

Closed book, closed notes (except for 1 sheet), closed neighbors, 48 minutes. Do ALL FOUR problems *in the exam booklets provided*. Problem (4) has a *choice*—you must do EXACTLY ONE of the options (a), (b), or (c). *Show all your work in the exam booklets*—this may help for partial credit. The exam totals 80 pts., subdivided as shown.

*Notation:* As in lectures,  $\#a(x)$  stands for the number of  $a$ 's in the string  $x$ , etc.

**(1) (3 × 4 = 12 pts.)** *True/False.*

Please write out the words **true** and/or **false** in full. No justifications are needed. *Be sure to write your answers in the exam books.*

- (a) The union of two context-free languages is always a context-free language.
- (b) There is a regular language  $L$  whose complement  $\tilde{L}$  is not a context-free language.
- (c) Let  $G = (V, \Sigma, \mathcal{R}, S)$  be a context-free grammar with the rule  $S \rightarrow SS$ . If a given string  $x$  is ambiguous in  $G$ , then so is  $xx$ .

**(2) (18 pts.)**

Let  $E = \{x \in \{a, b\}^* : \#b(x) - \#a(x) \text{ is a multiple of } 3\}$ . Let  $G$  be the context-free grammar

$$S \rightarrow aA \mid SbB \mid AB, \quad A \rightarrow aa \mid b \mid aB, \quad B \rightarrow SaS \mid AA.$$

Prove that  $L(G) \subseteq E$ . (Reasonable proof shortcuts are OK.)

**(3) (26 pts.)**

Let  $G$  be the context-free grammar

$$S \rightarrow AbS \mid BaS \mid A \mid B \mid \epsilon, \quad A \rightarrow aA \mid a, \quad B \rightarrow bB \mid b.$$

- (a) Prove that  $L(G) = \Sigma^*$ , where  $\Sigma = \{a, b\}$ . (Hint: Given  $x \in \Sigma^*$ ,  $x \neq \epsilon$ , you may (or may not) need 3 or 4 cases, depending on how you word things. You may use separate “lemmas” for the languages of strings derived by  $A$  and  $B$  without involving them in the overall induction, and some other reasonable proof shortcuts are OK. 18 pts.)
- (b) Is  $G$  ambiguous? If you say yes, find an ambiguous string  $x \in L(G)$  and give two different parse trees for  $x$ ; if you say no, prove your answer. (8 pts.)

**(4) (24 pts.)**

Define  $L = \{a^i b^j c^k : i = j + k, i, j, k \geq 0\}$ . Do EXACTLY ONE of options (a), (b), or (c). In whichever case you choose, you must write some prose comments that explain how your  $G$ ,  $M_1$ , or  $M_2$  works—and you need not prove correctness formally.

- (a) Design a context-free grammar  $G$  such that  $L(G) = L$ . Then modify your  $G$  into a context-free grammar  $G'$  such that  $L(G') = L \setminus \{\epsilon\}$  and  $G'$  has no  $\epsilon$ -rules. OR,
- (b) Design a single-tape deterministic Turing machine  $M_1$  such that  $L(M_1) = L$ . OR,
- (c) Design a deterministic pushdown automaton  $M_2$ , coded as a two-tape deterministic Turing machine that obeys the pushdown restrictions, such that  $L(M_2) = L$ .