sorted logic statically, order-sorted logic requires dynamic well-sortedness checking, which usually occurs during unification. Depending on the expressive power of the signature, sorted unification may be unitary, finitary, or infinitary (Walther, 1988; Schmidt-Schauss, 1989; Meseguer and co-workers, 1989). Allowing sets of sort constraints on variables or embedding a poset with non-unique glbs into a semilattice can turn the finitary into the unitary case. In some situations, sorted unification is not decidable. Schmidt-Schauss (1989) has investigated sorted unification under nonempty theories and presents a sorted paramodulation rule. If sort literals and overlapping are allowed then further inference rules are required to retain completeness (Cohn, 1987), for example, a rule to resolve two characteristic literals that do not have the same predicate symbol is needed; this rule is an instance of theory resolution (see RESOLUTION, THEORY). Often, a translation to unsorted logic is given (a relativisation) and a sort theorem proved, which shows that the sorted logic is no more expressive (although it may be more efficient). If the logic is substitutional (Frisch, in press), that is, if it obeys certain syntactic restrictions (in particular there are no sort literals), then a sound and complete inference procedure for an arbitrary sorted calculus can be automatically synthesized from an unsorted one. Frisch's approach to sorted deduction is not a sorted logic in the conventional sense because there is no signature, but rather a logical theory, which can be used to specify very general information about sorts and the sortal behavior of the nonlogical symbols. This theory does not affect the semantics as a signature does, and there is no notion of well-sorted formulas (only substitutions).

There are many other representation languages that treat taxonomic knowledge specially, in particular, semantic or associative networks and the KL-ONE family of languages. The TBOX of these latter languages is usually sufficiently expressive to make subsumption computationally intractable, whereas the signature of most sorted logics allows testing whether a term is of a particular sort (performing a subsumption test in KL-ONE terminology) to be cheap. Feature logics (Smolka and Aït-Kaci, 1989) are also closely related to sorted logics.

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A. G. COHN
University of Leeds

LOGIC, PREDICATE

Predicate logic—also known as predicate calculus or first-order (predicate) logic—is the study of inferences that can be made on the basis of an analysis of atomic sentences into "terms" (essentially, noun phrases) and "predicates" (essentially, verb phrases). It is an extension of propositional (or sentential) logic and is the modern descendant of Aristotle's logic of subjects and predicates (see LOGIC, PROPOSITIONAL). For discussions of traditional Aristotelian syllogistic logic see Kneale and Kneale (1962), Kneebone (1963), and Prior (1967a,b). For a general discussion of logic and references to other articles on logic in this *Encyclopedia*, see LOGIC. Secondarily, it is also the study of the representation of information (see KNOWLEDGE REPRESENTATION) by predicates and their terms. Because of the relationships of predicates and terms to noun phrases and verb phrases, predicate logic has often served as a foundation for natural-language syntax and semantics (see the NATURAL-LANGUAGE entries; PARSING).

In this article, the syntactic items that are used in the representation of information are called sentences, and the items in the "world" that sentences mean or express are called propositions. A "predicate" is, as suggested above, usually taken to be a verb phrase or the name of a property, relation, or class of objects. Thus, in the sentence

Roses are red

"(is) red" is the predicate; it can be taken to name the property or attribute of being red or of redness, or the class $\{x\colon x \text{ is red}\}$ or $\{x\colon x \text{ has redness}\}$. In addition to this "subatomic" analysis of the atomic sentences treated by propositional logic, predicate logic employs a machinery of variables and quantifiers that allows it to express how many objects fall under a given predicate. The adjective "first-order" indicates that the quantifiers only range over individuals, not properties, relations, or classes (ie, they range over the things represented by terms, not the things represented by predicates). Second-order logic (see below) quantifies over predicates; by extrapolation, propositional logic may be thought of as being of "zero order."

Although predicate logic is usually taken to be a way of analyzing propositions or declarative sentences, there are also predicate logics for other types of sentences (eg, quantified modal logic and quantified epistemic logic). In fact, the logic of some sentences, such as interrogatives (erotetic logic), only becomes interesting in the quantified case. (For discussions of epistemic and other modal logics, see LOGIC, MODAL; BELIEF REPRESENTATION SYSTEMS; Gabbay and Guenthner, 1984; Hintikka, 1962; Hughes and Cresswell, 1968; Nute, 1981; Prior, 1967c. For erotetic logic, see Belnap and Steel, 1976; Harrah, 1984; and Lambert, 1969.)

As is the case with propositional logic, the representational system of predicate logic is its underlying language, consisting essentially of terms, predicates, quantifiers, and truth-functional connectives, with a grammatical syntax and a semantics in terms of individuals and properties (or classes). The syntax is often extended to include functions (or term-producing operators), the identity predicate, and definite and indefinite description operators. The deductive system of predicate logic extends that of propositional logic to include axioms and rules for manipulating quantifiers.

THE LANGUAGE OF PREDICATE LOGIC

Informally, an atomic proposition is analyzed into a single verb phrase (the predicate) and a sequence of noun phrases (grammatically, its subjects and objects) called the *arguments* of the predicate. For example,

Socrates is Greek

consists of the predicate ". . . is Greek" together with its argument "Socrates"; and

Fredonia is between Erie and Buffalo

consists of the predicate ". . . is between . . . and . . ." together with its arguments "Fredonia," "Erie," and "Buffalo" (or the argument sequence, ⟨Fredonia, Erie, Buffalo⟩). In the first case the predicate names the property: being Greek, or the class: {x: x is Greek}; in the second case the predicate names the relation: being between . . . and . . ., or the class of ordered triples: $\{\langle x, y, z\rangle: x \text{ is between } y \text{ and } z\}$. Discussed below are important theoretical differences between the full first-order logic of relations and monadic first-order logic, which only has one-place predicates.

To be able to express propositions such as

All humans are mortal.
Some philosophers are computer scientists.
There are no unicorns.

quantifiers and variables are used. Thus, the first of these examples might be expressed using the universal quantifier ("for all"):

For all x, if x is human, then x is mortal.

and the second might be expressed using the existential quantifier ("for some" or "there exists"):

For some x, x is a philosopher and x is a computer scientist.

There exists an x such that x is a philosopher and x is a computer scientist.

Table 1. Alphabet of $\mathscr{L}$

n-place predicate symbols (n an integer)	A, . . . , Z; A_i, B_i, . . . (i an integer); any sequence of words separated by hyphens
n-place function symbols (n an integer)	f, g, h; f_i, g_i, h_i (i an integer); any sequence of words separated by hyphens
Individual variables	u, . . . , z; u_i, . . . , z_i (i an integer)
Individual constants	a, . . . , e; a_i, . . . , e_i (i an integer); any noun phrase (the words separated by hyphens)
Connectives	$\wedge$, $\neg$
Punctuation	",", "[", "]", "(", ")"
Quantifiers	
Universal	$\forall$
Existential	$\exists$

Syntax

A formal syntax for a language $\mathscr{L}$ of predicate logic can be presented by giving an alphabet, a recursive definition of *term,* and a recursive definition of *well-formed formula* (wff) (given in Tables 1–3). In order to define the notion of a sentence and to give the inference rules, the following definitions are necessary:

(D1) Let φ be a wff prefixed by a quantifier phrase (ie, either $\forall$v or $\exists$v). Then φ is the *scope of* the quantifier phrase. For example, the scope of $\forall x$ in $\forall x\varphi(x)$ is $\varphi(x)$, but the scope of $\exists y$ in $(\exists y\ \varphi(y) \vee \psi)$ is $\varphi(y)$.

Table 2. Terms of $\mathscr{L}$

(T1) All individual variables are terms.
(T2) All individual constants are terms.
(T3) If t_1, . . . , t_n are terms and f is an n-place function symbol, then $f(t_1, \ldots, t_n)$ is a term.
(T4) Nothing else is a term.

For example, each of the following is a term:

x
x_1
a
a_{23}
John
Mother-of(Bill)
Son-of(Harriet, Frank)

Table 3. Well-formed Formulas of $\mathscr{L}$

(wff.1) If $t_1, \ldots, t_n$ are terms and **P** is an n-place predicate symbol, then $\mathbf{P}(t_1, \ldots, t_n)$ is a(n atomic) well-formed formula.

(wff.2) If φ and ψ are well-formed formulas, **v** is an individual variable, and $\varphi(\mathbf{v}^*)$ is a well-formed formula containing zero or more occurrences of **v**, then

$$\neg\varphi$$
$$(\varphi \vee \psi)$$
$$\forall\mathbf{v}[\varphi(\mathbf{v}^*)]$$
$$\exists\mathbf{v}[\varphi(\mathbf{v}^*)]$$

are well-formed formulas.

(wff.3) Nothing else is a well-formed formula.

Parentheses and brackets will sometimes be omitted when no ambiguity results. For example, each of the following is a wff:

$$\mathrm{A}(x, y)$$
$$\text{In(Eiffel-tower, France)}$$
$$\neg\,\text{Republican(John-F.Kennedy)}$$
$$(\text{Capital(Albany, New-York)} \vee \mathrm{B})$$
$$\forall x \mathrm{F}x$$
$$\forall x[\neg\,\mathrm{Human}(x) \vee \mathrm{Mortal}(x)]$$
$$\neg\,\exists x\ \mathrm{Unicorn}(x)$$

(Note that a zero-place "predicate," like B, is an atomic wff.)

(D2) Let the variable in a quantified phrase be called its *variable of quantification*. Then:
- (a) An occurrence of an individual variable in a wff φ is *bound* means: the variable occurs in the scope of a quantifier phrase in φ that has that variable as its variable of quantification.
- (b) An occurrence of an individual variable in a wff φ is *free* means: the occurrence of that variable is not bound.
- (c) A variable is *bound* means: there is an occurrence of that variable that is bound.
- (d) A variable is *free* means: there is an occurrence of that variable that is free.

For example, in

$$(\mathrm{F}x \vee \forall x \mathrm{G}x)$$

the first occurrence of x is free and the second is bound; the variable x is both free and bound in this wff. Finally,

(D3) A *sentence* is a wff with no free variables.

For further discussion of the grammatical syntax of a first-order language and translations of natural-language sentences into it, see Kalish, Montague, and Mar (1980), Otto (1978), Schagrin (1979).

Semantics

Providing a semantics for such a first-order language is somewhat more problematic than it is in the propositional case. The main reason for this is that a decision must be made about the domain (or universe) of discourse. It was noted above that a predicate can name a property (or relation) or a class. But classes are extensional ("two" classes are identical if they have the same members), whereas properties are intensional (ie, nonextensional). Moreover, there are important questions about what counts as an individual:

1. Can properties or classes themselves be individuals? This is surely plausible; consider such propositions as:

   Red is a color.
   Colors are properties.
   $\{x$: x is a rational number$\}$ is countable.

   However, care must be taken to avoid paradox, as in Russell's (Whitehead and Russell, 1927) well-known example:

   $\{x: x \notin x\} \in \{x: x \notin x\}$ if and only if $\{x: x \notin x\} \notin \{x: x \notin x\}$

2. Must the individual actually exist? If variables and terms may only range over existents, how does one express such sentences as the following?

   There are no round squares.
   Santa Claus does not exist.
   All unicorns are white.

Thus, a semantics for a first-order language cannot be completely specified independently of an ontology—a precise specification of the domain. Nevertheless, the general form of such a semantics (often called *formal semantics*, see Nute (1981)) does not vary. Metatheoretical results are given here in terms of set-theoretic semantics (ie, in terms of an ontology of sets and their members), which is the way they are given in most of the literature.

Let **M** be the structure $\langle \mathbf{D}, \mathbf{R}, \mathbf{F}\rangle$, where **D** is a nonempty set, **R** is a set of n-place relations on the elements of **D**, and **F** is a set of n-place functions on the elements of **D**. An interpretation, I, on **M** for $\mathscr{L}$ is a function from the symbols of $\mathscr{L}$ to $\mathbf{D} \cup \mathbf{R} \cup \mathbf{F}$ such that:

If t is an individual constant or individual variable, then $I(t) \in \mathbf{D}$.

If f is a function symbol, then $I(f) \in \mathbf{F}$.

If f is an n-place function symbol and $t_1, \ldots, t_n$ are terms, then $I(f(t_1, \ldots, t_n)) = I(f)\langle I(t_1), \ldots, I(t_n)\rangle \in \mathbf{D}$.

If **P** is an n-place predicate symbol, then $I(\mathbf{P}) \in \mathbf{R}$.

The notion of "truth on an interpretation" (symbolized as: $\models_I$) can be defined recursively as follows:

1. If **P** is an n-place predicate symbol, and $t_1, \ldots, t_n$ are terms, then $\models_I \mathbf{P}(t_1, \ldots, t_n)$ if and only if $\langle I(t_1), \ldots, I(t_n)\rangle \in I(\mathbf{P})$.

2. If φ and ψ are wffs and $\mathbf{v}$ is an individual variable, then
   (a) $\models_I \neg \varphi$ if and only if not-$\models_I \varphi$;
   (b) $\models_I (\varphi \vee \psi)$ if and only if $\models_I \varphi$ or $\models_I \psi$;
   (c) $\models_I \forall \mathbf{v} \varphi$ if and only if $\models_{I'} \varphi$ for every interpretation I' that differs from I at most on what it assigns to $\mathbf{v}$;
   (d) $\models_I \exists \mathbf{v} \varphi$ if and only if $\models_{I'} \varphi$ for some interpretation I' that differs from I at most on what it assigns to $\mathbf{v}$.

Finally,

A wff φ is valid in $\mathbf{M}$ (written: $\mathbf{M} \models \varphi$) if and only if $\models_I \varphi$ for every interpretation I on $\mathbf{M}$.

A structure $\mathbf{M}$ is a model for a set $\mathbf{H}$ of wffs if and only if $\mathbf{M} \models \mathbf{H}_i$ for every wff $\mathbf{H}_i \in \mathbf{H}$.

Expressibility. As is the case with propositional logic, one can choose to employ either a small number of connectives and quantifiers (for elegance and metatheoretical simplicity) or a wide variety (for expressive power). Thus, on the one hand, the formal system presented above may be extended in a natural way to include the other truth-functional connectives or, on the other hand, restricted to using (say) only $\neg$, $\vee$, and $\forall$. The latter can be accomplished as in propositional logic, together with the following definition:

$$\exists \mathbf{v} \varphi =_{df} \neg \forall \mathbf{v} \neg \varphi$$

Another variation is to employ restricted quantifiers. Instead of translating

All *A*s are *B*s.
Some *A*s are *B*s.

as, respectively,

$$\forall x[\mathrm{A}x \rightarrow \mathrm{B}x] \text{ and } \exists x[\mathrm{A}x \wedge \mathrm{B}x]$$

with the noticeable change in syntactic structure, a family of restricted quantifiers can be introduced:

$$(\forall x\colon \varphi(x)) \text{ and } (\exists x\colon \varphi(x))$$

Using this notation, the translations become the more uniform-looking

$$(\forall x\colon \mathrm{A}x)\mathrm{B}x \text{ and } (\exists x\colon \mathrm{A}x)\mathrm{B}x$$

This notation has the additional advantage of being extendible to generalized quantifiers for handling such sentences as

Most *A*s are *B*s.
Many *A*s are *B*s.

as well as numerical quantifiers:

Exactly 4116 *A*s are *B*s.
Greater than 5 *A*s are *B*s.
Between 5 and 10 *A*s are *B*s.

Generalized and numerical quantifiers are, however, beyond the scope of first-order logic (for discussions of these issues, see Barwise and Cooper (1981), Brown (1984), McCawley (1981), Montague (1970), and Shapiro (1979).

Other alternatives to first-order languages and logics have been motivated by ontological concerns. As is seen below, when deduction is discussed, $\forall x \varphi(x)$ implies $\exists x \varphi(x)$ in a nonempty domain. But what about the empty domain? Why should logic imply that something exists? Shouldn't logic be independent of ontology? Attempts to broaden the scope of first-order logic have included free logics (ie, logics that are free of existence presuppositions) and Meinongian logics that allow [for representing and reasoning about nonexistents. Both of these kinds of logics often choose to represent existence by a special predicate, $E!$, rather than by trying to define existence in purely first-order terms (as, eg, "$\exists x[x = a]$" for "a exists") [for discussions of free logics, see Hintikka (1966), Lambert (1969, 1970, 1981, 1984), Leblanc and Thomason (1968), and Scott (1967) and for discussions of Meinongian logics, see Castañeda (1972), Parsons (1980), Rapaport (1978, 1981, 1984, 1985a,b), Routley (1979), and Zalta (1983)].

DEDUCTIVE SYSTEMS OF PREDICATE LOGIC

As with propositional logic, a deductive system for predicate logic can be presented axiomatically or as a natural deduction system.

Axiomatic Predicate Logic

In this section a set of axioms and rules of inference for predicate logic are presented using the terminology introduced in the article LOGIC, PROPOSITIONAL. As is done there, the wffs are restricted to those whose only connectives are $\neg$ and $\rightarrow$; and the only quantifier is the universal quantifier. All wffs of the following forms will be axiom schemata:

(A1) $(\varphi \rightarrow (\psi \rightarrow \varphi))$.
(A2) $((\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi)))$.
(A3) $((\neg \varphi \rightarrow \neg \psi) \rightarrow (\psi \rightarrow \varphi))$.
(A4) $(\forall \mathbf{v}[\varphi \rightarrow \psi] \rightarrow (\varphi \rightarrow \forall \mathbf{v} \psi))$, where $\mathbf{v}$ is not free in φ.
(A5) $(\forall \mathbf{v} \varphi(\mathbf{v}^*) \rightarrow \varphi(\mathbf{t}/\mathbf{v}))$, where $\varphi(\mathbf{t}/\mathbf{v})$ is the result of replacing all free occurrences of $\mathbf{v}$ in φ by any term $\mathbf{t}$ and where all variables in $\mathbf{t}$ are free at all locations in φ where $\mathbf{v}$ occurs freely.

There are two rules of inference:

Modus ponens: From φ and $(\varphi \rightarrow \psi)$, infer ψ.
Universal generalization: From φ, infer $\forall \mathbf{v} \varphi$.

A Natural-Deduction System for Predicate Logic

The natural-deduction system for propositional logic introduced in LOGIC, PROPOSITIONAL may be extended to predicate logic by providing introduction and elimination rules for the quantifiers. Because these rules involve the substitution of variables by constants, and vice versa, care must be taken not to accidentally bind a previously free variable or free a previously bound one. Consequently, the quantifier rules are not as "natural" as the rules for the connectives.

- $\forall$ *Elimination:* From $\forall \mathbf{v}\varphi(\mathbf{v}^*)$, infer $\varphi(\mathbf{c}/\mathbf{v})$, where $\varphi(\mathbf{c}/\mathbf{v})$ is the wff that results from $\varphi(\mathbf{v}^*)$ by replacing all free occurrences of the variable **v** by the constant **c**.
- $\forall$ *Introduction:* From $\varphi(\mathbf{c}^*)$, infer $\forall \mathbf{v}\varphi(\mathbf{v}/\mathbf{c})$, where $\varphi(\mathbf{v}/\mathbf{c})$ is the wff that results from $\varphi(\mathbf{c}^*)$ by replacing all occurrences of **c** by **v**, provided: **c** does not occur in a premise; if $\varphi(\mathbf{c}^*)$ occurs in a subproof, then no individual constant in $\varphi(\mathbf{c}^*)$ occurs in an assumption that is global to the subproof; and all new occurrences of **v** must be free after the replacement.
- $\exists$ *Introduction:* From $\varphi(\mathbf{c}^*)$, infer $\exists \mathbf{v}\varphi(\mathbf{v}/\mathbf{c}^*)$, where $\varphi(\mathbf{v}/\mathbf{c}^*)$ is the formula that results from $\varphi(\mathbf{c}^*)$ by replacing zero or more occurrences of **c** by **v**.
- $\exists$ *Elimination:* From $\exists \mathbf{v}\varphi(\mathbf{v}^*)$ and a subproof that begins with the assumption $\varphi(\mathbf{c}/\mathbf{v})$ and that ends with a proposition ψ not containing **c**, infer ψ, where **c** is an individual constant that has not been used before and $\varphi(\mathbf{c}/\mathbf{v})$ is as described above.

[These rules are adapted from Schagrin, Rapaport, and Dipert (1985). For further discussion and other sets of rules, see other standard introductory texts such as Copi (1979), Jeffrey (1981), Kalish, Montague, and Mar (1980), Mendelson (1979), Quine (1951, 1980, 1982).] As with the case of propositional logic, there is a form of the inference rule Resolution that has proved to be of importance in AI contexts (see Chang and Lee, 1973; Manna, 1974; Nilsson, 1971, 1980; Raphael, 1976; Rich, 1983; Winston, 1984 and THEOREM PROVING).

As an example of the use of the introduction and elimination rules, Figure 1 shows a translation and natural-deduction proof of the argument:

Horses are animals.
$\therefore$ Every head of a horse is a head of an animal.

The rules of $\rightarrow$ *Elimination* and $\rightarrow$ *Introduction* used on lines 7 and 11 can be derived from the rules for the connectives $\neg$ and $\wedge$ and the logical equivalence "material conditional"; the former rule is, essentially, *modus ponens* (see LOGIC, PROPOSITIONAL and Schagrin, Rapaport, and Dipert (1985) for details of these rules and the derivations).

Translation:

$$\forall x[\mathrm{Horse}(x) \rightarrow \mathrm{Animal}(x)] \vdash \forall y[\exists x[\mathrm{Horse}(x) \wedge \text{Head-of}(y, x)] \rightarrow \exists z[\mathrm{Animal}(z) \wedge \text{Head-of}(y, z)]]$$

Proof:

```
1. ∀x[Horse(x) → Animal(x)]          ; premise of argument
   ; BEGIN subproof using → Introduction to prove
   ;(∃x[Horse(x) ∧ Head-of(a, x)] → ∃z[Animal(z) ∧ Head-of(a, z)])
*  2. ∃x[Horse(x) ∧ Head-of(a, x)]     ; assumption for → Introduction
      ; BEGIN sub-subproof using ∃ Elimination to prove ∃z[Animal(z) ∧ Head-of(a, z)]
*  *  3. (Horse(b) ∧ Head-of(a, b))    ; from line 2 (assumption for ∃ Elimination)
*  *  4. ∀x[Horse(x) → Animal(x)]      ; sent in from line 1
*  *  5. (Horse(b) → Animal(b))        ; from line 4, by ∀ Elimination
*  *  6. Horse(b)                      ; from line 3, by ∧ Elimination
*  *  7. Animal(b)                     ; from lines 5 and 6, by → Elimination
*  *  8. Head-of(a, b)                 ; from line 3, by ∧ Elimination
*  *  9. (Animal(b) ∧ Head-of(a, b))   ; from lines 7 and 8, by ∧ Introduction
*  *  10.∃z[Animal(z) ∧ Head-of(a, z)]; from line 9, by ∃ Introduction
      ;END of sub-subproof that used ∃ Elimination to prove ∃z[Animal(z) ∧ Head-of(a, z)]
*  11.∃z[Animal(z) ∧ Head-of(a, z)]    ; returned to outer subproof from line 10 of innermost sub-subproof
*  12.(∃x[Horse(x) ∧ Head-of(a, x)] → ∃z[Animal(z) ∧ Head-of(a, z)])
                                       ; from lines 2 and 11, by → Introduction
   ; END of subproof that used → Introduction to prove
   ; (∃x[Horse(x) ∧ Head-of(a, x)] → ∃z[Animal(z) ∧ Head-of(a, z)])
13.(∃x[Horse(x) ∧ Head-of(a, x)] → ∃z[Animal(z) ∧ Head-of(a, z)])
                                       ; returned to main proof from line 12 of outer subproof
14.∀y[∃x[Horse(x) ∧ Head-of(y, x)] → ∃z[Animal(z) ∧ Head-of(y, z)]]
                                       ; from line 13, by ∀ Introduction
```

Figure 1. An example of *Introduction* and *Elimination* rules to prove the argument that if horses are animals, then every head of a horse is a head of an animal.

EXTENSIONS OF PREDICATE LOGIC

First-order languages are often extended by the addition of two important symbols: the two-place predicate symbol for identity, =, and the definite-description operator ι (in many AI and natural-language contexts, words such as *equal* and *the* are used instead). These additions to the representational power of the language also entail greater deductive power.

Identity

Syntactically, the identity predicate can be defined by adding the following to the definition of wff:

(wff.=) If t_1 and t_2 are terms, then $(t_1 = t_2)$ is a(n atomic) well-formed proposition.

Often, $(t_1 \neq t_2)$ is defined as an abbreviation for $\neg (t_1 = t_2)$. Semantically $\models_I (t_1 = t_2)$ if and only if $I(t_1) = I(t_2)$. The axiomatic formulation of predicate logic can then be extended by the following two axiom schemata:

(A6) $\forall \mathbf{v}[\mathbf{v} = \mathbf{v}]$.

(A7) $\forall \mathbf{v}_1 \forall \mathbf{v}_2[(\mathbf{v}_1 = \mathbf{v}_2) \rightarrow (\varphi(\mathbf{v}_1{}^*) \leftrightarrow \psi(\mathbf{v}_2/\mathbf{v}_1{}^*))]$, where $\psi(\mathbf{v}_2/\mathbf{v}_1{}^*)$ is the result of replacing $\mathbf{v}_2$ for $\mathbf{v}_1$ at zero or more of the free occurrences of $\mathbf{v}_1$ in φ where $\mathbf{v}_2$ would not be bound.

Descriptions

Definite Descriptions. Noun phrases such as

the first human on the Moon
the present King of France
the woman who wrote "The Story of an Hour"

can be treated as having the form

the x such that $\varphi(x)$.

Thus, the expressive capabilities of the first-order language (and hence the deductive capabilities of first-order logic) introduced here can be extended by introducing a new variable-binding operator in addition to the quantifiers. Unlike the quantifiers, which are wff-producing operators, the definite description operator ι is a term-producing operator. The definition of *term* can be augmented as follows:

(T5) If φ is a wff and $\mathbf{v}$ is an individual variable, then $\iota \mathbf{v}[\varphi]$ is a term.

There has been a great deal of controversy over the semantics of such terms. The approach due to B. Russell (1971) has become the standard logical one. According to Russell's analysis, sentences of the form $\psi(\iota x\, \varphi(x))$ should not be treated as subject–predicate sentences; that is, they should not be parsed as consisting of a noun phrase, $\iota x\, \varphi(x)$, and a verb phrase, ψ. Rather, they are to be analyzed as

$$\exists x[\varphi(x) \wedge \forall y[\varphi(y) \rightarrow y = x] \wedge \psi(x)]$$

For instance, to use Russell's famous example,

The present King of France is bald

is to be represented as

$$\exists x[\text{Present-King-of-France}(x) \wedge \forall y[\text{Present-King-of-France}(y) \rightarrow y = x] \wedge \text{Bald}(x)]$$

that is,

One and only one thing is a present King of France and he is bald.

It is a consequence of this analysis that the sentence comes out false, since there is no present King of France. Similarly,

The book that Knuth wrote is interesting

is false, since Knuth has written more than one book; and

The winged horse captured by Bellerophon is named "Pegasus"

is false, since the winged horse captured by Bellerophon does not exist.

The addition to the axiomatic formulation of predicate logic is straightforward: Simply add the axiom schema

(A8) $\psi(\iota \mathbf{v}_1 \varphi(\mathbf{v}_1)) \leftrightarrow \exists \mathbf{v}_1[\varphi(\mathbf{v}_1) \wedge \forall \mathbf{v}_2[\varphi(\mathbf{v}_2) \rightarrow \mathbf{v}_2 = \mathbf{v}_1] \wedge \psi(\mathbf{v}_1)]$.

Semantically, $\models_I \psi(\iota \mathbf{v} \varphi))$ if and only if

1. there is a unique element $d \in \mathbf{D}$ such that $d \in I(\varphi)$ and
2. $d \in I(\psi)$.

An alternative analysis, due to Strawson (1985) takes $\psi(\iota x \varphi(x))$ to be of subject–predicate form, but the interpretation I is taken to be a partial function: $\psi(\iota x \varphi(x))$ is neither true nor false on I if $I(\iota x \varphi(x))$ is undefined. That is, if $\iota x \varphi(x)$ does not denote a member of $\mathbf{D}$ (ie, if nothing satisfies the predicate φ), then $\psi(\iota x \varphi(x))$ is truth-valueless (for further discussion on truth-value gaps, see Lambert, 1969, 1970).

A third approach, stemming from work done by Meinong (1960), takes $\psi(\iota x \varphi(x))$ to be of subject–predicate form but chooses a universe of discourse that allows I to be total by providing an object for each definition description. This strategy can be made plausible if the universe of discourse is taken to consist of the objects of thought and, hence, is the most appropriate one for AI applications [for details see Castañeda (1977), Parsons (1980), Rapaport (1978, 1981, 1984, 1985a,b) and Routley (1979)].

Indefinite Descriptions. A noun phrase such as

a person I met today

can be treated as having the form

an x such that $\varphi(x)$.

The indefinite description operator ε, which is also variable binding and term producing, can be added to predicate logic in a manner similar to the addition of ι (for details, see Kaplan (1972) and Leisenring (1969)).

METATHEORETIC RESULTS

A few major metatheoretic results are worth mentioning briefly. As is the case for propositional logic, predicate logic is sound (all theorems are valid, ie, true on all interpretations—in symbols: if $\vdash\varphi$, then $\models\varphi$) and consistent (no wff φ is such that both $\vdash\varphi$ and $\vdash \neg\varphi$). And Gödel showed that it is complete (all valid wffs are theorems—if $\models\varphi$, then $\vdash\varphi$) (see COMPLETENESS).

Löwenheim (and, later, Skolem) (Kleene, 1950, p. 394) showed that monadic first-order logic (ie, first-order logic without relations) is decidable: for any wff φ, if there is a nonempty universe of discourse **D** and there is an interpretation I whose range is **D** and that is such that $\models_I\varphi$, then there is an interpretation I' whose range is the set of all positive integers and that is such that $\models_{I'}\varphi$. However, Church showed that the full first-order predicate calculus is undecidable (for details, see Blumberg (1967), Church (1956), and Jeffrey (1981)).

SECOND-ORDER LOGIC

If quantifiers are allowed to range over predicate variables, the resulting language allows the expression of such propositions as

There is a relation that holds between Bill and Hector,

which would seem to be a logical consequence of

Bill is a student of Hector.

In symbols,

Student-of(Hector, Bill)

implies

$$\exists \mathbf{P}\ \mathbf{P}(\text{Hector, Bill})$$

as well as

$$\exists x \exists y \exists \mathbf{P}\ \mathbf{P}xy$$

In such a language, identity can be defined by

$$\forall x \forall y[x = y \leftrightarrow \forall\varphi[\varphi(x) \leftrightarrow \varphi(y)]]$$

And, if predicates can be quantified over, then they can be the arguments of other, "higher-order" predicates. Thus, for example, that a relation is reflexive can be expressed as

$$\forall \mathbf{R}[\text{Reflexive}(\mathbf{R}) \leftrightarrow \forall x \mathbf{R}xx]$$

with **R** appearing in both subject and predicate position. Such a logic is termed second- or higher-order logic or the extended predicate calculus.

Although second-order logic clearly has greater expressive power than first-order logic, it also has some metatheoretic disadvantages. For one thing, a form of Russell's paradox can be developed:

$$\forall\varphi[\text{Self-referential}(\varphi) \leftrightarrow \varphi(\varphi)]$$

implies, by $\forall$ Elimination,

$$\text{Self-referential}(\neg\,\text{Self-referential}) \leftrightarrow \neg\,\text{Self-referential}(\neg\,\text{Self-referential})$$

For another, one version of Gödel's famous incompleteness theorem is that second-order logic is incomplete: There are true second-order wffs that are not theorems (for discussions of second-order logic, see Church (1956), Copi (1979), Jeffrey (1981), Kleene (1950), and Kneebone (1963, p. 110–118)).

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W. J. RAPAPORT
SUNY at Buffalo

LOGIC PROGRAMMING

Logic programming can be broadly understood as the use of logic to represent problems and problem-solving methods, together with the use of appropriate proof procedures for the effective solution of those problems. For the most part, logic programming today uses Horn-clause logic