility — to characterize uncertainty; in contrast, probability theory uses only one measure — the probability of an event or value. Belief and plausibility measures are statements about the likelihood related to sets of possible values. There is no need to distribute the evidence to individual values in the set. For example, evidence from experimental data or from expert opinion can be given for a parameter value to be within an interval. Such evidence makes no claim concerning any specific value within the interval or the likelihood of any one value compared with any other value in the interval. In other words, less information can be specified than the least information that is typically specified in applications of probability theory (e.g., the uniform likelihood of all values in the interval).

The second fundamental difference between evidence theory and the traditional application of probability theory is that in evidence theory, the evidential measure for an event and the evidential measure for the negation of an event do not have to sum to unity (i.e., certainty). In probability theory, the measure for an event plus the measure against an event (i.e., its negation) must be unity. This sum to unity implies that the absence in the evidence for an event must be equivalent to the evidence for the negation of the event. In evidence theory, this equivalence is rejected as excessively restrictive; that is, a weak statement of evidence can result in support for an event, but the evidence makes no inference for the support of the negation of the event.

As a final background comment concerning historical perspective of evidence theory, the theory is philosophically related to the approach of Bayesian estimation. Indeed, many of the originators of evidence theory viewed it as an offshoot of Bayesian estimation for the purpose of more properly dealing with subjective probabilities. There are, however, two key differences. First, evidence theory does not assign any prior distributions to a given state of knowledge (i.e., body of evidence) if none are given. The requirement for such a prior distribution is obviated because all possible probability distributions are allowed to describe the body of evidence. Second, evidence theory does not embody the theme of updating probabilities as new evidence becomes available. In Bayesian estimation, a dominant theme is toward continually improving statistical inference as new evidence becomes available; whereas in evidence theory, the emphasis is on precisely stating the present state of knowledge—not updating the statistical evidence.

10.1.3 Chapter Outline

In the following section (Section 10.2), the mathematical framework of evidence theory is explained and contrasted to the traditional application of probability theory. The definitions of belief and plausibility are given, along with the relationships between them. Also discussed is an important quantity called the basic probability assignment (BPA). A few simple examples are given illustrating the interpretation of the belief and plausibility functions and the BPA. Section 10.3 discusses the application of evidence theory to a simple example problem. The simple system is given by an algebraic equation with two uncertain input parameters and one system response variable. The example is analyzed using the traditional application of probability theory and evidence theory. The discussion of each solution approach stresses the mathematical and procedural steps needed to compute uncertainty bounds in the system response, as well as the similarities and differences between each approach. The analyses using traditional probabilistic and evidence theory approaches are compared with regard to their assessment for the system yielding an unsafe response. The presentation concludes in Section 10.4 with a brief discussion of important research and practical issues hindering the widespread application of evidence theory to large-scale engineering systems. For example, the critical issue of propagating BPAs through a "black box" computer code using input/output sampling techniques is discussed.

10.2 Fundamentals of Evidence Theory

10.2.1 Belief, Plausibility, and BPA Functions

Evidence theory provides an alternative to the traditional manner in which probability theory is used to represent uncertainty by allowing less restrictive statements about "likelihood" than is the case with a full probabilistic specification of uncertainty. Evidence theory can be viewed as a generalization of the traditional application of probability theory. By "generalization" we mean that when probability distributions are specified, evidence theory yields the same measures of likelihood as the traditional application of probability theory. Evidence theory involves two specifications of likelihood—a belief and a plausibility—for each subset of the universal set under consideration. Formally, an application of evidence theory involves the specification of a triple $(\mathcal{S}, \mathbb{S}, m)$, where (i) $\mathcal{S}$ is a set that contains everything that could occur in the particular universe under consideration, typically referred to as the sample space or universal set; (ii) $\mathbb{S}$ is a countable collection of subsets of $\mathcal{S}$, typically referred to as the set of focal elements of $\mathcal{S}$; and (iii) m is a function defined on subsets of $\mathcal{S}$ such that

$$m(\mathcal{E}) > 0 \quad \text{if} \quad \mathcal{E} \in \mathbb{S}, m(\mathcal{E}) = 0 \quad \text{if} \quad \mathcal{E} \subset \mathcal{S} \text{ and } \mathcal{E} \notin \mathbb{S}, \tag{10.1}$$

and

$$\sum_{\mathcal{E}\in\mathbb{S}} m(\mathcal{E}) = 1. \tag{10.2}$$

The quantity $m(\mathcal{E})$ is referred to as the basic probability assignment (BPA), or the mass function, associated with the subset $\mathcal{E}$ of $\mathcal{S}$.

The sets $\mathcal{S}$ and $\mathbb{S}$ are similar to probability theory where one has the specification of a triple ($\mathcal{S}$, $\mathbb{S}$, p) called a probability space, where (i) $\mathcal{S}$ is a set that contains everything that could occur in the particular universe under consideration, (ii) $\mathbb{S}$ is a suitably restricted set of subsets of $\mathcal{S}$, and (iii) p is the function that defines probability for elements of $\mathbb{S}$ (see [75], Section IV.4). In probability theory, the set $\mathbb{S}$ is required to have the properties that (i) if $\mathcal{E}\in\mathbb{S}$, then $\mathcal{E}^C\in\mathbb{S}$, where $\mathcal{E}^C$ is the complement of $\mathcal{E}$, and (ii) if $\mathcal{E}_1$, $\mathcal{E}_2$, … is a sequence of elements of $\mathbb{S}$, then $\cup_i\mathcal{E}_i\in\mathbb{S}$ and $\cap_i\mathcal{E}_i\in\mathbb{S}$. Further, p is required to have the properties that (i) if $\mathcal{E}\in\mathbb{S}$, then $0\le p(\mathcal{E})\le 1$, (ii) $p(\mathcal{S}) = 1$, and (iii) if $\mathcal{E}_1$, $\mathcal{E}_2$, … is a sequence of disjoint sets from $\mathbb{S}$, then $p(\cup_i\mathcal{E}_i)=\sum_i p(\mathcal{E}_i)$ (see [75], Section IV.3). In the terminology of probability theory, $\mathcal{S}$ is called the sample space or universal set; elements of $\mathcal{S}$ are called elementary events; subsets of $\mathcal{S}$ contained in $\mathbb{S}$ are called events; the set $\mathbb{S}$ itself has the properties of what is called a σ-algebra (see [75], Section IV.3); and p is called a probability measure.

The sample space $\mathcal{S}$ plays the same role in both probability theory and evidence theory. However, the $\mathbb{S}$ set has a different character in the two theories. In probability theory, $\mathbb{S}$ has special algebraic properties fundamental to the development of probability and contains all subsets of $\mathcal{S}$ for which probability is defined (see [75], Section IV.3). In evidence theory, $\mathbb{S}$ has no special algebraic properties (i.e., $\mathbb{S}$ is not required to be a σ-algebra, as is the case in probability theory) and contains the subsets of $\mathcal{S}$ with nonzero BPAs. In probability theory, the function p actually defines the probabilities for elements of $\mathbb{S}$, with these probabilities being the fundamental measure of likelihood. In evidence theory, the function m is *not* the fundamental measure of likelihood. Rather, there are two measures of likelihood, called belief and plausibility, that are obtained from m as described in the next paragraph. The designation BPA for $m(\mathcal{E})$ is almost universally used, but, unfortunately, m does not define probabilities except under very special circumstances. Given the requirement in Equation 10.2, the set $\mathbb{S}$ of focal elements associated with an evidence space ($\mathcal{S}$, $\mathbb{S}$, m) can contain at most a countable number of elements; in contrast, the set $\mathbb{S}$ of events associated with a probability space ($\mathcal{S}$, $\mathbb{S}$, p) can, and essentially always does, contain an uncountable number of elements.

The belief, $Bel(\mathcal{E})$, and plausibility, $Pl(\mathcal{E})$, for a subset $\mathcal{E}$ of $\mathcal{S}$ are defined by

$$Bel(\mathcal{E}) = \sum_{\mathcal{U}\subset\mathcal{E}} m(\mathcal{U}) \tag{10.3}$$

and

$$Pl(\mathcal{E}) = \sum_{\mathcal{U}\cap\mathcal{E}\neq\varnothing} m(\mathcal{U}). \tag{10.4}$$

Conceptually, $m(\mathcal{U})$ is the amount of likelihood that is associated with a set $\mathcal{U}$ that cannot be further assigned to specific subsets of $\mathcal{U}$. Specifically, no specification is implied concerning how this likelihood is apportioned over $\mathcal{U}$. Given the preceding conceptualization of $m(\mathcal{U})$, the belief $Bel(\mathcal{E})$ can be viewed as the minimum amount of likelihood that *must* be associated with $\mathcal{E}$. Stated differently, $Bel(\mathcal{E})$ is the amount of likelihood that must be associated with $\mathcal{E}$ because the summation in Equation 10.3 involves all $\mathcal{U}$ that satisfy $\mathcal{U}\subset\mathcal{E}$. Similarly, the plausibility $Pl(\mathcal{E})$ can be viewed as the maximum amount of likelihood that *could* be associated with $\mathcal{E}$. Stated differently, $Pl(\mathcal{E})$ is the maximum amount of likelihood that could possibly be associated with $\mathcal{E}$ because the summation in Equation 10.4 involves all $\mathcal{U}$ that intersect $\mathcal{E}$. From the perspective of making informed decisions, the information provided by beliefs and plausibilities is more useful than the information provided by BPAs. This statement is made because a BPA only provides likelihood information that can be attributed to a set, but to none of its subsets. In contrast, a belief provides likelihood information about a set and all its subsets, and a plausibility provides likelihood information about a set and all sets that intersect it.

Belief and plausibility satisfy the equality

$$Bel(\mathcal{E}) + Pl(\mathcal{E}^c) = 1 \tag{10.5a}$$

for every subset $\mathcal{E}$ of $\mathcal{S}$. In words, the belief in the occurrence of an event (i.e., $Bel(\mathcal{E})$) and the plausibility of the nonoccurrence of an event (i.e., $Pl(\mathcal{E}^c)$) must sum to one. In contrast, probability theory relies on the analogous equation

$$p(\mathcal{E}) + p(\mathcal{E}^c) = 1. \tag{10.5b}$$

As is well known, Equation 10.5b states that the likelihood in the occurrence of an event and the likelihood of the nonoccurrence of an event must sum to one. Stated differently, in probability theory, the likelihood for an event is the complement of the likelihood against an event, whereas in evidence theory, there is no such assumption of symmetry.

In evidence theory, it can be shown that

$$Bel(\mathcal{E}) + Bel(\mathcal{E}^c) \leq 1. \tag{10.6}$$

The specification of belief is capable of incorporating a lack of assurance that is manifested in the sum of the beliefs in the occurrence (i.e., $Bel(\mathcal{E})$) and nonoccurrence (i.e., $Bel(\mathcal{E}^c)$) of an event $\mathcal{E}$ being less than one. Stated differently, the belief function is a likelihood measure that allows evidence for and against an event to be inconclusive. For the plausibility function, it can be shown that

$$Pl(\mathcal{E}) + Pl(\mathcal{E}^c) \geq 1. \tag{10.7}$$

The specification of plausibility is capable of incorporating a recognition of alternatives that is manifested in the sum of the plausibilities in the occurrence (i.e., $Pl(\mathcal{E})$) and nonoccurrence (i.e., $Pl(\mathcal{E}^c)$) of an event $\mathcal{E}$ being greater than one. Stated differently, the plausibility function is a likelihood measure that allows evidence for and against an event to be redundant.

10.2.2 Cumulative and Complementary Cumulative Functions

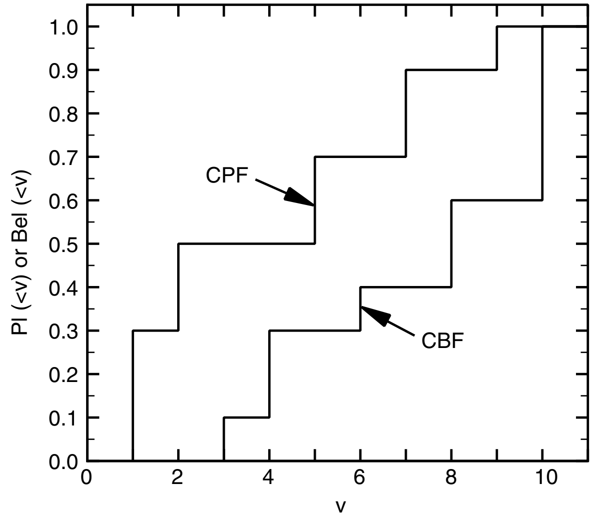
In probability theory, the cumulative distribution function (CDF) and the complementary cumulative distribution function (CCDF) are commonly used to provide summaries of the information contained in a probability space $(\mathcal{S}, \mathbb{S}, p)$. The CCDF is also referred to as the *exceedance risk* curve in risk assessment analyses. Similarly in evidence theory, cumulative belief functions (CBFs), complementary cumulative belief functions (CCBFs), cumulative plausibility functions (CPFs), and complementary cumulative plausibility functions (CCPFs) can be used to summarize beliefs and plausibilities. Specifically, CBFs, CCBFs, CPFs, and CCPFs are defined by the sets of points

$$\mathcal{CBF} = \left\{\left[v, Bel\left(\mathcal{S}_v^c\right)\right], v \in \mathcal{S}\right\} \tag{10.8}$$

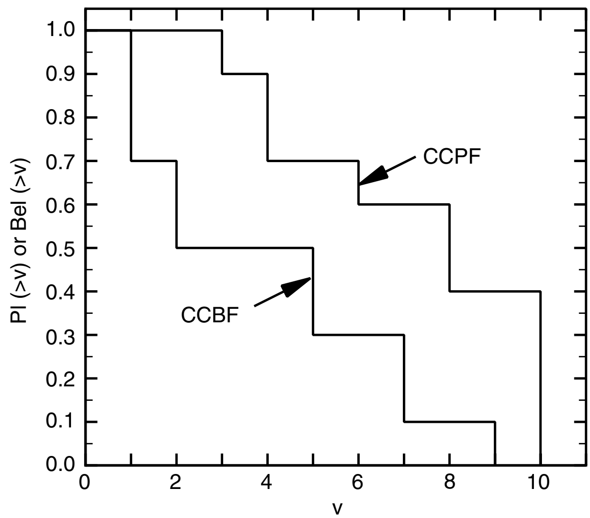
$$\mathcal{CCBF} = \left\{[v, Bel(\mathcal{S}_v)], v \in \mathcal{S}\right\} \tag{10.9}$$

$$\mathcal{CPF} = \left\{\left[v, Pl\left(\mathcal{S}_v^c\right)\right], v \in \mathcal{S}\right\} \tag{10.10}$$

$$\mathcal{CCPF} = \left\{[v, Pl(\mathcal{S}_v)], v \in \mathcal{S}\right\}, \tag{10.11}$$



(a) CBF and CPF



(b) CCBF and CCPF

FIGURE 10.1 Example of (a) cumulative belief and plausibility functions and (b) complementary cumulative belief and plausibility functions.

where $\mathcal{S}_v$ is defined as

$$\mathcal{S}_v = \{x : x \in \mathcal{S} \text{ and } x > v\}. \tag{10.12}$$

Plots of the points in the preceding sets produce CBFs, CCBFs, CPFs, and CCPFs (Figure 10.1).

As grouped in Figure 10.1, a CBF and the corresponding CPF occur together naturally as a pair because, for a given value v on the abscissa, (i) the value of the CBF (i.e., $Bel(\mathcal{S}_v^c)$) is the smallest probability for $\mathcal{S}_v^c$ that is consistent with the information characterized by $(\mathcal{S}, \mathbb{S}, m)$ and (ii) the value of the CPF (i.e., $Pl(\mathcal{S}_v^c)$) is the largest probability for $\mathcal{S}_v^c$ that is consistent with the information characterized by $(\mathcal{S}, \mathbb{S}, m)$. A similar interpretation holds for the CCBF and CCPF. Indeed, this bounding relationship occurs for any subset $\mathcal{E}$ of $\mathcal{S}$, and thus $Pl(\mathcal{E})$ and $Bel(\mathcal{E})$ can be thought of as defining upper and lower probabilities for $\mathcal{E}$ [61].

10.2.3 Input/Output Uncertainty Mapping

The primary focus in many, if not most, engineering problems involving uncertainty estimation is on functions

$$y = f(\boldsymbol{x}), \tag{10.13}$$

where

$$\boldsymbol{x} = [x_1, x_2, \ldots, x_n]$$

and the uncertainty in each x_i is characterized by a probability space $(\mathcal{X}_i, \mathbb{X}_i, p_i)$ or an evidence space $(\mathcal{X}_i, \mathbb{X}_i, m_i)$. The elements x_i of $\boldsymbol{x}$ are used to construct the input sample space

$$\mathcal{X} = \{x : x = [x_1, x_2, \ldots, x_n] \in \mathcal{X}_1 \times \mathcal{X}_2 \times \ldots \times \mathcal{X}_n\}. \tag{10.14}$$

When the uncertainty in each x_i is characterized by a probability space $(\mathcal{X}_i, \mathbb{X}_i, p_i)$, this leads to a probability space $(\mathcal{X}, \mathbb{X}, p_X)$ that characterizes the uncertainty in $\boldsymbol{x}$, where $\mathbb{X}$ is developed from the sets contained in

$$\mathbb{C} = \{\mathcal{E} : \mathcal{E} = \mathcal{E}_1 \times \mathcal{E}_2 \times \ldots \times \mathcal{E}_n \in \mathbb{X}_1 \times \mathbb{X}_2 \times \ldots \times \mathbb{X}_n\} \tag{10.15}$$

(see [75], Section IV.6, and [76], Section 2.6) and p_X is developed from the probability functions (i.e., measures) p_i. Specifically, if the x_i are independent and d_i is the density function associated with p_i, then

$$p_X(\mathcal{E}) = \int_{\mathcal{E}} d(\boldsymbol{x})\, dV \tag{10.16}$$

for $\mathcal{E} \in \mathbb{X}$ and

$$d(\boldsymbol{x}) = \prod_{i=1}^{n} d_i(x_i) \tag{10.17}$$

for $\boldsymbol{x} = [x_1, x_2, \ldots, x_n] \in \mathcal{X}$.

In engineering practice, f is often a set of nonlinear partial differential equations that are numerically solved on a computer. The dimensionality of the vector $\boldsymbol{x}$ of inputs can be high in practical problems (e.g., on the order of 50). The analysis outcome y is also often a vector of high dimensionality, but is indicated in Equation 10.13 as being a single system response for notational convenience.

The sample space $\mathcal{X}$ constitutes the domain for the function f in Equation 10.13. In turn, the range of f is given by the set

$$\mathcal{Y} = \{y : y = f(\boldsymbol{x}), \boldsymbol{x} \in \mathcal{X}\}. \tag{10.18}$$

The uncertainty in the values of y contained in $\mathcal{Y}$ derives from the probability space $(\mathcal{X}, \mathbb{X}, p_X)$ that characterizes the uncertainty in $\boldsymbol{x}$ and from the properties of the function f. In concept, $(\mathcal{X}, \mathbb{X}, p_X)$ and f induce a probability space $(\mathcal{Y}, \mathbb{Y}, p_Y)$. The probability p_Y is defined for a subset $\mathcal{E}$ of $\mathcal{Y}$ by

$$p_Y(\mathcal{E}) = p_X(f^{-1}(\mathcal{E})), \tag{10.19}$$

where

$$f^{-1}(\mathcal{E}) = \{x : x \in \mathcal{X} \text{ and } y = f(x) \in \mathcal{E}\}. \tag{10.20}$$

The uncertainty in y characterized by the probability space $(\mathcal{Y}, \mathbb{Y}, p_Y)$ is typically presented as a CDF or CCDF.

Equation 10.19 describes the mapping of probabilities from the input space to probabilities in the space of the system outcome y when the uncertainty associated with the domain of the function is characterized by a probability space $(\mathcal{X}, \mathbb{X}, p_X)$. When the uncertainty in each x_i is characterized by an evidence space $(\mathcal{X}_i, \mathbb{X}_i, m_i)$, the uncertainty in $\boldsymbol{x}$ is characterized by an evidence space $(\mathcal{X}, \mathbb{X}, m_X)$, where (i) $\mathcal{X}$ is defined by Equation 10.14, (ii) $\mathbb{X}$ is the same as the set $\mathbb{C}$ defined in Equation 10.15, and (iii) under the assumption that the x_i are independent, m_X is defined by

$$m_X(\mathcal{E}) = \begin{cases} \prod_{i=1}^{n} m_i(\mathcal{E}_i) & \text{if } \mathcal{E} = \mathcal{E}_1 \times \mathcal{E}_2 \times \ldots \times \mathcal{E}_n \in \mathbb{X} \\ 0 & \text{otherwise} \end{cases} \tag{10.21}$$

for subsets $\mathcal{E}$ of $\mathcal{X}$.

The development is more complex when the x_i are not independent. For a vector $\boldsymbol{x}$ of the form defined in conjunction with Equation 10.13, the structure of $\mathbb{X}$ for the evidence space $(\mathcal{X}, \mathbb{X}, m_X)$ is much simpler than the structure of $\mathbb{X}$ for an analogous probability space $(\mathcal{X}, \mathbb{X}, p_X)$. In particular, the set $\mathbb{X}$ for the evidence space $(\mathcal{X}, \mathbb{X}, m_X)$ is the same as the set $\mathbb{C}$ in Equation 10.15. In contrast, the set $\mathbb{X}$ for an analogous probability space $(\mathcal{X}, \mathbb{X}, p_X)$ is constructed from $\mathbb{C}$ and in general contains an uncountable number of elements rather than the finite number of elements usually contained in $\mathbb{C}$.

For evidence theory, relations analogous to Equation 10.19 define belief and plausibility for system outcomes when the uncertainty associated with the domain of the function is characterized by an evidence space $(\mathcal{X}, \mathbb{X}, m_X)$. In concept, $(\mathcal{X}, \mathbb{X}, m_X)$ and f induce an evidence space $(\mathcal{Y}, \mathbb{Y}, m_Y)$. In practice, m_Y is not determined. Rather, the belief $Bel_Y(\mathcal{E})$ and plausibility $Pl_Y(\mathcal{E})$ for a subset $\mathcal{E}$ of $\mathcal{Y}$ are determined from the BPA m_X associated with $(\mathcal{X}, \mathbb{X}, m_X)$. In particular,

$$Bel_Y(\mathcal{E}) = Bel_X[f^{-1}(\mathcal{E})] = \sum_{\mathcal{U} \subset f^{-1}(\mathcal{E})} m_X(\mathcal{U}) \tag{10.22}$$

and

$$Pl_Y(\mathcal{E}) = Pl_X[f^{-1}(\mathcal{E})] = \sum_{\mathcal{U} \cap f^{-1}(\mathcal{E}) \neq \varnothing} m_X(\mathcal{U}), \tag{10.23}$$

where Bel_X and Pl_X represent belief and plausibility defined for the evidence space $(\mathcal{X}, \mathbb{X}, m_X)$.

Similarly to the use of CDFs and CCDFs in probability theory, the uncertainty in y characterized by the evidence space $(\mathcal{Y}, \mathbb{Y}, m_Y)$ can be summarized with CBFs, CCBFs, CPFs, and CCPFs. In particular, the CBF, CCBF, CPF, and CCPF for y are defined by the sets of points

$$\mathcal{CBF} = \left\{\left[v, Bel_Y\left(\mathcal{Y}_v^c\right)\right], v \in \mathcal{Y}\right\} = \left\{\left[v, Bel_X\left(f^{-1}\left(\mathcal{Y}_v^c\right)\right)\right], v \in \mathcal{Y}\right\} \tag{10.24}$$

$$\mathcal{CCBF} = \{[v, Bel_Y(\mathcal{Y}_v)], v \in \mathcal{Y}\} = \{[v, Bel_X(f^{-1}(\mathcal{Y}_v))], v \in \mathcal{Y}\} \tag{10.25}$$

$$\mathcal{CPF} = \left\{\left[v, Pl_Y\left(\mathcal{Y}_v^c\right)\right], v \in \mathcal{Y}\right\} = \left\{\left[v, Pl_X\left(f^{-1}\left(\mathcal{Y}_v^c\right)\right)\right], v \in \mathcal{Y}\right\} \tag{10.26}$$

$$\mathcal{CCPF} = \{[v, Pl_Y(\mathcal{Y}_v)], v \in \mathcal{Y}\} = \{[v, Pl_X(f^{-1}(\mathcal{Y}_v))], v \in \mathcal{Y}\}, \tag{10.27}$$

where

$$\mathcal{Y}_v = \{y : y \in \mathcal{Y} \text{ and } y > v\} \tag{10.28}$$

$$\mathcal{Y}_v^c = \{y : y \in \mathcal{Y} \text{ and } y \leq v\}. \tag{10.29}$$

Plots of the points contained in $\mathcal{CBF}$, $\mathcal{CCBF}$, $\mathcal{CPF}$, and $\mathcal{CCPF}$ produce a figure similar to Figure 10.1 and provide a visual representation of the uncertainty in y in terms of belief and plausibility.

The beliefs and plausibilities appearing in Equation 10.24 through Equation 10.27 are defined by sums of BPAs for elements of $\mathbb{X}$. For notational convenience, let $\mathcal{E}_j$ denote the jth element of $\mathbb{X}$ for the evidence space $(\mathcal{X}, \mathbb{X}, m_X)$. Such a numbering is possible because $\mathbb{X}$ is countable due to the constraint imposed by Equation 10.2. For $v \in \mathcal{Y}$, let

$$\mathcal{ICBF}_v = \left\{ j : \mathcal{E}_j \subset f^{-1}\left(\mathcal{Y}_v^c\right) \right\} \tag{10.30}$$

$$\mathcal{ICCBF}_v = \{ j : \mathcal{E}_j \subset f^{-1}(\mathcal{Y}_v) \} \tag{10.31}$$

$$\mathcal{ICPF}_v = \left\{ j : \mathcal{E}_j \cap f^{-1}\left(\mathcal{Y}_v^c\right) \neq \varnothing \right\} \tag{10.32}$$

$$\mathcal{ICCPF}_v = \{ j : \mathcal{E}_j \cap f^{-1}(\mathcal{Y}_v) \neq \varnothing \}. \tag{10.33}$$

In turn, the beliefs and plausibilities in Equation 10.24 through Equation 10.27 are defined by

$$Bel_Y\left(\mathcal{Y}_v^c\right) = Bel_X\left(f^{-1}\left(\mathcal{Y}_v^c\right)\right) = \sum_{j \in \mathcal{ICBF}_v} m_X(\mathcal{E}_j) \tag{10.34}$$

$$Bel_Y(\mathcal{Y}_v) = Bel_X(f^{-1}(\mathcal{Y}_v)) = \sum_{j \in \mathcal{ICCBF}_v} m_X(\mathcal{E}_j) \tag{10.35}$$

$$Pl_Y\left(\mathcal{Y}_v^c\right) = Pl_X\left(f^{-1}\left(\mathcal{Y}_v^c\right)\right) = \sum_{j \in \mathcal{ICPF}_v} m_X(\mathcal{E}_j) \tag{10.36}$$

$$Pl_Y(\mathcal{Y}_v) = Pl_X(f^{-1}(\mathcal{Y}_v)) = \sum_{j \in \mathcal{ICCPF}_v} m_X(\mathcal{E}_j). \tag{10.37}$$

The summations in Equation 10.34 through Equation 10.37 provide formulas by which the CBF, CCBF, CPF, and CCPF defined in Equation 10.24 through Equation 10.27 can be calculated. In practice, determination of the sets in Equation 10.24 through Equation 10.27 can be computationally demanding due to the computational complexity of determining f^{-1}. For example, if f corresponds to the numerical solution of a system of nonlinear PDEs, there will be no closed form representation for f^{-1} and the computation of approximate representations for f^{-1} will require many computationally demanding evaluations of f. Section 10.3.3.4 gives a detailed example of how Equation 10.24 through Equation 10.27 are calculated in a simple example. The issue of convergence of sampling was recently addressed in [77], where it was established that, as the number of samples increases, even with minimal assumptions concerning the nature of f, convergence to the correct belief and plausibility of the system response is ensured.

10.2.4 Simple Conceptual Examples

The theoretical fundamentals given above can be rather impenetrable, even to those well grounded in the theoretical aspects of probability theory. We believe two of the reasons evidence theory is difficult to grasp are the following. First, evidence theory has *three* likelihood measures — belief, plausibility, and BPA functions—any one of which can determine the other two. In probability theory there is only one, the probability measure p. Second, the conversion of data or information into a BPA in evidence theory seems rather nebulous and confusing compared to the construction of probability measures in probability theory. In practice, this conversion and its representation in probability theory is simplified by using probability density functions (PDFs) as surrogates for the corresponding probability measures. For example, in probability theory if one assumes a "noninformative prior" then a uniform PDF is chosen. Or, if one has a histogram of experimental data, it is rather straightforward to construct a PDF. To aid in understanding evidence theory and how it compares with probability theory, the following two simple conceptual examples (similar to those given in [53], p. 75–76) are given.

Example 1

Children are playing with black and white hollow plastic Easter eggs. The children place a chocolate candy in each black Easter egg and put nothing into each white Easter egg. A parent appears and says, "I will secretly put one or more of your various Easter eggs in a paper bag. Tell me what is the probability (these are very precocious children) of you drawing out a black Easter egg from the paper bag (which you cannot see into)?"

The universal set X contains the two possible outcomes: $\{B, W\}$, where B indicates a black egg with chocolate, and W indicates a white egg without chocolate.

A probabilistic solution would traditionally assume that, according to the Principle of Insufficient Reason, there is an equal likelihood of drawing out a black egg and a white egg. Therefore, the answer to the question would be: "There is a 0.5 probability of getting a black egg."

An evidence theory solution would assign the BPA of the universal set a value of 1, which can be written as $m(X) = 1$. Because nothing is known about what this parent might do concerning putting white vs. black eggs in the bag, a BPA of zero is assigned to each possible event: $m(B) = 0$ and $m(W) = 0$. Using Equations 10.3 and 10.4, one computes that $Bel(B) = 0$ and $Pl(B) = 1$ for drawing out a black egg. Therefore, an evidence theory approach would answer the question: "The probability is between 0 and 1 for getting a black egg."

Example 2

The same children are playing with plastic Easter eggs and the same parent shows up. However, this time, the parent paints some of the plastic Easter eggs gray and places chocolates into some of the gray eggs, out of sight of the children. The parent now says, "I have put ten Easter eggs into a paper bag, which you cannot see into. In the bag there are two black eggs (with chocolates), three white eggs (without chocolates), and five gray eggs that may or may not contain a chocolate. Tell me what is the probability of drawing out an egg with a chocolate?"

The universal set X can be written as $\{B, W, G\}$; B and W are as before, and G is a gray egg which may or may not have a chocolate. A probabilistic solution could be based on the probabilities

$$p(B) = 0.2,\ p(W) = 0.3,\ p(G) = 0.5\,.$$

In addition, according to the Principle of Insufficient Reason, each gray egg containing a chocolate, G_w, has a probability of 0.5, and each gray egg without a chocolate, G_{wo}, has a probability of 0.5. Therefore, they would assign a probability of 0.25 to the likelihood that a gray egg with chocolate would be drawn. The probabilistic answer to the question would then be: "The probability of getting a chocolate from the bag is $p(\mathrm{B}) + 0.25 = 0.45$."

The evidence theory solution would assign the following BPAs

$$m(B) = 0.2,\ m(W) = 0.3,\ m(G_w, G_{wo}) = 0.5\,.$$

With the use of Equations 10.3 and 10.4, the belief for getting a chocolate, $Bel(C)$, and plausibility for getting a chocolate, $Pl(C)$, can be computed:

$$Bel(C) = Bel(B, G_w) = 0.2,\ Pl(C) = Pl(B, G_w) = 0.2 + 0.5 = 0.7.$$

Therefore, an evidence theory answer to the question would be: "There is a probability between 0.2 and 0.7 of getting a chocolate from the bag."

10.2.4.1 Observations

First, on these simple examples, it can be seen, hopefully, that evidence theory accurately represents the range in probabilities that are consistent with the given data; no additional assumptions concerning the given data are imposed. Stated differently, traditional application of probability theory leads one to assign probabilities to all events of the universal set, thereby forcing one to make assumptions that are not supported by the evidence. Evidence theory allows one to assign basic probabilities to *sets* of elements in the universal set, thus avoiding unjustified assumptions. Concern might be expressed that the range of probabilities with evidence theory is so large that little useful information is gained with the approach to aid in the decision-making process. The response to this is that when large epistemic uncertainty is present, the decision maker should be clearly aware of the range of probabilities, rather than having assumptions buried in the analysis disguise the probabilities. If high-consequence decisions are involved instead of chocolates, it is imperative that the decision maker understand the probabilities and resulting risks. If the highest possible risks are unacceptable to the decision maker, then resources must be made available to reduce the epistemic uncertainty.

Second, in these examples the inappropriate use of the Principle of Insufficient Reason, particularly in Example 1, is obvious. However, the assumption, without justification, of a uniform PDF in engineering analyses is very common. Sometimes the assumption is made with the caveat that, "For the first pass through the analysis, a uniform PDF is assumed." If the decision maker acts on the "first pass analysis" and a refined analysis is never conducted, inappropriate risks could be the result. Or, the more common situation might occur: "The risks appear acceptable based on the first pass analysis, and if the funds and schedule permit, we will conduct a more refined analysis in the future." Commonly, the funds and schedule are consumed with "more pressing issues."

10.3 Example Problem

The topics covered in this section are the following. First, an algebraic equation will be given (Section 10.3.1), which is a model for describing the response of a simple nondeterministic system. The nondeterministic character of the system is due to uncertainty in the parameters embodied in the algebraic model of the system. Second, the uncertainty of the response of the system will be estimated using the traditional application of probability theory (Section 10.3.2) and evidence theory (Section 10.3.3). Comparisons will be made (Section 10.3.4) concerning the representation of uncertainty using each approach. Third, in the solution procedure using evidence theory, the steps are described for converting the information concerning the uncertain parameters into input structures usable by evidence theory. A detailed discussion will be given for propagating input uncertainties represented by BPAs through the system response function and computing belief and plausibility measures in the output space.

10.3.1 Problem Description

The following equation is a simple special case of the input/output mapping indicated in Equation 10.13:

$$y = f(a,b) = (a+b)^{a}. \tag{10.38}$$

The parameters a and b are the analysis inputs and are the only uncertain quantities affecting the system response. Parameters a and b are independent; that is, knowledge about the value of one parameter implies nothing about the value of the other. Multiple expert sources provide information concerning a and b, but the precision of the information is relatively poor. Stated differently, there is scarce and conflicting information concerning a and b, resulting in large epistemic uncertainty. All of the sources for a and b are considered equally credible.

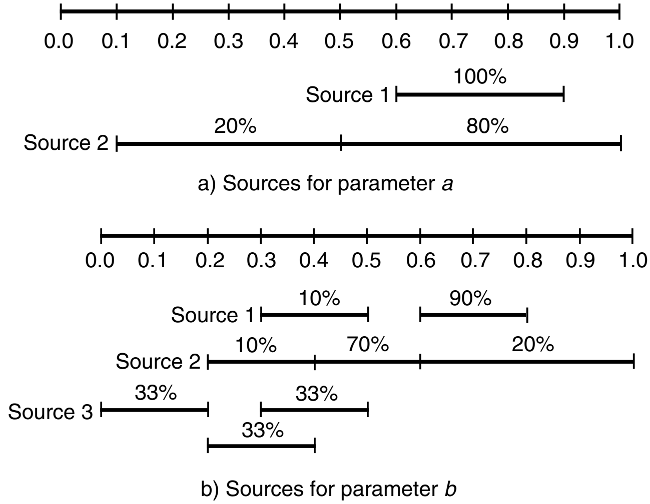


FIGURE 10.2 Information from each source for parameters a (upper) and b (lower). (Originially published in *Investigation of Evidence Theory for Engineering Applications,* Oberkampf, W.L. and Helton, J.C., 4th Nondeterministic Approaches Forum, Denver, AIAA-2002-1569. Copyright © 2002 by the American Institute of Aeronautics and Astronautics, Inc. Reprinted with permission.)

For parameter a, two sources provide information. Source 1 states that he believes that the actual, or true, value lies in the interval [0.6, 0.9]. Source 2 states that, in her opinion, the actual value is in one of two contiguous intervals: in the interval [0.1, 0.5] with a 20% level of subjective belief, or in the interval [0.5, 1.0] with a 80% level of subjective belief. (Note: For clarity, an adjective such as "subjective" or "graded" with the term "belief" is used to designate the common language meaning for "belief." When "belief" is used without such adjectives, reference is being made to the belief function in evidence theory.)

For parameter b, three sources provide information. Source 1 states that the actual value could lie in one of two disjoint intervals: in the interval [0.3, 0.5] with a 10% level of subjective belief, or in the interval [0.6, 0.8] with a 90% level of subjective belief. Source 2 states that the actual value could be in one of three contiguous intervals: [0.2, 0.4] with a 10% level of subjective belief, [0.4, 0.6] with a 70% level of subjective belief, and [0.6, 1.0] with a 20% level of subjective belief. Source 3 states that he believes that three experimental measurements he is familiar with should characterize the actual value of b. The three experimental realizations yielded: 0.1 ± 0.1, 0.3 ± 0.1, and 0.4 ± 0.1. He chooses to characterize these measurements, that is, his input to the uncertainty analysis, as three intervals: [0.0, 0.2], [0.2, 0.4] and [0.3, 0.5], all with equal levels of subjective belief. (See [74] for a description of more complex subjective belief statements.)

The input data for a and b for each source are shown graphically in Figure 10.2a and b, respectively.

To complete the statement of the mathematical model of the system, the system response y is considered to be unsafe for values of y larger than 1.7. It is desired that both the traditional and evidence theory approaches be used to assess what can be said about the occurrence of $y > 1.7$. Although the example problem may appear quite simple to some, the solution is not simple, or even unique, because of the poor information given for the parameters. It will be seen that both methods require additional assumptions to estimate the occurrence of $y > 1.7$. Some of these assumptions will be shown to dominate the estimated safety of the system.

Figure 10.3 presents a three-dimensional representation of $y = f(a, b)$ over the range of possible values of a and b. Several level curves, or response contours, of y are shown (i.e., loci of $[a, b]$ values that produce equal y values). The rectangle defined by $0.1 \le a \le 1$ and $0 \le b \le 1$ is referred to as the input product space.

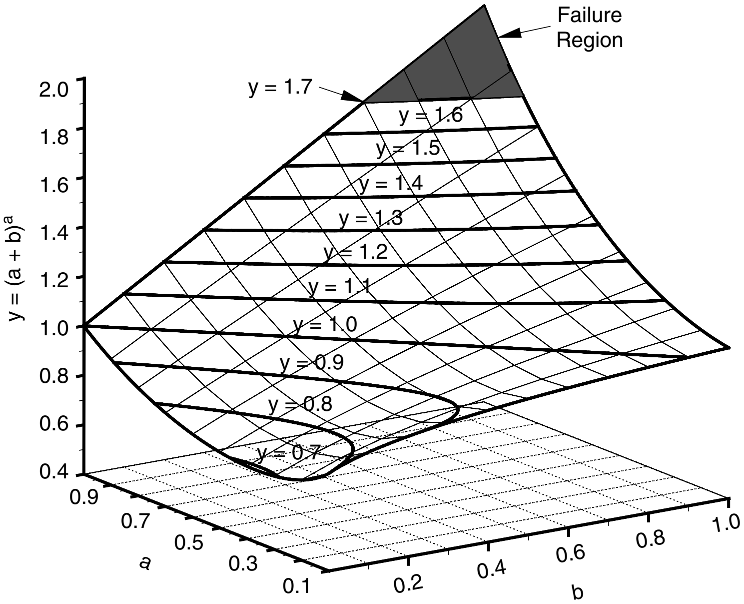


FIGURE 10.3 Three-dimensional representation of $y = (a + b)^a$ on the rectangle defined by $0.1 \le a \le 1$ and $0 \le b \le 1$. (Originially published in *Investigation of evidence theory for engineering applications,* Oberkampf, W.L. and Helton, J.C., 4th Non-Deterministic Approaches Forum, Denver, AIAA-2002-1569. Copyright © 2002 by the American Institute of Aeronautics and Astronautics, Inc. Reprinted with permission.)

10.3.2 Traditional Analysis Using Probability Theory

10.3.2.1 Combination of Evidence

The information concerning a from the two sources is written as

$$\mathcal{A}_1 = \{a : 0.6 \le a \le 0.9\} \tag{10.39}$$

$$\mathcal{A}_2 = \begin{Bmatrix} a : 0.1 \le a \le 0.5 \text{ with 20\% belief} \\ a : 0.5 \le a \le 1.0 \text{ with 80\% belief} \end{Bmatrix}. \tag{10.40}$$

Thus, the set

$$\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 = \{a : 0.1 \le a \le 1\} \tag{10.41}$$

contains all specified values for a.

Similarly, the information for b from the three sources is written as

$$\mathcal{B}_1 = \begin{Bmatrix} b : 0.3 \le b \le 0.5 \text{ with 10\% belief} \\ b : 0.6 \le b \le 0.8 \text{ with 90\% belief} \end{Bmatrix} \tag{10.42}$$

$$\mathcal{B}_2 = \begin{Bmatrix} b : 0.2 \le b \le 0.4 \text{ with 10\% belief} \\ b : 0.4 \le b \le 0.6 \text{ with 70\% belief} \\ b : 0.6 \le b \le 1.0 \text{ with 20\% belief} \end{Bmatrix} \tag{10.43}$$

$$\mathcal{B}_2 = \begin{Bmatrix} b : 0.0 \le b \le 0.2 \text{ with 33\% belief} \\ b : 0.2 \le b \le 0.4 \text{ with 33\% belief} \\ b : 0.3 \le b \le 0.5 \text{ with 33\% belief} \end{Bmatrix}. \tag{10.44}$$

Thus, the set

$$\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 = \{b : 0 \le b \le 1\} \tag{10.45}$$

contains all specified values for b.

Given that each of the sources of information specifies only intervals of values, the traditional probabilistic analysis is implemented by assuming that a and b are uniformly distributed over each of their specified intervals. This is a significant assumption beyond what was given for the state of knowledge concerning a and b. The only claim from the expert sources is that the actual value is contained in specified intervals, *not* that all values over each of the intervals are equally likely. Other specifications of probability are possible, e.g., the intervals specified by each source could be divided into a finite number of subintervals, and probability density distributions (PDFs) could be defined for each of the subintervals. However, this is essentially never done. The specification of uniform distributions over each interval is the most common technique used to convert intervals of possible values to PDFs.

Let $[r_i, s_i]$, $i = 1, 2, \dots n$, denote the specified intervals for a parameter from a given source. Let g_i be the graded level of belief associated with the interval $[r_i, s_i]$, expressed as a decimal. Then the resultant density function d is given by

$$d(v) = \sum_{i=1}^{n} \delta_i(v) g_i/(s_i - r_i), \tag{10.46}$$

where

$$\delta_i(v) = \begin{cases} 1 & \text{for } v \in [r_i, s_i] \\ 0 & \text{otherwise.} \end{cases} \tag{10.47}$$

For parameter a, n has values of 1 and 2 for sources 1 and 2, respectively. For parameter b, n has values of 2, 3, and 3 for sources 1, 2, and 3, respectively.

Because it is given that each source of information is equally credible, the resultant density functions for each source are simply averaged. Thus, if $d_{A1}(a)$ and $d_{A2}(a)$ denote the density functions for sources 1 and 2 for variable a, the resultant combined density function is

$$d_A(a) = \sum_{i=1}^{2} d_{Ai}(a)/2. \tag{10.48}$$

Similarly, the resultant combined density function for b is

$$d_B(b) = \sum_{i=1}^{3} d_{Bi}(b)/3, \tag{10.49}$$

where $d_{B1}(b)$, $d_{B2}(b)$, and $d_{B3}(b)$ denote the density functions from sources 1, 2, and 3, respectively. The density functions $d_A(a)$ and $d_B(b)$ in Equation 10.48 and Equation 10.49 effectively define the probability space used to characterize the uncertainty in a and b, respectively.

10.3.2.2 Construction of Probabilistic Response

Each possible value for y in Equation 10.38 derives from multiple vectors $\mathbf{c} = [a, b]$ of possible values for a and b. For example, each contour line in Figure 10.3 derives from multiple values of $\mathbf{c}$ that produce the same value of y. The set

$$\mathcal{C} = \mathcal{A} \times \mathcal{B} = \{\mathbf{c} = [a, b] : a \in \mathcal{A}, b \in \mathcal{B}\} \tag{10.50}$$

contains all possible values for **c**. Because it is given that a and b are independent, the PDF for $\mathcal{C}$ is given by

$$d_C(\mathbf{c}) = d_A(a)d_B(b) \tag{10.51}$$

for $\mathbf{c} = [a, b] \in \mathcal{C}$. For notational convenience, the probability space used to characterize the uncertainty in $\mathbf{c} = [a, b]$ will be represented by $(\mathcal{C}, \mathbb{C}, p_C)$.

Possible values for a and b give rise to possible values for y through the relationship in Equation 10.38. In particular, the set $\mathcal{Y}$ of all possible values for y is given by

$$\mathcal{Y} = \{y : y = f(a, b) = (a+b)^a, [a, b] \in \mathcal{C}\}. \tag{10.52}$$

With a probability-based approach, the uncertainty in y is represented by defining a probability distribution over $\mathcal{Y}$. Ultimately, the probability distribution associated with $\mathcal{Y}$ derives from the nature of the mapping $y = f(a, b)$ and the probability distributions over $\mathcal{A}$ and $\mathcal{B}$.

The probability $p_Y(\mathcal{E})$ of a subset $\mathcal{E}$ of $\mathcal{Y}$ can be formally represented by

$$p_Y(\mathcal{E}) = \int_{\mathcal{E}} d_Y(y)dy = p_C(f^{-1}(\mathcal{E})), \tag{10.53}$$

where d_Y denotes the density function associated with the distribution of y, $p_C(f^{-1}(\mathcal{E}))$ denotes the probability of the set $f^{-1}(\mathcal{E})$ in $\mathcal{C} = \mathcal{A} \times \mathcal{B}$, and

$$f^{-1}(\mathcal{E}) = \{\mathbf{c} : \mathbf{c} = [a, b] \in \mathcal{C} = \mathcal{A} \times \mathcal{B} \text{ and } y = f(a, b) \in \mathcal{E}\}. \tag{10.54}$$

A closed-form representation for the density function d_Y can be derived from f, d_A and d_B. However, in real analysis problems, this is rarely done due to the complexity of the function and distributions involved. The relations in Equation 10.53 are presented to emphasize that probabilities for subsets of $\mathcal{Y}$ result from probabilities for subsets of $\mathcal{C}$.

An alternative representation for $p_Y(\mathcal{E})$ is given by

$$\begin{aligned} p_Y(\mathcal{E}) &= \int_{\mathcal{Y}} \delta_E(y) d_Y(y)dy \\ &= \int_{\mathcal{C}} \delta_E[f(a, b)] d_A(a) d_B(b) da\, db, \end{aligned} \tag{10.55}$$

where

$$\delta_E(y) = \begin{cases} 1 & \text{if } y \in \mathcal{E} \\ 0 & \text{otherwise.} \end{cases} \tag{10.56}$$

The preceding representation underlies the calculations carried out when a Monte Carlo or Latin Hypercube sampling procedure is used to estimate $p_Y(\mathcal{E})$ [78–80].

As indicated in Section 10.2.2, CCDFs provide a standard way to summarize probability distributions and provide an answer to the following commonly asked question: "How likely is y to be this large or larger?" The defining equation for a CCDF is analogous to Equation 10.25 and Equation 10.27 for CCBFs and CCPFs, respectively, in evidence theory. Specifically, the CCDF for y is defined by the set of points

$$\mathcal{CCDF} = \{[v, p_Y(\mathcal{Y}_v)], v \in \mathcal{Y}\} = \{[v, p_C(f^{-1}(\mathcal{Y}_v))], v \in \mathcal{Y}\} \tag{10.57}$$

for $\mathcal{Y}_v$ defined in Equation 10.28. The CCDF for the function $y = f(a, b)$ defined in Equation 10.38 and the probability space $(\mathcal{C}, \mathbb{C}, p_C)$ introduced in this section is presented in Section 10.3.4.

10.3.3 Analysis Using Evidence Theory

10.3.3.1 Construction of Basic Probability Assignments for Individual Inputs

The BPAs for $\mathcal{A}$ and $\mathcal{B}$ are obtained by first defining BPAs for $\mathcal{A}_1$ and $\mathcal{A}_2$, and $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_3$. This is done using the information for a and b from each source (see Equation 10.39, 10.40 and Equation 10.42 through Equation 10.44). A convenient representational device for BPAs associated with interval data can be obtained with the use of lower triangular matrices. This representation, an extension of that used in [81], is constructed as follows. For each uncertain parameter, an interval is identified that contains the range of the parameter when all of the sources of information for that parameter are combined. The range of each parameter is divided into as many contiguous subintervals as needed to describe the interval value information from each of the sources; that is, any specified interval of values for the parameter is equal to the union of a subset of these contiguous intervals. The columns of the lower triangular matrix are indexed by taking the lower value of each subinterval, and the rows are indexed by taking the upper value of each subinterval.

To represent this lower triangular matrix of potential nonzero BPAs, let $l_1, l_2, \ldots l_n$ be the lower values for the n subintervals, where $l_1 \le l_2 \ldots \le l_n$. Let $u_1, u_2, \ldots u_n$ be the upper values of the subintervals, where $u_1 \le u_2 \ldots \le u_n$. Consistent with the contiguous assumption, the intervals can be expressed as $[l_1, l_2]$ $[l_2, l_3], \ldots [l_n, u_n]$ with $l_n \le u_n$, or equivalently as $[l_1, u_1], [u_1, u_2], \ldots [u_{n-1}, u_n]$ with $l_1 \le u_1$. Let $m([l_i, u_j])$ be the BPA for the subinterval $[l_i, u_j]$. The $n \times n$ lower triangular matrix can then be written as

$$\begin{array}{cccccc} & l_1 & l_2 & l_3 & \cdots & l_n \\ u_1 & m([l_1,u_1]) & & & & \\ u_2 & m([l_1,u_2]) & m([l_2,u_2]) & & & \\ u_3 & m([l_1,u_3]) & m([l_2,u_3]) & m([l_3,u_3]) & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ u_n & m([l_1,u_n]) & m([l_2,u_n]) & m([l_3,u_n]) & \cdots & m([l_n,u_n]). \end{array} \tag{10.58}$$

Specifically, m in the preceding matrix defines a BPA for $\mathcal{S} = \{x : l_1 \le x \le u_n\}$ provided (i) the values for m in the matrix are nonnegative and sum to 1, and (ii) $m(\mathcal{E}) = 0$ if $\mathcal{E} \subset \mathcal{S}$ and $\mathcal{E}$ does not correspond to one of the intervals with a BPA in the matrix. In essence, this representation provides a way to define a BPA over an interval when all noncontiguous subintervals are given a BPA of zero. When only the diagonal elements of the matrix are nonzero, the resultant BPA assignment is equivalent to the specification of a discrete probability space. For this space, the set $\mathbb{S}$ contains the null set and all sets that can be generated by forming unions of the intervals $[l_1, l_2), [l_2, l_3), \ldots [l_n, u_n]$. Note that half-open intervals are assumed so that the intersection of any two intervals will be the null set. Conversely, from the structure of the matrix it can be seen that the precision of the information decreases as the distance from the diagonal increases. For example, the least precise statement of information appears in the lower-left element of the matrix with the definition of the BPA $m[l_1, u_n]$.

In the present example, the sets $\mathcal{A}$ and $\mathcal{B}$ correspond to the intervals [0.1, 1] and [0, 1], respectively. The corresponding matrices for $\mathcal{A}$ and $\mathcal{B}$ are

$$\begin{array}{ccccc} & 0.1 & 0.5 & 0.6 & 0.9 \\ 0.5 & m_A([0.1,0.5]) & & & \\ 0.6 & m_A([0.1,0.6]) & m_A([0.5,0.6]) & & \\ 0.9 & m_A([0.1,0.9]) & m_A([0.5,0.9]) & m_A([0.6,0.9]) & \\ 1.0 & m_A([0.1,1.0]) & m_A([0.5,1.0]) & m_A([0.6,1.0]) & m_A([0.9,1.0]) \end{array} \tag{10.59}$$

and

$$\begin{array}{cccccccc}
 & 0.0 & 0.2 & 0.3 & 0.4 & 0.5 & 0.6 & 0.8 \\
0.2 & m_B([0.0, 0.2]) & & & & & & \\
0.3 & m_B([0.0, 0.3]) & m_B([0.2, 0.3]) & & & & & \\
0.4 & m_B([0.0, 0.4]) & m_B([0.2, 0.4]) & m_B([0.3, 0.4]) & & & & \\
0.5 & m_B([0.0, 0.5]) & m_B([0.2, 0.5]) & m_B([0.3, 0.5]) & m_B([0.4, 0.5]) & & & \\
0.6 & m_B([0.0, 0.6]) & m_B([0.2, 0.6]) & m_B([0.3, 0.6]) & m_B([0.4, 0.6]) & m_B([0.5, 0.6]) & & \\
0.8 & m_B([0.0, 0.8]) & m_B([0.2, 0.8]) & m_B([0.3, 0.8]) & m_B([0.4, 0.8]) & m_B([0.5, 0.8]) & m_B([0.6, 0.8]) & \\
1.0 & m_B([0.0, 1.0]) & m_B([0.2, 1.0]) & m_B([0.3, 1.0]) & m_B([0.4, 1.0]) & m_B([0.5, 1.0]) & m_B([0.6, 1.0]) & m_B([0.8, 1.0]),
\end{array} \tag{10.60}$$

respectively.

The information for a provided by the two sources (see Equation 10.39 and Equation 10.40) can be summarized by the following matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ of the form shown in Equation 10.59:

$$\mathbf{A}_1 = \begin{bmatrix} 0 & & & \\ 0 & 0 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{A}_2 = \begin{bmatrix} 0.2 & & & \\ 0 & 0 & & \\ 0 & 0 & 0 & \\ 0 & 0.8 & 0 & 0 \end{bmatrix}. \tag{10.61}$$

Similarly, the information for b provided by the three sources (see Equation 10.42 through Equation 10.44) can be summarized by the following matrices $\mathbf{B}_1$, $\mathbf{B}_2$, and $\mathbf{B}_3$ of the form shown in Equation 10.60:

$$\mathbf{B}_1 = \begin{bmatrix} 0 & & & & & & \\ 0 & 0 & & & & & \\ 0 & 0 & 0 & & & & \\ 0 & 0 & 0.1 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & 0.9 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B}_2 = \begin{bmatrix} 0 & & & & & & \\ 0 & 0 & & & & & \\ 0 & 0.1 & 0 & & & & \\ 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0.7 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0.2 & 0 \end{bmatrix},$$
$$\text{and } \mathbf{B}_3 = \begin{bmatrix} 0.333 & & & & & & \\ 0 & 0 & & & & & \\ 0 & 0.333 & 0 & & & & \\ 0 & 0 & 0.333 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{10.62}$$

The BPAs for all unspecified subsets of $\mathcal{A}$ and $\mathcal{B}$ are zero.

10.3.3.2 Combination of Evidence

Combining the BPAs for the $\mathcal{A}_i$'s to obtain a BPA for $\mathcal{A}$ and combining the BPAs for the $\mathcal{B}_i$'s to obtain a BPA for $\mathcal{B}$ is now considered. The issue of combination of evidence, also referred to as *aggregation of evidence* or *data fusion*, is a topic closely related to evidence theory itself. After the introduction of evidence theory by Dempster in 1967 [61], many researchers believed evidence theory and the issue of combination of evidence were essentially the same topic. This view was reinforced by the close relationship between evidence theory and Dempster's rule of combination of evidence. In fact, much of the criticism of evidence theory has actually been directed at Dempster's rule of combination. It is now recognized that combination of evidence is a separate topic of growing importance in many fields [50, 52, 64, 82–89]. Combination of evidence takes on even more importance in newer theories of uncertainty, such as evidence theory, because many of the newer theories can deal more directly with large epistemic uncertainty than the traditional application of probability theory. The manner in which conflicting evidence is combined can have a large impact on the results of an uncertainty analysis, particularly when evidence theory is used.

In this presentation, the emphasis is on comparing uncertainty estimation results from the traditional application of probability theory and evidence theory. As a result, evidence from the various sources is combined in the same manner as was done for the solution using probability theory. The equivalent formulation to Equation 10.48 and Equation 10.49 for the matrices $\mathbf{A}_i$ and $\mathbf{B}_i$ is

$$\mathbf{A} = \sum_{i=1}^{2} \mathbf{A}_i/2 \quad \text{and} \quad \mathbf{B} = \sum_{i=1}^{3} \mathbf{B}_i/3. \tag{10.63}$$

Applying these equations to Equation 10.61 and Equation 10.62, respectively, we have

$$\mathbf{A} = \begin{bmatrix} 0.1 & & & \\ 0 & 0 & & \\ 0 & 0 & 0.5 & \\ 0 & 0.4 & 0 & 0 \end{bmatrix} \tag{10.64}$$

$$\text{and } \mathbf{B} = \begin{bmatrix} 0.111 & & & & & & \\ 0 & 0 & & & & & \\ 0 & 0.144 & 0 & & & & \\ 0 & 0 & 0.144 & 0 & & & \\ 0 & 0 & 0 & 0.233 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & 0.3 & \\ 0 & 0 & 0 & 0 & 0 & 0.067 & 0 \end{bmatrix}, \tag{10.65}$$

with the additional specification that $m_A(\mathcal{E}) = 0$ if $\mathcal{E}$ is a subset of $\mathcal{A}$ without an assigned BPA in $\mathbf{A}$ and $m_B(\mathcal{E}) = 0$ if $\mathcal{E}$ is a subset of $\mathcal{B}$ without an assigned BPA in $\mathbf{B}$.

10.3.3.3 Construction of Basic Probability Assignments for the Product Space

The variables a and b are specified as being independent. With the use of Equation 10.21, the BPA $m_C(\mathcal{E})$ defined on $\mathcal{C} = \mathcal{A} \times \mathcal{B}$ is given by

$$m_C(\mathcal{E}) = \begin{cases} m_A(\mathcal{E}_A)m_B(\mathcal{E}_B) & \text{if } \mathcal{E}_A \subset \mathcal{A}, \mathcal{E}_B \subset \mathcal{B} \text{ and } \mathcal{E} = \mathcal{E}_A \times \mathcal{E}_B \\ 0 & \text{otherwise} \end{cases} \tag{10.66}$$

for $\mathcal{E} \subset \mathcal{C}$. The resultant nonzero BPAs and associated subsets of $\mathcal{C}$ are summarized in Table 10.1.

Subset 11 is used to illustrate the entries in Table 10.1. This subset corresponds to the interval [0.5, 1.0] for a, and the interval [0.4, 0.6] for b. From Equation 10.66,

$$m_C(\mathcal{E}) = m_A([0.5, 1.0])\,m_B([0.4, 0.6]), \tag{10.67}$$

TABLE 10.1 Summary of the Nonzero Values of the BPA m_C for $\mathcal{C} = \mathcal{A} \times \mathcal{B}$

	$m_A([0.1, 0.5]) = 0.1$	$m_A([0.5, 1.0]) = 0.4$	$m_A([0.6, 0.9]) = 0.5$
$m_B([0., 0.2]) = 0.111$	0.0111 (subset 1)	0.0444 (subset 2)	0.0555 (subset 3)
$m_B([0.2, 0.4]) = 0.144$	0.0144 (subset 4)	0.0576 (subset 5)	0.0720 (subset 6)
$m_B([0.3, 0.5]) = 0.144$	0.0144 (subset 7)	0.0576 (subset 8)	0.0720 (subset 9)
$m_B([0.4, 0.6]) = 0.233$	0.0233 (subset 10)	0.0932 (subset 11)	0.1165 (subset 12)
$m_B([0.6, 0.8]) = 0.300$	0.0300 (subset 13)	0.1200 (subset 14)	0.1500 (subset 15)
$m_B([0.6, 1.0]) = 0.067$	0.0067 (subset 16)	0.0268 (subset 17)	0.0335 (subset 18)

where the values for $m_A([0.5, 1.0])$ and $m_B([0.4, 0.6])$ appear in the column and row designators associated with subset 11 (i.e., $\mathcal{E} = [0.5, 1.0] \times [0.4, 0.6]$) and are defined by entries in the matrices **A** and **B** in Equation 10.64 and Equation 10.65, respectively. This yields

$$m_C(\mathcal{E}) = (0.4)(0.233) = 0.0932. \tag{10.68}$$

The magnitude of m_C shown for subsets of $\mathcal{C}$ in Table 10.1 indicates the likelihood that can be assigned to a given set, but *not* to any proper subset of that set. As required in the definition of a BPA, the BPAs in Table 10.1 sum to unity.

10.3.3.4 Construction of Belief and Plausibility for the System Response

The belief and plausibility measures for the system outcome y can now be computed. The goal is to obtain an assessment of the likelihood, in the context of evidence theory, that y will be in the failure region. As previously indicated, the threshold of the failure region is $v = 1.7$. Thus, the system failure question is: "How likely is it that y will have a value in the set $\mathcal{Y}_{1.7}$ defined in Equation 10.28?" As indicated in Equation 10.35 and Equation 10.37,

$$Bel_Y(\mathcal{Y}_v) = \sum_{j \in \mathcal{ICCBF}_v} m_C(\mathcal{E}_j) \tag{10.69}$$

and

$$Pl_Y(\mathcal{Y}_v) = \sum_{j \in \mathcal{ICCPF}_v} m_C(\mathcal{E}_j), \tag{10.70}$$

where $\mathcal{E}_j$, $j = 1, 2, \ldots 18$, are the subsets of $\mathcal{C}$ with nonzero BPAs given in Table 10.1. The sets $\mathcal{ICCBF}_v$ and $\mathcal{ICCPF}_v$ are defined in Equation 10.31 and Equation 10.33. In turn, the sets

$$\mathcal{CCBF} = \{[v, Bel_Y(\mathcal{Y}_v)] : v \in \mathcal{Y}\} \tag{10.71}$$

and

$$\mathcal{CCPF} = \{[v, Pl_Y(\mathcal{Y}_v)] : v \in \mathcal{Y}\} \tag{10.72}$$

define the CCBF and the CCPF for y.

The quantities $\mathcal{ICCBF}_v$, $\mathcal{ICCPF}_v$, $Bel_Y(\mathcal{Y}_v)$, and $Pl_Y(\mathcal{Y}_v)$ for the example problem are summarized in Table 10.2. The contour lines of $y = (a+b)^a$ are shown in Figure 10.4a to aid in understanding where the jumps in the CCBF and CCPF occur. Along the edges of Figure 10.4a are the values of y at equal increments

TABLE 10.2 Determination of the CCBF and CCPF for $y = f(a, b)$ with Equation 10.67 and Equation 10.68 and the BPAs in Table 10.1.

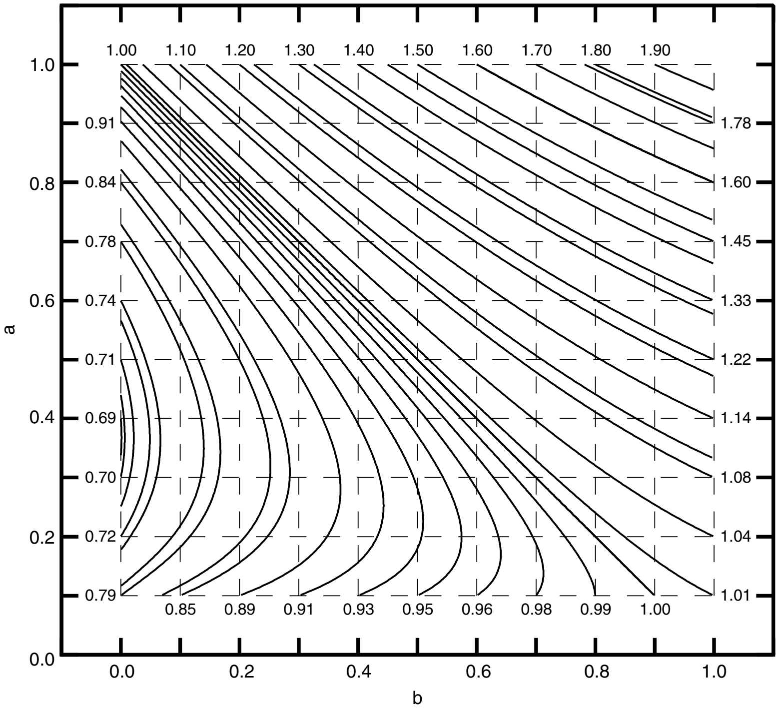
v	$\mathcal{ICCBF}_v$	$\mathcal{ICCPF}_v$	$Bel_Y(\mathcal{Y}_v)$	$Pl_Y(\mathcal{Y}_v)$
0.69220	1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18	1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18	1.00000	1.00000
[0.69220, 0.70711]	2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18	1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18	0.98889	1.00000
[0.70711, 0.73602]	3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18	1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18	0.94444	1.00000
[0.73602, 0.81096]	4,5,6,7,8,9,10,11,12,13,14,15,16,17,18	1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18	0.88889	1.00000
[0.81096, 0.83666]	5,6,7,8,9,10,11,12,13,14,15,16,17,18	1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18	0.87444	1.00000
[0.83666, 0.85790]	6,7,8,9,10,11,12,13,14,15,16,17,18	1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18	0.81667	1.00000
[0.85790, 0.87469]	6,8,9,10,11,12,13,14,15,16,17,18	1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18	0.80222	1.00000
[0.87469, 0.88657]	8,9,10,11,12,13,14,15,16,17,18	1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18	0.73000	1.00000
[0.88657, 0.89443]	8,9,10,11,12,13,14,15,16,17,18	2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18	0.73000	0.98889
[0.89443, 0.89751]	9,10,11,12,13,14,15,16,17,18	2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18	0.67222	0.98889
[0.89751, 0.93874]	9,11,12,13,14,15,16,17,18	2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18	0.64889	0.98889
[0.93874, 0.94868]	11,12,13,14,15,16,17,18	2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18	0.57667	0.98889
[0.94868, 0.95620]	12,13,14,15,16,17,18	2,3,5,6,7,8,9,10,11,12,13,14,15,16,17,18	0.48333	0.97444
[0.95620, 1.00000]	12,14,15,17,18	2,3,5,6,7,8,9,10,11,12,13,14,15,16,17,18	0.44667	0.97444
[1.00000, 1.04881]	14,15,17,18	2,3,5,6,8,9,10,11,12,13,14,15,16,17,18	0.33000	0.96000
[1.04881, 1.08957]	15,18	2,3,5,6,8,9,11,12,13,14,15,16,17,18	0.18333	0.93667
[1.08957, 1.11560]	15,18	2,5,6,8,9,11,12,13,14,15,16,17,18	0.18333	0.88111
[1.11560, 1.14018]		2,5,6,8,9,11,12,13,14,15,16,17,18	0.00000	0.88111
[1.14018, 1.20000]		2,5,6,8,9,11,12,14,15,16,17,18	0.00000	0.85111
[1.20000, 1.22474]		5,6,8,9,11,12,14,15,16,17,18	0.00000	0.80667
[1.22474, 1.26634]		5,6,8,9,11,12,14,15,17,18	0.00000	0.80000
[1.26634, 1.35368]		5,8,9,11,12,14,15,17,18	0.00000	0.72778
[1.35368, 1.40000]		5,8,11,12,14,15,17,18	0.00000	0.65556
[1.40000, 1.44040]		8,11,12,14,15,17,18	0.00000	0.59778
[1.44040, 1.50000]		8,11,14,15,17,18	0.00000	0.48111
[1.50000, 1.60000]		11,14,15,17,18	0.00000	0.42333
[1.60000, 1.61214]		14,15,17,18	0.00000	0.33000
[1.61214, 1.78188]		14,17,18	0.00000	0.18000
[1.78188, 1.80000]		14,17	0.00000	0.14667
[1.80000, 2.00000]		17	0.00000	0.02667
2.00000			0.00000	0.00000

of a and b along the boundary of their domain. Also note that these y values are only given to two significant figures. These contour lines approximately define the sets $f^{-1}(\mathcal{Y}_v)$ for selected values of v. For example, $f^{-1}(\mathcal{Y}_{1.04})$ is indicated in Figure 10.4b through Figure 10.4d. The sets $\mathcal{ICCBF}_v$ and $\mathcal{ICCPF}_v$ are obtained by determining the j for which $\mathcal{E}_j \subset f^{-1}(\mathcal{Y}_v)$ and $\mathcal{E}_j \cap f^{-1}(\mathcal{Y}_v) \neq \varnothing$, respectively. Thus, as an examination of Figure 10.4b through Figure 10.4d shows,

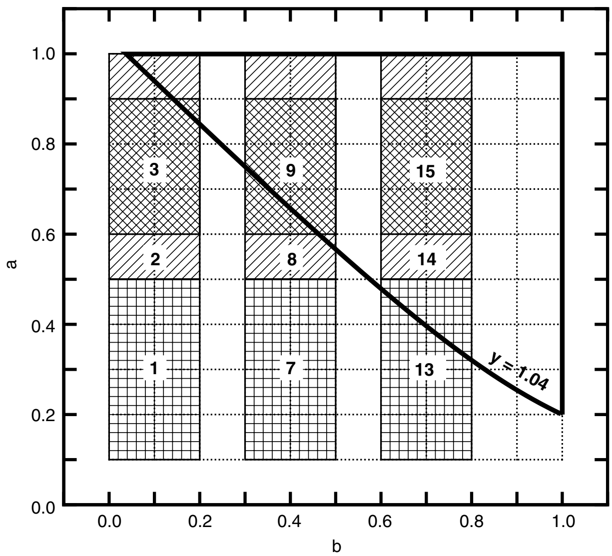
$$\mathcal{ICCBF}_{1.04} = \{14, 15, 17, 18\} \tag{10.73}$$

and

$$\mathcal{ICCPF}_{1.04} = \{2, 3, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18\}, \tag{10.74}$$

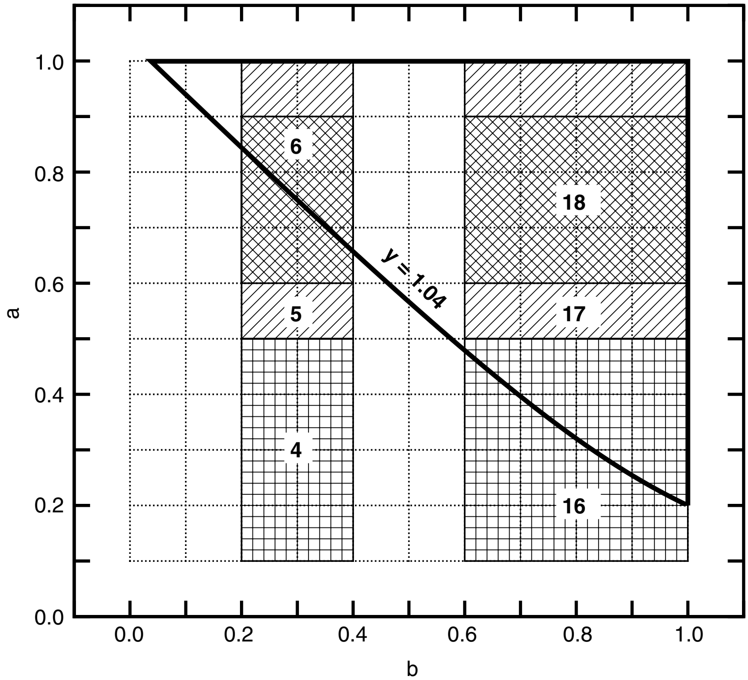


(a) Contour Lines of $y = (a + b)^a$

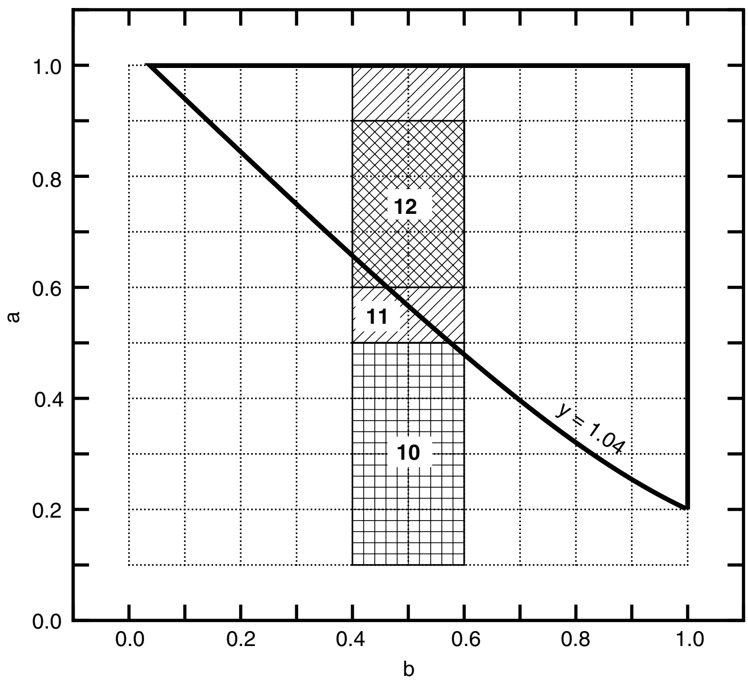


(b) Subsets 1, 2, 3, 7, 8, 9, 13, 14, and 15

FIGURE 10.4 Contour lines of $y = (a + b)^a$ and subsets 1, 2,…, 18 of $\mathcal{C} = \mathcal{A} \times \mathcal{B}$ with nonzero BPAs indicated in Table 10.1: (a) contour lines of $y = (a + b)^a$; (b) subsets 1, 2, 3, 7, 8, 9, 13, 14, and 15; (c) subsets 4, 5, 6, 16, 17, and 18; and (d) subsets 10, 11, and 12. (Originially published in *Investigation of Evidence Theory for Engineering Applications,* Oberkampf, W.L. and Helton, J.C., 4th Nondeterministic Approaches Forum, Denver, AIAA-2002-1569. Copyright © 2002 by the American Institute of Aeronautics and Astronautics, Inc. Reprinted with permission.)



(c) Subsets 4, 5, 6, 16, 17, and 18



(d) Subsets 10, 11, and 12

FIGURE 10.4 (*Continued*)

which correspond to the values indicated in Table 10.2 for $\mathcal{ICCBF}_{1.04}$ and $\mathcal{ICCPF}_{1.04}$. In turn,

$$\begin{aligned} Bel_{1.04}(\mathcal{Y}_v) &= \sum_{j \in \mathcal{ICCBF}_{1.04}} m_C(\mathcal{E}_j) \\ &= m_C(\mathcal{E}_{14}) + m_C(\mathcal{E}_{15}) + m_C(\mathcal{E}_{17}) + m_C(\mathcal{E}_{18}) \\ &= 0.1200 + 0.1500 + 0.0268 + 0.0335 \\ &= 0.3303 \end{aligned} \tag{10.75}$$

and

$$\begin{aligned} Pl_Y(\mathcal{Y}_{1.04}) &= \sum_{j \in \mathcal{ICCPF}_{1.04}} m_C(\mathcal{E}_j) \\ &= m_C(\mathcal{E}_2) + m_C(\mathcal{E}_3) + m_C(\mathcal{E}_5) + m_C(\mathcal{E}_6) + m_C(\mathcal{E}_8) + \ldots + m_C(\mathcal{E}_{18}) \\ &= 0.9591. \end{aligned} \tag{10.76}$$

The resultant CCBF and CCPF are provided by plots of the points contained in the sets $\mathcal{CCBF}$ and $\mathcal{CCPF}$ defined in Equation 10.71 and Equation 10.72 and are shown in Figure 10.5. We stress that the jumps, or discontinuities, in the CCBF and CCPF shown in Figure 10.5 are accurate and consistent with evidence theory. Using Monte Carlo sampling for traditional probability theory, one occasionally sees jumps or stair-steps in the CCDF. However, these jumps are typically numerical artifacts, that is, numerical approximations due to use of a finite number of samples to compute the CCDF. In evidence theory, the jumps are *not* numerical artifacts but are due to discontinuous assignments of BPAs across the boundaries of subsets.

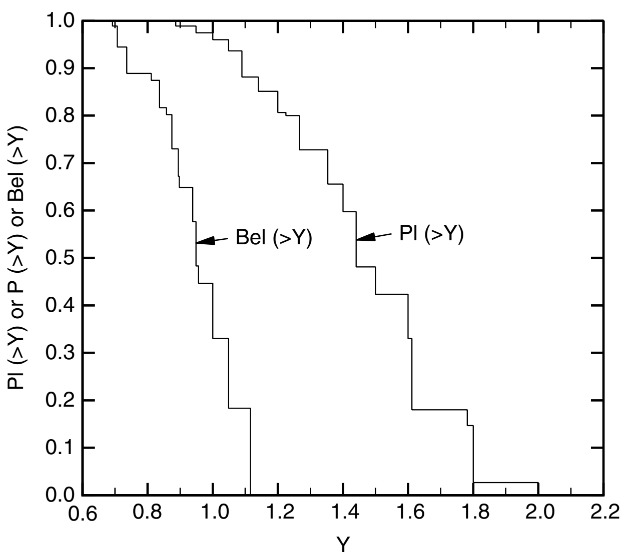


FIGURE 10.5 CCBF and CCPF for example problem $y = (a + b)^a$. (Originially published in *Investigation of Evidence Theory for Engineering Applications,* Oberkampf, W.L. and Helton, J.C., 4th Nondeterministic Approaches Forum, Denver, AIAA-2002-1569. Copyright © 2002 by the American Institute of Aeronautics and Astronautics, Inc. Reprinted with permission.)

10.3.4 Comparison and Interpretation of Results

For the present example, the solution using probability theory is obtained by numerically evaluating Equation 10.55 using Monte Carlo sampling. A random sample of size of one million was used so that an accurate representation of the CCDF could be obtained for probabilities as low as 10^{-4}. The CCDF for y is shown in Figure 10.6. In essence, the CCDF is constructed by plotting the pairs $[v, p_Y(\mathcal{Y}_v)]$ for an increasing sequence of values for v. The resultant CCDF indicates that the probability of the unsafe region, $y > 1.7$, is 0.00646.

Also shown in Figure 10.6 is the CCPF and the CCBF from evidence theory for the example problem. It can be seen in Figure 10.6, and also in Table 10.2, that the highest and lowest probabilities for the unsafe region using evidence theory are 0.18 and 0.0, respectively. That is, the absolutely highest probability that is consistent with the interval data for a and b is 0.18, and the absolutely lowest probability that is consistent with the interval data is 0.0. Stated differently, given the large epistemic uncertainty in the values for a and b, the probability of the unsafe region can *only* be bounded by 0.0 and 0.18. By comparing these interval valued probabilities with the traditional probabilistic result, it is seen that evidence theory states that the likelihood of the unsafe region could be 28 times higher than that indicated by the traditional analysis, or possibly as low as zero.

The key difference between the results obtained with probability theory and evidence theory is that in the probability-based analysis, it was assumed that all values in each specified interval for a and b were equally likely. Stated differently, the probability-based analysis assumed that the probability density function (PDF) was given by a piecewise uniform distribution; whereas in the evidence theory analysis, no additional assumptions were made beyond the uncertainty information supplied by the original sources. In essence, evidence theory permits the specification of partial (i.e., not completely defined)

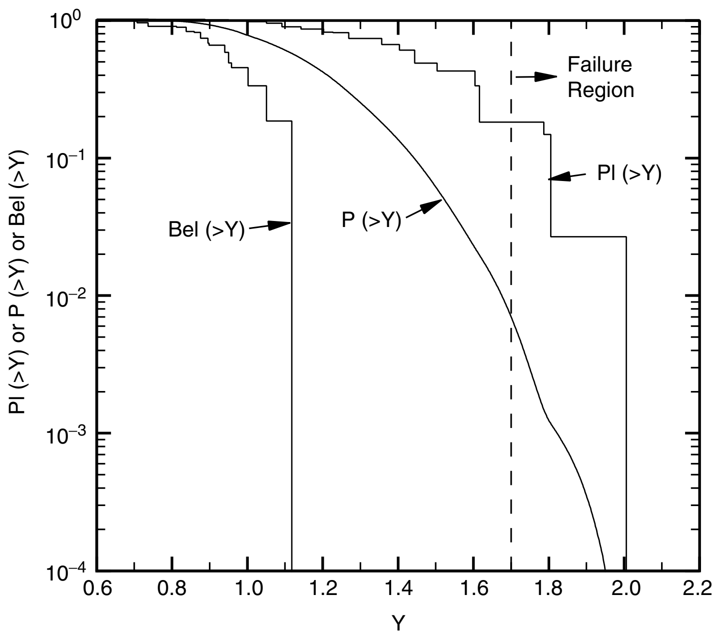


FIGURE 10.6 CCDF, CCPF, and CCBF for example problem $y = (a + b)^a$. (Originially published in *Investigation of Evidence Theory for Engineering Applications,* Oberkampf, W.L. and Helton, J.C., 4th Nondeterministic Approaches Forum, Denver, AIAA-2002-1569. Copyright © 2002 by the American Institute of Aeronautics and Astronautics, Inc. Reprinted with permission.)

probability distributions for a and b. Thus, evidence theory can be viewed as allowing the propagation of partially specified PDFs through the model of the system, in this case $(a + b)^a$, resulting in a range of likelihoods for y. The structure of evidence theory allows the incomplete specification of probability distributions, or more precisely, the complete determination of all possible probability distributions consistent with the input data.

As a final comment, both solution approaches relied on one additional common assumption: namely, that the information from each of the sources of data for a and b could be combined by a simple averaging procedure. As mentioned, a variety of methods have been developed for combining evidence. The results of an uncertainty analysis can strongly depend on which combination method is chosen for use. The selection of an appropriate combination method is an important open issue in uncertainty estimation. Research is needed to provide guidance for the appropriate selection of a combination method given the particular characteristics of a specific analysis and the type of information. Whatever method of combination is chosen in a given situation, the choice should primarily depend on the nature of the information to be combined. For a recent review of methods for combination of evidence, see [89, 90].

10.4 Research Topics in the Application of Evidence Theory

Although evidence theory possesses some advantages compared to the traditional application of probability theory, there are several prominent open issues that must be investigated and resolved before evidence theory can be confidently and productively used in large-scale engineering analyses. First, consider the use sampling techniques to propagate basic probability assignments (BPAs) through "black-box" computational models. An important practical issue arises: what is the convergence rate of approximations to belief and plausibility in the output space as a function of the number of samples? Our preliminary experience using traditional sampling techniques, such as Monte Carlo or Latin Hypercube, indicates that these techniques are reliable for evidence theory, but they are also expensive from the standpoint of function evaluations. Possibly faster convergence techniques could be developed for a wide range of black-box functions. Second, how can sensitivity analyses be conducted in the context of evidence theory? Rarely is a nondeterministic analysis conducted simply for the purpose of estimating the uncertainty in specified system response variables. A more common question is: what are the primary contributors to the uncertainty in the system response? Thus, sensitivity analysis procedures must be available for use in conjunction with uncertainty propagation procedures. Third, for situations of pure epistemic uncertainty (i.e., no aleatory uncertainty) in input parameters, such as the present example problem with interval data, how does one properly interpret belief and plausibility in the output space as minimum and maximum probabilities, respectively? A closely related question is: how does the averaging technique for aggregating conflicting expert opinion, which is the common technique in classical probabilistic thinking, affect the interpretation of interval valued probabilities in the output space?

The issue of combination of evidence from multiple sources is a separate issue from evidence theory itself. Evidence theory has been criticized in the past because there is no unique method for combining multiple sources of evidence. However, combination of evidence in probability theory is also nonunique and open to question. Further research is needed into methods of combining different types of evidence, particularly highly conflicting evidence from different sources. We believe that the method of combination of evidence chosen in a given situation should be context dependent. Stated differently, there is no single method appropriate for combining all types of evidence in all situations dealing with epistemic uncertainty.

Finally, evidence theory may have advantages over the traditional application of probability theory with regard to representing model form uncertainty (i.e., uncertainty due to lack of knowledge of the physical process being mathematically modeled). Because model form uncertainty is just a special case of epistemic uncertainty, any benefits that evidence theory has in the representation of epistemic uncertainty in general should also apply to model form uncertainty. Although this topic was not specifically addressed in this presentation, evidence theory has the capability to leave unspecified the probability assigned to any given mathematical model among alternative candidate models.

Acknowledgments

We thank Jay Johnson of ProStat, Inc., for computation of the results and generation of the figures. This work was performed at Sandia National Laboratories, which is a multiprogram laboratory operated by Sandia Corporation, a Lockheed Martin Company, for the United States Department of Energy's National Nuclear Security Administration under contract DE-AC04-AL85000.

References

1. AIAA, Guide for the verification and validation of computational fluid dynamics simulations, American Institute of Aeronautics and Astronautics, Reston, VA, AIAA-G-077-1998, 1998.
2. Kleijnen, J.P.C., Verification and validation of simulation models, *European Journal of Operational Research*, 82, 145–162, 1995.
3. Oberkampf, W.L. and Trucano, T.G., Verification and validation in computational fluid dynamics, *Progress in Aerospace Sciences*, 38, 209–272, 2002.
4. Oberkampf, W.L. and Trucano, T.G., Verification, validation, and predictive capability in computational engineering and physics, Sandia National Laboratories, Albuquerque, NM, SAND2003-3769, 2003.
5. Knupp, P. and Salari, K., *Verification of Computer Codes in Computational Science and Engineering*, Chapman & Hall/CRC, Boca Raton, FL, 2002.
6. Roache, P.J., *Verification and Validation in Computational Science and Engineering*, Hermosa Publishers, Albuquerque, NM, 1998.
7. Hora, S.C. and Iman, R.L., Expert opinion in risk analysis: the NUREG-1150 methodology, *Nuclear Science and Engineering*, 102, 323–331, 1989.
8. Hauptmanns, U. and Werner, W., *Engineering Risks Evaluation and Valuation*, 1st ed., Springer-Verlag, Berlin, 1991.
9. Beckjord, E.S., Cunningham, M.A., and Murphy, J.A., Probabilistic safety assessment development in the United States 1972–1990, *Reliability Engineering and System Safety*, 39, 159–170, 1993.
10. Modarres, M., *What Every Engineer Should Know about Reliability and Risk Analysis*, Marcel Dekker, New York, 1993.
11. Kafka, P., Important issues using PSA technology for design of new systems and plants, *Reliability Engineering and System Safety*, 45, 205–213, 1994.
12. Breeding, R.J., Helton, J.C., Murfin, W.B., Smith, L.N., Johnson, J.D., Jow, H.-N., and Shiver, A.W., The NUREG-1150 probabilistic risk assessment for the Surry Nuclear Power Station, *Nuclear Engineering and Design*, 135, 29–59, 1992.
13. Kumamoto, H. and Henley, E.J., *Probabilistic Risk Assessment and Management for Engineers and Scientists*, 2nd ed., IEEE Press, New York, 1996.
14. Helton, J.C., Uncertainty and sensitivity analysis in performance assessment for the Waste Isolation Pilot Plant, *Computer Physics Communications*, 117, 156–180, 1999.
15. Paté-Cornell, M.E., Conditional uncertainty analysis and implications for decision making: the case of WIPP, *Risk Analysis*, 19, 1003–1016, 1999.
16. Helton, J.C. and Breeding, R.J., Calculation of reactor accident safety goals, *Reliability Engineering and System Safety*, 39, 129–158, 1993.
17. Ross, T.J., *Fuzzy Logic with Engineering Applications*, McGraw-Hill, New York, 1995.
18. Melchers, R.E., *Structural Reliability Analysis and Prediction*, 2nd ed., John Wiley & Sons, New York, 1999.
19. Hoffman, F.O. and Hammonds, J.S., Propagation of uncertainty in risk assessments: the need to distinguish between uncertainty due to lack of knowledge and uncertainty due to variability, *Risk Analysis*, 14, 707–712, 1994.
20. Rowe, W.D., Understanding uncertainty, *Risk Analysis*, 14, 743–750, 1994.
21. Helton, J.C., Treatment of uncertainty in performance assessments for complex systems, *Risk Analysis*, 14, 483–511, 1994.

22. Ayyub, B.M., The Nature of Uncertainty in Structural Engineering, in *Uncertainty Modelling and Analysis: Theory and Applications*, B. M. Ayyub and M. M. Gupta, Eds., 1st ed., Elsevier, New York, 1994, 195–210.
23. Hora, S.C., Aleatory and epistemic uncertainty in probability elicitation with an example from hazardous waste management, *Reliability Engineering and System Safety*, 54, 217–223, 1996.
24. Frey, H.C. and Rhodes, D.S., Characterizing, simulating, and analyzing variability and uncertainty: an illustration of methods using an air toxics emissions example, *Human and Ecological Risk Assessment*, 2, 762–797, 1996.
25. Ferson, S. and Ginzburg, L.R., Different methods are needed to propagate ignorance and variability, *Reliability Engineering and System Safety*, 54, 133–144, 1996.
26. Ferson, S., What Monte Carlo methods cannot do, *Human and Ecological Risk Assessment*, 2, 990–1007, 1996.
27. Rai, S.N., Krewski, D., and Bartlett, S., A general framework for the analysis of uncertainty and variability in risk assessment, *Human and Ecological Risk Assessment*, 2, 972–989, 1996.
28. Parry, G.W., The characterization of uncertainty in probabilistic risk assessments of complex systems, *Reliability Engineering and System Safety*, 54, 119–126, 1996.
29. Helton, J.C., Uncertainty and sensitivity analysis in the presence of stochastic and subjective uncertainty, *Journal of Statistical Computation and Simulation*, 57, 3–76, 1997.
30. Paté-Cornell, M.E., Uncertainties in risk analysis: six levels of treatment, *Reliability Engineering and System Safety*, 54, 95–111, 1996.
31. Cullen, A.C. and Frey, H.C., *Probabilistic Techniques in Exposure Assessment: A Handbook for Dealing with Variability and Uncertainty in Models and Inputs*, Plenum Press, New York, 1999.
32. Frank, M.V., Treatment of uncertainties in space nuclear risk assessment with examples from Cassini Mission applications, *Reliability Engineering and System Safety*, 66, 203–221, 1999.
33. Haimes, Y.Y., *Risk Modeling, Assessment, and Management*, John Wiley & Sons, New York, 1998.
34. Ang, A.H.S. and Tang, W.H., *Probability Concepts in Engineering Planning and Design: Vol. I Basic Principles*, 1st ed., John Wiley & Sons, New York, 1975.
35. Ditlevsen, O., *Uncertainty Modeling with Applications to Multidimensional Civil Engineering Systems*, 1st ed., McGraw-Hill, New York, 1981.
36. Ang, A.H.S. and Tang, W.H., *Probability Concepts in Engineering Planning and Design: Vol. II Decision, Risk, and Reliability*, John Wiley & Sons, New York, 1984.
37. Neelamkavil, F., *Computer Simulation and Modelling*, 1st ed., John Wiley & Sons, New York, 1987.
38. Haldar, A. and Mahadevan, S., *Probability, Reliability, and Statistical Methods in Engineering Design*, John Wiley & Sons, New York, 2000.
39. Oberkampf, W.L., DeLand, S.M., Rutherford, B.M., Diegert, K.V., and Alvin, K.F., Error and uncertainty in modeling and simulation, *Reliability Engineering and System Safety*, 75, 333–357, 2002.
40. Oberkampf, W.L., DeLand, S.M., Rutherford, B.M., Diegert, K.V., and Alvin, K.F., Estimation of total uncertainty in computational simulation, Sandia National Laboratories, Albuquerque, NM, SAND2000-0824, 2000.
41. Oberkampf, W.L., Helton, J.C., and Sentz, K., Mathematical representation of uncertainty, *3rd Non-deterministic Approaches Forum,* Seattle, WA, AIAA-2001-1645, 2001.
42. Zadeh, L.A., Fuzzy sets as a basis for a theory of possibility, *Fuzzy Sets and Systems*, 1, 3–28, 1978.
43. Manton, K.G., Woodbury, M.A., and Tolley, H.D., *Statistical Applications Using Fuzzy Sets*, John Wiley & Sons, New York, 1994.
44. Onisawa, T. and Kacprzyk, J., Eds., *Reliability and Safety Analyses under Fuzziness*, Physica-Verlag, Heidelberg, 1995.
45. Klir, G.J., St. Clair, U., and Yuan, B., *Fuzzy Set Theory: Foundations and Applications*, Prentice Hall PTR, Upper Saddle River, NJ, 1997.
46. Dubois, D. and Prade, H., *Fundamentals of Fuzzy Sets*, Kluwer Academic, Boston, 2000.
47. Moore, R.E., *Methods and Applications of Interval Analysis*, SIAM, Philadelphia, PA, 1979.

48. Kearfott, R.B. and Kreinovich, V., Eds., *Applications of Interval Computations*, Kluwer Academic, Boston, 1996.
49. Shafer, G., *A Mathematical Theory of Evidence*, Princeton University Press, Princeton, NJ, 1976.
50. Guan, J. and Bell, D.A., *Evidence Theory and Its Applications*, Vol. I., North Holland, Amsterdam, 1991.
51. Krause, P. and Clark, D., *Representing Uncertain Knowledge: An Artificial Intelligence Approach*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1993.
52. Kohlas, J. and Monney, P.-A., *A Mathematical Theory of Hints — An Approach to the Dempster-Shafer Theory of Evidence*, Springer, Berlin, 1995.
53. Almond, R.G., *Graphical Belief Modeling*, 1st ed., Chapman & Hall, London, 1995.
54. Klir, G.J. and Wierman, M.J., *Uncertainty-Based Information: Elements of Generalized Information Theory*, Physica-Verlag, Heidelberg, 1998.
55. Kramosil, I., *Probabilistic Analysis of Belief Functions*, Kluwer, New York, 2001.
56. Dubois, D. and Prade, H., *Possibility Theory: An Approach to Computerized Processing of Uncertainty*, Plenum Press, New York, 1988.
57. de Cooman, G., Ruan, D., and Kerre, E.E., *Foundations and Applications of Possibility Theory*, World Scientific Publishing Co., Singapore, 1995.
58. Walley, P., *Statistical Reasoning with Imprecise Probabilities*, Chapman & Hall, London, 1991.
59. Klir, G.J. and Yuan, B., *Fuzzy Sets and Fuzzy Logic*, Prentice Hall, Saddle River, NJ, 1995.
60. Klir, G.J. and Smith, R.M., On measuring uncertainty and uncertainty-based information: recent developments, *Annals of Mathematics and Artificial Intelligence*, 32, 5–33, 2001.
61. Dempster, A.P., Upper and lower probabilities induced by a multivalued mapping, *Annals of Mathematical Statistics*, 38, 325–339, 1967.
62. Wasserman, L.A., Belief functions and statistical inference, *The Canadian Journal of Statistics*, 18, 183–196, 1990.
63. Halpern, J.Y. and Fagin, R., Two views of belief: belief as generalized probability and belief as evidence, *Artifical Intelligence*, 54, 275–317, 1992.
64. Yager, R.R., Kacprzyk, J., and Fedrizzi, M., Eds., *Advances in Dempster-Shafer Theory of Evidence*, John Wiley & Sons, New York, 1994.
65. Dong, W.-M. and Wong, F.S., From uncertainty to approximate reasoning. 1. Conceptual models and engineering interpretations, *Civil Engineering Systems*, 3, 143–154, 1986.
66. Dong, W.-M. and Wong, F.S., From uncertainty to approximate reasoning. 3. Reasoning with conditional rules, *Civil Engineering Systems*, 4, 45–53, 1987.
67. Lai, K.-L. and Ayyub, B.M., Generalized uncertainty in structural reliability assessment, *Civil Engineering Systems*, 11, 81–110, 1994.
68. Tonon, F. and Bernardini, A., A random set approach to the optimization of uncertain structures, *Computers & Structures*, 68, 583–600, 1998.
69. Tanaka, K. and Klir, G.J., A design condition for incorporating human judgement into monitoring systems, *Reliability Engineering and System Safety*, 65, 251–258, 1999.
70. Tonon, F., Bernardini, A., and Elishakoff, I., Concept of random sets as applied to the design of structures and analysis of expert opinions for aircraft crash, *Chaos, Solitons, & Fractals*, 10, 1855–1868, 1999.
71. Rakar, A., Juricic, D., and Ball, P., Transferable belief model in fault diagnosis, *Engineering Applications of Artificial Intelligence*, 12, 555–567, 1999.
72. Fetz, T., Oberguggenberger, M., and Pittschmann, S., Applications of possibility and evidence theory in civil engineering, *International Journal of Uncertainty*, 8, 295–309, 2000.
73. Oberkampf, W.L. and Helton, J.C., Investigation of evidence theory for engineering applications, *4th Nondeterministic Approaches Forum*, Denver, CO, AIAA-2002-1569, 2002.
74. Helton, J.C., Oberkampf, W.L., and Johnson, J.D., Competing failure risk analysis using evidence theory, *5th Nondeterministic Approaches Forum*, Norfolk, VA, AIAA-2003-1911, 2003.
75. Feller, W., *An Introduction to Probability Theory and Its Applications*, Vol. 2, John Wiley & Sons, New York, 1971.

76. Ash, R.B. and Doléans-Dade, C.A., *Probability and Measure Theory*, 2nd ed., Harcourt/Academic Press, New York, 2000.
77. Joslyn, C. and Kreinovich, V., Convergence properties of an interval probabilistic approach to system reliability estimation, *International Journal of General Systems*, in press.
78. McKay, M.D., Beckman, R.J., and Conover, W.J., A comparison of three methods for selecting values of input variables in the analysis of output from a computer code, *Technometrics*, 21, 239–245, 1979.
79. Iman, R.L., Uncertainty and sensitivity analysis for computer modeling applications, *Reliability Technology*, 28, 153–168, 1992.
80. Helton, J.C. and Davis, F.J., Latin hypercube sampling and the propagation of uncertainty in analyses of complex systems, *Reliability Engineering and System Safety*, 81, 23–69, 2003.
81. Luo, W.B. and Caselton, W., Using Dempster-Shafer theory to represent climate change uncertainties, *Journal of Environmental Management*, 49, 73–93, 1997.
82. Kacprzyk, J. and Fedrizzi, M., *Multiperson Decision Making Models Using Fuzzy Sets and Possibility Theory*, Kluwer Academic Publishers, Boston, 1990.
83. Abidi, M.A. and Gonzalez, R.C., *Data Fusion in Robotics and Machine Intelligence*, Academic Press, San Diego, 1992.
84. Goodman, I.R., Mahler, R.P.S., and Nguyen, H.T., *Mathematics of Data Fusion,* Kluwer Academic Publishers, Boston, 1997.
85. Goutsias, J., Mahler, R.P.S., and Nguyen, H.T., *Random Sets, Theory and Applications*, Springer, New York, 1997.
86. Slowinski, R., *Fuzzy Sets in Decision Analysis, Operations Research and Statistics*, Kluwer Academic Publishers, Boston, 1998.
87. Bouchon-Meunier, B., Aggregation and fusion of imperfect information, in *Studies in Fuzziness and Soft Computing*, J. Kacprzyk, Ed., Springer-Verlag, New York, 1998.
88. Ferson, S., Kreinovich, V., Ginzburg, L., Myers, D.S., and Sentz, K., Constructing probability boxes and Dempster-Shafer structures, Sandia National Laboratories, Albuquerque, NM, SAND2003-4015, 2003.
89. Sentz, K. and Ferson, S., Combination of evidence in Dempster-Shafer theory, Sandia National Laboratories, Albuquerque, NM, SAND2002-0835, 2002.