A Dependent LP-rounding Approach for the k-Median Problem

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Abstract. In this paper, we revisit the classical k-median problem: Given n points in a metric space, select k centers so as to minimize the sum of distances of points to their closest center. Using the standard LP relaxation for k-median, we give an efficient algorithm to construct a probability distribution on sets of k centers that matches the marginals specified by the optimal LP solution. Our algorithm draws inspiration from clustering and randomized rounding approaches that have been used previously for k-median and the closely related facility location problem, although ensuring that we choose at most k centers requires a careful dependent rounding procedure.

Analyzing the approximation ratio of our algorithm presents significant technical difficulties: we are able to show an upper bound of 3.25. While this is worse than the current best known $3 + \epsilon$ guarantee of [2], our approach is interesting because: (1) The random choice of the k centers given by the algorithm keeps the marginal distributions and satisfies the negative correlation, leading to 3.25 approximation algorithms for some generalizations of the k-median problem, including the k-UFL problem introduced in [8], (2) our algorithm runs in $\tilde{O}(k^3n^2)$ time compared to the $O(n^8)$ time required by the local search algorithm of [2] to guarantee a 3.25 approximation, and (3) our approach has the potential to beat the decade old bound of $3 + \epsilon$ for k-median by a suitable instantiation of various parameters in the algorithm.

We also give a 34-approximation for the knapsack median problem, which greatly improves the approximation constant in [11]. Besides the improved approximation ratio, both our algorithm and analysis are simple, compared to [11]. Using the same technique, we also give a 9-approximation for matroid median problem introduced in [9], improving on their 16-approximation.

1 Introduction

In this paper, we present a novel LP rounding algorithm for the metric k-median problem which achieves approximation ratio 3.25. For the k-median problem, we are given a finite metric space $(\mathcal{F} \cup \mathcal{C}, d)$ and an integer $k \geq 1$, where $\mathcal{F}$ is a set of facility locations and $\mathcal{C}$ is a set of clients. Our goal is to select k facilities to open, such that the total connection cost for all clients in $\mathcal{C}$ is minimized, where the connection cost of a client is its distance to its nearest open facility. When $\mathcal{F} = \mathcal{C} = X$, the set of points with the same nearest open facility is known as a cluster and thus the sum measures how well X can be partitioned into k clusters. The k-median problem has numerous applications, starting from clustering to data mining [3], to assigning efficient sources of supplies to minimize the transportation cost([10, 13]).

The problem is NP-hard and has received a lot of attention. The first constant factor approximation is due to [5]. Based on LP rounding, their algorithm produces a $6\frac{2}{3}$-approximation. The best known approximation algorithm is the local search algorithm given by [2]. They showed that if there is a solution $\mathcal{F}'$, where any p swaps of the centers can not improve the solution, then $\mathcal{F}'$ is a $3 + 2/p$ approximation. This leads to a $3 + \epsilon$ approximation in $n^{2/\epsilon}$ running time. Jain, Mahdian and Saberi [7] proved that the k-median problem is $1 + 2/e \approx 1.736$-hard to approximate.

Our algorithm (like several previous ones) is based on the following natural LP relaxation:

$$\text{LP(1)} \qquad \min \quad \sum_{i\in\mathcal{F},j\in\mathcal{C}} d(i,j)x_{i,j} \qquad \text{s.t.}$$

$$\sum_{i\in\mathcal{F}} x_{i,j} = 1, \quad \forall j \in \mathcal{C}; \qquad x_{i,j} \le y_i, \qquad \forall i \in \mathcal{F}, j \in \mathcal{C};$$

$$\sum_{i\in\mathcal{F}} y_i \le k; \qquad x_{i,j}, y_i \in [0,1], \quad \forall i \in \mathcal{F}, j \in \mathcal{C}$$

It is known that the LP has an integrality gap of 2. On the positive side, [1] showed that the integrality gap is at most 3 by giving an exponential time rounding algorithm that requires to solve maximum independent set.

Very recently, Kumar [11] gave a (large) constant-factor approximation algorithm for a generalization of the k-median problem, which is called knapsack median problem. In this problem, each facility $i \in \mathcal{F}$ has an opening cost f_i and we are given a budget M. The goal is to open a set of facilities such that their total opening cost is at most M , and minimize the total connection cost of all clients. When $M = k$ and $f_i = 1$ for all facilities $i \in \mathcal{F}$, the problem is k-median problem.

Another generalization of the k-median problem is the matroid-median problem, introduced by Krishnaswamy et al. [9]. In the problem, the set of open facilities has to form an independent set of some given matroid. [9] gave a 16-approximation for this problem, assuming there is a separation oracle for the matroid polytope.

1.1 Our results

We give a simple and efficient rounding procedure. Given a LP solution, we open a set of k facilities from some distribution and connect each client j to its nearest open facility, such that the expected connection cost of j is at most 3.25 times its fractional connection cost. This leads to a 3.25 approximation for the k-median algorithm. Though the provable approximation ratio is worse than that of the current best algorithm, we believe the algorithm (and particularly our approach) is interesting for the following reasons:

Firstly, our algorithm is more efficient than the $3 + \epsilon$-approximation algorithm with the same approximation guarantee. The bottleneck of our algorithm is solving the LP. Using Young's $(1 + \epsilon)$-approximation for the k-median LP [15] (for $\epsilon = O(1/k)$), we get a running time of $\tilde{O}(k^3n^2)$. By comparison, the local search algorithm of [2] requires $O(n^8)$ time to achieve a 3.25 approximation.

Secondly, our approach has the potential to beat the decade old $3+\epsilon$-approximation algorithm for k-median. In spite of the simplicity of our algorithm, we are unable to exploit its full potential due to technical difficulties in the analysis. Our upper bound of 3.25 is not tight. The algorithm has some parameters which we have instantiated for ease of analysis. It is possible that the algorithm with these specific choices gives an approximation ratio strictly better than 3; further there is additional room for improvement by making a judicious choice of algorithm parameters.

The distribution of solutions produced by the algorithm has two nice properties that make it applicable for some variants of the k-median problem: (1) The probability that a facility i is open is exactly y_i, and (2) The events that facilities are open are negatively related. Consequently, the algorithm can be easily extended to solve the k-median problem with facility costs and the k-median problem with multiple types of facilities, both introduced in [8]. The techniques of this paper yield a factor 3.25 algorithm for the two generalizations.

Based on our techniques for the k-median problem, we give a 34-approximation algorithm for the knapsack median problem, which greatly improves the constant approximation given by [11].(The constant was 2700.) Besides the improved approximation ratio, both our algorithm and analysis are simpler compared to those in [11]. Following the same line of the algorithm, we can give a 9-approximation for the matroid-median problem, improving on the 16-approximation in [9].

2 The approximation algorithm for the k-median problem

Our algorithm is inspired by the $6\frac{2}{3}$-approximation for k-median by [5] and the clustered rounding approach of Chudak and Shmoys [6] for facility location as well as the analysis of the 1.5-approximation for UFL problem by [4]. In particular, we are able to save the additive factor of 4 that is lost at the beginning of the $6\frac{2}{3}$-approximation algorithm by [5], using some ideas from the rounding approaches for facility location.

We first give with a high level overview of the algorithm. A simple way to match the marginals given by the LP solution is to interpret the y_i variables as probabilities of opening facilities and sample independently for each i. This has the problem that with constant probability, a client j could have no facility opened close to j. In order to address this, we group fractional facilities into bundles, each containing a total fractional of between $1/2$ and 1. At most one facility is opened in each bundle and the probability that some facility in a bundle is picked is exactly the volume, i.e. the sum of y_i values for the bundle.

Creating bundles reduces the uncertainty of the sampling process. E.g. if the facilities in a bundle of volume $1/2$ are sampled independently, with probability $e^{-1/2}$ in the worst case, no facility will be open; while sampling the bundle as a single entity reduces the probability to $1/2$. The idea of creating bundles alone does not reduce the approximation ratio to a constant, since still with some non-zero probability, no nearby facilities are open.

In order to ensure that clients always have an open facility within expected distance comparable to their LP contribution, we pair the bundles. Each pair now has at least a total fraction of 1 facility and we ensure that the rounding procedure always picks one facility in each pair. The randomized rounding procedure makes independent choices for each pair of bundles and for fractional facilities that are not in any bundle.

This produces k facilities in expectation. We get exactly k by replacing the independent rounding by a dependent rounding procedure with negative correlation properties so that our analysis need only consider the independent rounding procedure. One final detail: In order to obtain a faster running time, we use a procedure that yields an approximately optimal LP solution with at most $(1+\epsilon)k$ fractional facilities. Setting $\epsilon = \delta/k$, and applying our dependent rounding approach, we get at most k facilities with high probability with a small additive loss in the guarantee.

Now we proceed to give more details. We solve LP(1) to obtain a fractional solution (x, y). By splitting one facility into many if necessary, we can assume $x_{i,j} \in \{0, y_i\}$. We remove all facilities i from $\mathcal{C}$ that have $y_i = 0$. Let $\mathcal{F}_j = \{i \in \mathcal{F} : x_{i,j} > 0\}$. So, instead of using x and y, we shall use $(y, \{\mathcal{F}_j | j \in \mathcal{C}\})$ to denote a solution.

For a subset of facilities $\mathcal{F}' \subseteq \mathcal{F}$, define $\mathsf{vol}(\mathcal{F}') = \sum_{i \in \mathcal{F}'} y_i$ to be the *volume* of $\mathcal{F}'$. So, $\mathsf{vol}(\mathcal{F}_j) = 1, \forall j \in \mathcal{C}$. W.L.O.G, we assume $\mathsf{vol}(\mathcal{F}) = k$. Denote by $d(j, \mathcal{F}')$ the average distance from j to $\mathcal{F}'$ w.r.t weights y, i.e, $d(j, \mathcal{F}') = \sum_{i \in \mathcal{F}'} y_i d(j, i)/\mathsf{vol}(\mathcal{F}')$. Unless otherwise stated, $d(j, \emptyset) = 0$. Define $d_{av}(j) = \sum_{i \in \mathcal{F}_j} y_i d(i, j)$ to be the connection cost of j in the fractional solution. For a client j, let $B(j, r)$ denote the set of facilities that have distance strictly smaller than r to j.

Our rounding algorithm consists of 4 phases, which we now describe.

2.1 Filtering phase

We begin our algorithm with a filtering phase, where we select a subset $\mathcal{C}' \subseteq \mathcal{C}$ of clients. $\mathcal{C}'$ has two properties: (1) The clients in $\mathcal{C}'$ are far away from each other. With this property, we can guarantee that each client in $\mathcal{C}'$ can be assigned an exclusive set of facilities with large volume. (2) A client in $\mathcal{C} \setminus \mathcal{C}'$ is close to some client in $\mathcal{C}'$, so that its connection cost is bounded in terms of the connection cost of its neighbour in $\mathcal{C}'$. So, $\mathcal{C}'$ captures the connection requirements of $\mathcal{C}$ and also has a nice structure. After this filtering phase, our algorithm is independent of the clients in $\mathcal{C} \setminus \mathcal{C}'$. Following is the filtering phase.

Initially, $\mathcal{C}' = \emptyset, \mathcal{C}'' = \mathcal{C}$. At each step, we select the client $j \in \mathcal{C}''$ with the minimum $d_{av}(j)$, breaking ties arbitrarily, add j to $\mathcal{C}'$ and remove j and all j's that $d(j, j') \le 4 d_{av}(j')$ from $\mathcal{C}''$. This operation is repeated until $\mathcal{C}'' = \emptyset$.

Lemma 1. *The following statements hold:*

1. *For any $j, j' \in \mathcal{C}', j \ne j', d(j, j') > 4 \max\{d_{av}(j), d_{av}(j')\}$;*
2. *For any $j' \in \mathcal{C} \setminus \mathcal{C}'$, there is a client $j \in \mathcal{C}'$ such that $d_{av}(j) \le d_{av}(j'), d(j, j') \le 4 d_{av}(j')$.*

Proof. Consider two different clients $j, j' \in \mathcal{C}'$. W.L.O.G, assume j is added to $\mathcal{C}'$ first. So, $d_{av}(j) \le d_{av}(j')$. Since j' is not removed from $\mathcal{C}''$, we have $d(j, j') > 4 d_{av}(j') = 4 \max\{d_{av}(j), d_{av}(j')\}$.

Now we prove the second statement. Since j' is not in $\mathcal{C}'$, it must be removed from $\mathcal{C}''$ when some j is added to $\mathcal{C}'$. Since j was picked instead of j', $d_{av}(j) \le d_{av}(j')$; since j' was removed from $\mathcal{C}''$, $d(j, j') \le 4 d_{av}(j')$.

2.2 Bundling phase

Since clients in $\mathcal{C}'$ are far away from each other, each client $j \in \mathcal{C}'$ can be assigned a set of facilities with large volume. To be more specific, for a client $j \in \mathcal{C}'$, we define a set $\mathcal{U}_j$ as follows. Let $R_j = \frac{1}{2}\min_{j' \in \mathcal{C}', j' \neq j} d(j, j')$ be half the distance of j to its nearest neighbour in $\mathcal{C}'$, and $\mathcal{F}'_j = \mathcal{F}_j \cap B(j, 1.5R_j)$ to be the set of facilities that serve j and are at most $1.5R_j$ away.[1] A facility i which belongs to at least one $\mathcal{F}'_j$ is *claimed* by the nearest $j \in \mathcal{C}'$ such that $i \in \mathcal{F}'_j$, breaking ties arbitrarily. Then, $\mathcal{U}_j \subseteq \mathcal{F}_j$ is the set of facilities claimed by j.

Lemma 2. *The following two statements are true:*
(1) $1/2 \leq \mathsf{vol}(\mathcal{U}_j) \leq 1, \forall j \in \mathcal{C}'$, *and* ***(2)*** $\mathcal{U}_j \cap \mathcal{U}_{j'} = \emptyset, \forall j, j' \in \mathcal{C}', j \neq j'$.

Proof. Statement 2 is trivial; we only consider the first one. Since $\mathcal{U}_j \subseteq \mathcal{F}'_j \subseteq \mathcal{F}_j$, we have $\mathsf{vol}(\mathcal{U}_j) \leq \mathsf{vol}(\mathcal{F}_j) = 1$. For a client $j \in \mathcal{C}'$, the closest client $j' \in \mathcal{C}' \setminus \{j\}$ to j has $d(j, j') > 4d_{av}(j)$ by lemma 1. So, $R_j > 2d_{av}(j)$ and the facilities in $\mathcal{F}_j$ that are at most $2d_{av}(j)$ away must be claimed by j. The set of these facilities has volume at least $1 - d_{av}(j)/(2d_{av}(j)) = 1/2$. Thus, $\mathsf{vol}(\mathcal{U}_j) \geq 1/2$.

The sets $\mathcal{U}_j$'s are called *bundles*. Each bundle $\mathcal{U}_j$ is treated as a single entity in the sense that at most 1 facility from it is open, and the probability that 1 facility is open is exactly $\mathsf{vol}(\mathcal{U}_j)$. From this point, a bundle $\mathcal{U}_j$ can be viewed as a single facility with $y = \mathsf{vol}(\mathcal{U}_j)$, except that it does not have a fixed position. We will use the phrase "opening the bundle $\mathcal{U}_j$" the operation that opens 1 facility randomly from $\mathcal{U}_j$, with probabilities $y_i/\mathsf{vol}(\mathcal{U}_j)$.

2.3 Matching phase

Next, we construct a matching $\mathcal{M}$ over the bundles (or equivalently, over $\mathcal{C}'$). If two bundles $\mathcal{U}_j$ and $\mathcal{U}_{j'}$ are matched, we sample them using a joint distribution. Since each bundle has volume at least $1/2$, we can choose a distribution such that with probability 1, at least 1 bundle is open.

We construct the matching $\mathcal{M}$ using a greedy algorithm. While there are at least 2 unmatched clients in $\mathcal{C}'$, we choose the closest pair of unmatched clients $j, j' \in \mathcal{C}'$ and match them.

2.4 Sampling phase

Following is our sampling phase.

[1] It is worthwhile to mention the motivation behind the choice of the scalar 1.5 in the definition of $\mathcal{F}'_j$. If we were only aiming at a constant approximation ratio smaller than 4, we could replace 1.5 with 1, in which case the analysis is simpler. On the other hand, we believe that changing 1.5 to ∞ would give the best approximation, in which case the algorithm also seems cleaner (since $\mathcal{F}'_j = \mathcal{F}_j$). However, if the scalar were ∞, the algorithm is hard to analyze due to some technical reasons. So, the scalar 1.5 is selected so that we don't lose too much in the approximation ratio and yet the analysis is still manageable.

for each pair $(j, j') \in \mathcal{M}$ **do**
  With probability $1 - \mathsf{vol}(\mathcal{U}_{j'})$, open $\mathcal{U}_j$; with probability $1 - \mathsf{vol}(\mathcal{U}_j)$, open $\mathcal{U}_{j'}$; and with probability $\mathsf{vol}(\mathcal{U}_j) + \mathsf{vol}(\mathcal{U}_{j'}) - 1$, open both $\mathcal{U}_j$ and $\mathcal{U}_{j'}$;
end for
If some $j \in \mathcal{C}'$ is not matched in $\mathcal{M}$, open $\mathcal{U}_j$ randomly and independently with probability $\mathsf{vol}(\mathcal{U}_j)$;
For each facility i not in any bundle $\mathcal{U}_j$, open it independently with probability y_i.

It is easy to see that our sampling process opens k facilities in expectation, since each facility i is open with probability y_i. It does not always open k facilities as we promised.[2] We shall describe in Appendix B how to modify our algorithm so that it opens exactly k facilities, without sacrificing the approximation ratio. We also show in Appendix B that the distribution for the k open facilities satisfies a negative correlation property so that our approximation ratio analysis need only consider the simpler sampling process above. Along with the marginal probability property, we show how our algorithm can be extended to solve variants of the k-median problem in Appendix B.

We shall outline the proof of the 3.25 approximation ratio for the above algorithm in section 3. As a warmup, we conclude this section with a much weaker result:

Lemma 3. *The algorithm gives a constant approximation for the k-median problem.*

Proof. It is enough to show that the ratio between the expected connection cost of j and $d_{av}(j)$ is bounded, for any $j \in \mathcal{C}$. Moreover, it suffices to consider a client $j \in \mathcal{C}'$. Indeed, if $j \notin \mathcal{C}'$, there is a client $j_1 \in \mathcal{C}'$ such that $d_{av}(j_1) \leq d_{av}(j), d(j, j_1) \leq 4d_{av}(j)$, by the second property of lemma 1. So the expected connection cost for j is at most the expected connection cost for j_1 plus $4d_{av}(j)$. Thus, the ratio for j is bounded by the ratio for j_1 plus 4. So, it suffices to consider the expected connection cost of j_1.

W.L.O.G, assume $d_{av}(j_1) = 1$. Let j_2 be the client in $\mathcal{C}' \setminus \{j_1\}$ that is closest to j_1. Consider the case where j_1 is not matched with j_2 (this is worse than the case where they are matched). Then, j_2 must be matched with another client, say $j_3 \in \mathcal{C}'$, before j_1 is matched, and $d(j_2, j_3) \leq d(j_1, j_2)$. The sampling process guarantees that there must be a open facility in $\mathcal{U}_{j_2} \cup \mathcal{U}_{j_3}$. It is true that j_2 and j_3 may be far away from j_1. However, if $d(j_1, j_2) = 2R$ (thus, $d(j_1, j_3) \leq 4R, d_{av}(j_2), d_{av}(j_3) \leq R/2$), the volume of $\mathcal{U}_{j_1}$ is at least $1 - 1/R$. That means with probability at least $1 - 1/R$, j_1 will be connected to a facility that serves it in the fractional solution; only with probability $1/R$, j_1 will be connected to a facility that is $O(R)$ away. So, the algorithm will give a constant approximation.

3 Outline of the proof of the 3.25 approximation ratio

If we analyze the algorithm as in the proof of lemma 3, an additive factor of 4 is lost at the first step. This additive factor can be avoided,[3] if we notice that there is a set $\mathcal{F}_j$

[2] One could argue that with high probability, the number of open facilities is at most $k + O(\sqrt{k})$ by applying Chernoff bound. This argument, however, can only lead to a pseudo-approximation.

[3] this is inspired by the analysis for the facility location problem in [6, 4, 12].

of facilities of volume 1 around j. Hopefully with some probability, some facility in $\mathcal{F}_j$ is open. It is not hard to show that this probability is at least $1-1/e$. So, only with probability $1/e$, we are going to pay the additive factor of 4. Even if there are no open facilities in $\mathcal{F}_j$, the facilities in $\mathcal{F}_{j_1}$ and $\mathcal{F}_{j_2}$ can help to reduce the constant.

A natural style of analysis is: focus on a set of "potential facilities", and consider the expected distance between j and the closest open facility in this set. An obvious candidate for the potential set is $\mathcal{F}_j \cup \mathcal{F}_{j_1} \cup \mathcal{F}_{j_2} \cup \mathcal{F}_{j_3}$. However, we are unable to analyze this complicated system.

Instead, we will consider a different potential set. Observing that $\mathcal{U}_{j_1}, \mathcal{U}_{j_2}, \mathcal{U}_{j_3}$ are disjoint, the potential set $\mathcal{F}_j \cup \mathcal{U}_{j_1} \cup \mathcal{U}_{j_2} \cup \mathcal{U}_{j_3}$ is much more tractable. Even with this simplified potential set, we still have to consider the intersection between $\mathcal{F}_j$ and each of $\mathcal{U}_{j_1}, \mathcal{U}_{j_2}$ and $\mathcal{U}_{j_3}$. Furthermore, we tried hard to reduce the approximation ratio at the cost of complicating the analysis(recall the argument about the choice of the scalar 1.5). With the potential set $\mathcal{F}_j \cup \mathcal{U}_{j_1} \cup \mathcal{U}_{j_2} \cup \mathcal{U}_{j_3}$, we can only prove a worse approximation ratio. To reduce it to 3.25, different potential sets are considered for different bottleneck cases.

W.L.O.G, we can assume $j \notin \mathcal{C}'$, since we can think of the case $j \in \mathcal{C}'$ as $j \notin \mathcal{C}'$ and there is another client $j_1 \in \mathcal{C}'$ with $d(j, j_1) = 0$. We also assume $d_{av}(j) = 1$. Let $j_1 \in \mathcal{C}'$ be the client such that $d_{av}(j_1) \le d_{av}(j) = 1, d(j, j_1) \le 4d_{av}(j) = 4$. Let j_2 be the closest client in $\mathcal{C}' \setminus \{j_1\}$ to j_1, thus $d(j_1, j_2) = 2R_{j_1}$. Then, either j_1 is matched with j_2, or j_2 is matched with a different client $j_3 \in \mathcal{C}'$, in which case we will have $d(j_2, j_3) \le d(j_1, j_2) = 2R_{j_1}$. We only consider the second case. Readers can verify this is indeed the bottleneck case.

For the ease of notation, define $2R := d(j_1, j_2) = 2R_{j_1}, 2R' := d(j_2, j_3) \le 2R, d_1 := d(j, j_1), d_2 := d(j, j_2)$ and $d_3 := d(j, j_3)$.

At the top level, we divide the analysis into two cases : the case $2 \le d_1 \le 4$ and the case $0 \le d_1 \le 2$(Notice that we assumed $d_{av}(j) = 1$ and thus $0 \le d_1 \le 4$). For some technical reason, we can not include the whole set $\mathcal{F}_j$ in the potential set for the case $2 \le d_1 \le 4$. Instead we only include a subset $\mathcal{F}'_j$ (notice that $j \notin \mathcal{C}'$ and thus $\mathcal{F}'_j$ was not defined before). $\mathcal{F}'_j$ is defined as $\mathcal{F}_j \cap B(j, d_1)$.

The case $2 \le d_1 \le 4$ is further divided into 2 sub-cases : $\mathcal{F}'_j \cap \mathcal{F}'_{j_1} \subseteq \mathcal{U}_{j_1}$ and $\mathcal{F}'_j \cap \mathcal{F}'_{j_1} \not\subseteq \mathcal{U}_{j_1}$. Thus, we will have 3 cases :

1. $2 \le d_1 \le 4, \mathcal{F}'_j \cap \mathcal{F}'_{j_1} \subseteq \mathcal{U}_{j_1}$. In this case, we consider the potential set $\mathcal{F}'' = \mathcal{F}'_j \cup \mathcal{F}'_{j_1} \cup \mathcal{U}_{j_2} \cup \mathcal{U}_{j_3}$. Notice that $\mathcal{F}'_j = \mathcal{F}_j \cap B(j, d_1)$, $\mathcal{F}'_{j_1} = \mathcal{F}_{j_1} \cap B(j_1, 1.5R)$. In Appendix C.1, we show that the expected connection cost of j in this case is bounded by 3.243.
2. $2 \le d_1 \le 4, \mathcal{F}'_j \cap \mathcal{F}'_{j_1} \not\subseteq \mathcal{U}_{j_1}$. In this case, some facility i in $\mathcal{F}'_j \cap \mathcal{F}'_{j_1}$ must be claimed by some client $j' \ne j_1$. Since $d(j, i) \le d_1, d(j_1, i) \le 1.5R$, we have

$$d(j, j') \le d(j, i) + d(j', i) \le d(j, i) + d(j_1, i) \le d_1 + 1.5R.$$

   If $j' \notin \{j_2, j_3\}$, we can include $\mathcal{U}_{j'}$ in the potential set and thus the potential set is $\mathcal{F}'' = \mathcal{F}'_j \cup \mathcal{F}'_{j_1} \cup \mathcal{U}_{j_2} \cup \mathcal{U}_{j_3} \cup \mathcal{U}_{j'}$. If $j \in \{j_2, j_3\}$, then we know j and j_2, j_3 are close. So, we either have a "larger" potential set, or small distances between j and j_2, j_3. Intuitively, this case is unlikely to be the bottleneck case. In Appendix C.2, we prove that the expected connection cost of j in this case is bounded by 3.189.

3. $0 \le d_1 \le 2$. In this case, we consider the potential set $\mathcal{F}'' = \mathcal{F}_j \cup \mathcal{U}_{j_1} \cup \mathcal{U}_{j_2} \cup \mathcal{U}_{j_3}$. In Appendix C.3, we show that the expected connection cost of j in this case is bounded by 3.25.

3.1 Running time of the algorithm

Let's now analyze the running time of our algorithm in terms of $n = |\mathcal{F} \cup \mathcal{C}|$. The bottleneck of the algorithm is solving the LP. Indeed, generating the set $\mathcal{C}'$, creating bundles and constructing the matching $\mathcal{M}$ all take time $O(n^2)$. Then new sampling algorithm takes time $O(n)$ and computing the nearest open facility for all the clients takes time $O(n^2)$. Thus, the total time to round a fractional solution is $O(n^2)$.

To solve the LP, we use the $(1+\epsilon)$ approximation algorithm for the fractional k-median problem in [15]. The algorithm gives a fractional solution which opens $(1+\epsilon)k$ facilities with connection cost at most $1+\epsilon$ times the fractional optimal in time $O(kn^2 \ln(n/\epsilon)/\epsilon^2)$. To apply it here, we set $\epsilon = \delta/k$ for some small constant δ. Then, our rounding procedure will open k facilities with probability $1-\delta$ and $k+1$ facilities with probability δ. The expected connection cost of the integral solution is at most $3.25(1+\delta/k)$ times the fractional optimal. Conditioned on the rounding procedure opening k facilities, the expected connection cost is at most $3.25(1+\delta/k)/(1-\delta) \le 3.25(1+O(\delta))$ times the optimal fractional value.

Theorem 1. *For any $\delta > 0$, there is a $3.25(1+\delta)$-approximation algorithm for k-median problem that runs in $\tilde{O}\left((1/\delta^2)k^3n^2\right)$ time.*

3.2 Generalization of the algorithm to variants of *k*-median problems

The distribution of k open facilities that our algorithm produces has two good properties. First, the probability that a facility i is open is exactly y_i. Second, the events that facilities are open are negatively correlated, as stated in Lemma 10. The two properties allow our algorithm to be extended to some variants of the k-median problem, which the local search algorithm seems hard to apply to.

The first variant is a common generalization of the k-median problem and the UFL problem introduced in [8]. In the generalized problem, we have both upper bound on the number of facilities we can open and facility costs. For this problem, the LP is the same as LP(1), except that the objective function contains a term for the opening cost. After solving the LP, we can use our rounding procedure to get an integral solution. The expected opening cost of the integral solution is exactly the fractional opening cost in the LP solution, while the expected connection cost is at most 3.25 times the fractional connection cost.

Another generalization introduced in [8] is the k-median problem with multiple types of facilities. Suppose we have t types of facilities, and the total number of the t types of facilities we can open is at most k so that each client is connected to one facility of each type. The goal is to minimize the total connection cost. Our techniques yield a 3.25 approximation for this problem as well. We first solve the natural LP for this problem and then round t instances of the k-median problem separately. We can again use the dependence rounding technique described in Appendix B to guarantee the cardinality constraint.

4 Approximation algorithms for knapsack-median problem and matroid-median problem

The LP for Knapsack-median problem is the same as LP (1), except that we change the cardinality constraint $\sum_{i\in\mathcal{F}} y_i \leq k$ to the knapsack constraint $\sum_{i\in\mathcal{F}} f_i y_i \leq M$.

As showed in [11], the LP has unbounded integrality gap. To amend this, we do the same trick as in [11]. Suppose we know the optimal cost OPT for the knapsack median problem. For a client j, let L_j be its connection cost. Then, for some other client j', its connection cost is at least $\max\{0, L_j - d(j,j')\}$. This suggests

$$\sum_{j'\in\mathcal{C}} \max\{0, L_j - d(j,j')\} \leq \mathsf{OPT}. \tag{1}$$

Thus, knowing OPT, we can get an upper bound L_j on the connection cost of j: L_j is the largest number such that the above inequality is true. We solve the LP with the additional constraint that $x_{i,j} = 0$ if $d(i,j) > L_j$. Then, the LP solution, denoted by LP, must be at most OPT. By binary searching, we find the minimum OPT so that $\mathsf{LP} \leq \mathsf{OPT}$. Let $\left(x^{(1)}, y^{(1)}\right)$ be the fractional solution given by the LP. We use $\mathsf{LP}_j = d_{av}(j) = \sum_{i\in\mathcal{F}} d(i,j)x^{(1)}_{i,j}$ to denote the contribution of the client j to LP.

Then we select a set of filtered clients $\mathcal{C}'$ as we did in the algorithm for the k-median problem. For a client $j \in \mathcal{C}$, let $\pi(j)$ be a client $j' \in \mathcal{C}'$ such that $d_{av}(j') \leq d_{av}(j), d(j,j') \leq 4d_{av}(j)$. Notice that for a client $j \in \mathcal{C}'$, we have $\pi(j) = j$. This time, we can not save the additive factor of 4; instead, we move the connection demand on each client $j \notin \mathcal{C}'$ to $\pi(j)$. For a client $j' \in \mathcal{C}'$, let $w_{j'} = \left|\pi^{-1}(j')\right|$ be its connection demand. Let $\mathsf{LP}^{(1)} = \sum_{j'\in\mathcal{C}', i\in\mathcal{F}} w_{j'} x_{i,j'} d(i,j') = \sum_{j'\in\mathcal{C}'} w_{j'} d_{av}(j')$ be the cost of the solution $\left(x^{(1)}, y^{(1)}\right)$ to the new instance. For a client $j \in \mathcal{C}$, let $\mathsf{LP}^{(1)}_j = d_{av}(\pi(j))$ be the contribution of j to $\mathsf{LP}^{(1)}$. (The amount $w_{j'} d_{av}(j')$ is evenly spread among the $w_{j'}$ clients in $\pi^{-1}(j')$.) Since $\mathsf{LP}_j = d_{av}(j) \leq d_{av}(j) \leq \mathsf{LP}^{(1)}_j$, we have $\mathsf{LP}^{(1)} \leq \mathsf{LP}$.

For any client $j \in \mathcal{C}'$, let $2R_j = \min_{j'\in\mathcal{C}', j'\neq j} d(j,j')$, if $\mathsf{vol}(B(j,R_j)) \leq 1$; otherwise let R_j be the smallest number such that $\mathsf{vol}(B(j,R_j)) = 1$. ($\mathsf{vol}(S)$ is defined as $\sum_{i\in S} y^{(1)}_i$.) Let $B_j = B(j,R_j)$ for the ease of notation. If $\mathsf{vol}(B_j) = 1$, we call B_j a full ball; otherwise, we call B_j a partial ball. Notice that we always have $\mathsf{vol}(B_j) \geq 1/2$. Notice that $R_j \leq L_j$ since $x^{(1)}_{i,j} = 0$ for all facilities i with $d_{i,j} > L_j$.

We find a matching $\mathcal{M}$ over the partial balls as in Section 2: while there are at least 2 unmatched partial balls, match the two balls B_j and $B_{j'}$ with the smallest $d(j,j')$.

Consider the following LP.

$$\text{LP(2)} \qquad \min \quad \sum_{j'\in\mathcal{C}'} w_{j'} \left(\sum_{i\in B_{j'}} d(i,j')y_i + \left(1 - \sum_{i\in B_j} y_i\right) R_{j'}\right) \quad \text{s.t.}$$

$$\sum_{i\in B_{j'}} y_i = 1, \quad \forall j' \in \mathcal{C}', B_{j'} \text{ full}; \qquad \sum_{i\in B_{j'}} y_i \leq 1, \quad \forall j' \in \mathcal{C}', B_{j'} \text{ partial};$$

$$\sum_{i\in B_j} y_i + \sum_{i\in B_{j'}} y_i \geq 1, \quad \forall (B_j, B_{j'}) \in \mathcal{M}; \qquad \sum_{i\in\mathcal{F}} f_i y_i \leq M;$$

$$y_i \geq 0, \quad \forall i \in \mathcal{F}$$

Let $y^{(2)}$ be an optimal *basic solution* of LP (2) and let $\mathsf{LP}^{(2)}$ be the value of LP(2). For a client $j \in \mathcal{C}$ with $\pi(j) = j'$, let $\mathsf{LP}_j^{(2)} = \sum_{i \in B_{j'}} d(i, j')y_i + \left(1 - \sum_{i \in B_{j'}} y_i\right) R_{j'}$ be the contribution of j to $\mathsf{LP}^{(2)}$. Then we prove

Lemma 4. $\mathsf{LP}^{(2)} \le \mathsf{LP}^{(1)}$.

Proof. It is easy to see that $y^{(1)}$ is a valid solution for LP(2). By slightly abusing the notation, we can think of $\mathsf{LP}^{(2)}$ is the cost of $y^{(1)}$ to LP(2). We compare the contribution of each client $j \in \mathcal{C}$ with $\pi(j) = j'$ to $\mathsf{LP}^{(2)}$ and to $\mathsf{LP}^{(1)}$. If $B_{j'}$ is a full ball, j' contributes the same to $\mathsf{LP}^{(2)}$ and as to $\mathsf{LP}^{(1)}$. If $B_{j'}$ is a partial ball, j' contributes $\sum_{i \in \mathcal{F}_{j'}} d(i, j')y_i^{(1)}$ to $\mathsf{LP}^{(1)}$ and $\sum_{i \in B_{j'}} d(i, j')y_i^{(1)} + (1 - \sum_{i \in B_{j'}} y_i^{(1)})R_{j'}$ to $\mathsf{LP}^{(2)}$. Since $B_{j'} = B(j', R_{j'}) \subseteq \mathcal{F}_{j'}$ and $\mathsf{vol}(\mathcal{F}_{j'}) = 1$, the contribution of j' to $\mathsf{LP}^{(2)}$ is at most that to $\mathsf{LP}^{(1)}$. So, $\mathsf{LP}^{(2)} \le \mathsf{LP}^{(1)}$.

Notice that LP(2) only contains y-variables. We show that any basic solution y^* of LP(2) is almost integral. In particular, we prove the following lemma in Appendix A.

Lemma 5. *Any basic solution y^* of LP(2) contains at most 2 fractional values. Moreover, if it contains 2 fractional values $y_i^*, y_{i'}^*$, then $y_i^* + y_{i'}^* = 1$ and either there exists some $j \in \mathcal{C}'$ such that $i, i' \in B_j$ or there exists a pair $(B_j, B_{j'}) \in \mathcal{M}$ such that $i \in B_j, i' \in B_{j'}$.*

Let $y^{(3)}$ be the integral solutin obtained from $y^{(2)}$ as follows. If $y^{(2)}$ contains at most 1 fractional value, we zero-out the fractional value. If $y^{(2)}$ contains 2 fractional values $y_i^{(2)}, y_{i'}^{(2)}$, let $y_i^{(3)} = 1, y_{i'}^{(3)} = 0$ if $f_i \le f_{i'}$ and let $y_i^{(3)} = 0, y_{i'}^{(3)} = 1$ otherwise. Notice that since $y_i^{(2)} + y_{i'}^{(2)} = 1$, this modification does not increase the budget. Let SOL be the cost of the solution $y^{(3)}$ to the original instance.

We leave the proof of the following lemma to Appendix A.

Lemma 6. $\sum_{i \in B(j', 5R_{j'})} y_i^{(2)} \ge 1$ *and* $\sum_{i \in B(j', 5R_{j'})} y_i^{(3)} \ge 1$. *i.e, there is an open facility (possibly two facilities whose opening fractions sum up to 1) inside $B(j', 5R_{j'})$ in both the solution $y^{(2)}$ and the solution $y^{(3)}$.*

Lemma 7. $\mathsf{SOL} \le 34\mathsf{OPT}$.

Proof. Let $\tilde{i}$ be the facility that $y_{\tilde{i}}^{(2)} > 0, y_{\tilde{i}}^{(3)} = 0$, if it exists; let $\tilde{j}$ be the client that $\tilde{i} \in B_{\tilde{j}}$.

Now, we focus on a client $j \in \mathcal{C}$ with $\pi(j) = j'$. Then, $d(j, j') \le 4d_{av}(j) = 4\mathsf{LP}_j$. Assume that $j' \ne \tilde{j}$. Then, to obtain $y^{(3)}$, we did not move or remove an open facility from $B_{j'}$. In other words, for every $i \in B_{j'}$, $y_i^{(3)} \ge y_i^{(2)}$. In this case, we show

$$\mathsf{SOL}_{j'} \le \sum_{i \in B_{j'}} d(i, j')y_i^{(2)} + (1 - \sum_{i \in B_{j'}} y_i^{(2)}) \times 5R_{j'}.$$

If there is no open facility in $B_{j'}$ in $y^{(3)}$, then there is also no open facility in $B_{j'}$ in $y^{(2)}$. Then, by Lemma 6, $\mathsf{SOL}_{j'} = 5R_{j'} =$ right-side. Otherwise, there is exactly one open facility in $B_{j'}$ in $y^{(3)}$. In this case, $\mathsf{SOL}_{j'} = \sum_{i \in B_{j'}} d(j', i) y_i^{(3)} \leq$ right-side since $y_i^{(3)} \geq y_i^{(2)}$ and $d(i, j') \leq 5R_{j'}$ for every $i \in B_{j'}$.

Observing that the right side of the inequality is at most $5\mathsf{LP}_j^{(2)}$, we have $\mathsf{SOL}_j \leq 4\mathsf{LP}_j + \mathsf{SOL}_{j'} \leq 4\mathsf{LP}_j + 5\mathsf{LP}_j^{(2)}$.

Now assume that $j' = \tilde{j}$. Since there is an open facility in $B(j', 5R_{j'})$ by Lemma 6, we have $\mathsf{SOL}_j \leq 4\mathsf{LP}_j + 5R_{j'}$. Consider the set $\pi^{-1}(j')$ of clients. Notice that we have $R_{j'} \leq L_{j'}$ since $x_{i,j'}^{(1)} = 0$ for facilities i such that $d(i, j') > L_{j'}$. Also by Inequality (1), we have $\sum_{j \in \pi^{-1}(j')} (R_{j'} - d(j, j')) \leq \sum_{j \in \pi^{-1}(j')} (L_{j'} - d(j, j')) \leq \mathsf{OPT}$. Then, since $d(j, j') \leq 4\mathsf{LP}_j$ for every $j \in \pi^{-1}(j')$, we have

$$\begin{aligned}
\sum_{j \in \pi^{-1}(j')} \mathsf{SOL}_j &\leq \sum_j (4\mathsf{LP}_j + 5R_{j'}) \leq 4 \sum_j \mathsf{LP}_j + 5 \sum_j R_{j'} \\
&\leq 4 \sum_j \mathsf{LP}_j + 5\Big(\mathsf{OPT} + \sum_j d(j, j')\Big) \leq 24 \sum_j \mathsf{LP}_j + 5\mathsf{OPT},
\end{aligned}$$

where the sums are all over clients $j \in \pi^{-1}(j')$. Summing up all clients $j \in \mathcal{C}$, we have

$$\begin{aligned}
\mathsf{SOL} = \sum_{j \in \mathcal{C}} \mathsf{SOL}_j &= \sum_{j \notin \pi^{-1}(\tilde{j})} \mathsf{SOL}_j + \sum_{j \in \pi^{-1}(\tilde{j})} \mathsf{SOL}_j \\
&\leq \sum_{j \notin \pi^{-1}(\tilde{j})} (4\mathsf{LP}_j + 5\mathsf{LP}_j^{(2)}) + 24 \sum_{j \in \pi^{-1}(\tilde{j})} \mathsf{LP}_j + 5\mathsf{OPT} \\
&\leq 24 \sum_{j \in \mathcal{C}} \mathsf{LP}_j + 5 \sum_{j \in \mathcal{C}} \mathsf{LP}_j^{(2)} + 5\mathsf{OPT} \leq 24\mathsf{LP} + 5\mathsf{LP}^{(2)} + 5\mathsf{OPT} \leq 34\mathsf{OPT},
\end{aligned}$$

where the last inequality follows from the fact that $\mathsf{LP}^{(2)} \leq \mathsf{LP}^{(1)} \leq \mathsf{LP} \leq \mathsf{SOL}$. Thus, we proved

Theorem 2. *There is an efficient 34-approximation algorithm for the knapsack-median problem.*

It is not hard to change our algorithm so that it works for the matroid median problem. The analysis for the matroid median problem is simpler, since $y^{(2)}$ will already be an integral solution. We leave the proof of the following theorem to Appendix A.

Theorem 3. *There is an efficient 9-approximation algorithm for the matroid median problem, assuming there is an efficient oracle for the input matroid.*

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A Proofs omitted from Section 4

Proof (of Lemma 5). Focus on an independent set of tight constraints defining y^*. We make sure that if $y_i^* = 0$, then the constraint $y_i = 0$ is in the independent set. Any tight constraint other than the knapsack constraint and the constraints $y_i = 0$ is defined by a set S, which is either B_j for some $j \in C'$ or $B_j \cup B_{j'}$ for some $(B_j, B_{j'}) \in \mathcal{M}$. The constraint for S is $\sum_{i \in S} y_i = 1$. Let $\mathcal{S}$ be the set of subsets S whose correspondent constraint is in the independent set.

We show that sets in $\mathcal{S}$ are disjoint. This is not true only if there is some pair $(B_j, B_{j'}) \in \mathcal{M}$ such that $B_j \in \mathcal{S}, B_j \cup B_{j'} \in \mathcal{S}$. However, this would imply that $y_i = 0$ for every $i \in B_{j'}$. Thus, the two constraints for B_j and $B_j \cup B_{j'}$ are not independent. Thus, sets in $\mathcal{S}$ are disjoint.

Consider the matrix A defined by the set of tight constraints, where each rows represent constraints and columns represent variables. Focus on set S^* of columns (facilities) correspondent to the fractional values in y^*. Then, the sub-matrix A_{Y^*} defined by Y^* must have rank $|Y^*|$. Also, if some $S \in \mathcal{S}$ contains elements in S^*, it must contain at least 2 elements in S^*. Noticing that sets in $\mathcal{S}$ are disjoint, if $|S^*| \geq 3$, the rank of A_{Y^*} can be at most $\lfloor |S^*|/2 \rfloor + 1 < |S^*|$. Thus, $|S^*| \leq 2$. If $|S^*| = 2$, then there must be a set $S \in \mathcal{S}$ containing S^*. This implies that the two fractional value $y_i^*, y_{i'}^*$ satisfies $y_i^* + y_{i'}^* = 1$ and either $i, i' \in B_j$ for some $j \in mC'$, or $i \in B_j, i' \in B_{j'}$ for some $(B_j, B_{j'}) \in \mathcal{M}$.

Proof (of Lemma 6). Focus on the solution $y^{(2)}$. Consider the nearest neighbour j_2 of j' in $\mathcal{C}'$. If $B_{j'}$ is a full ball, then there is an open facility inside $B_{j'} = B(j', R_{j'})$; we assume $B_{j'}$ is a partial ball and thus $d(j', j_2) = 2R_{j'}, R_{j_2} \leq R_{j'}$. If B_{j_2} is a full ball, there is an open facility in $B_{j_2} \subseteq B(j', 2R_{j'} + R_{j_2}) \subseteq B(j', 3R_{j'})$. We assume B_{j_2} is a partial ball. If $B_{j'}$ is matched with B_{j_3}, then there is an open facility inside $B_{j'} \cup B_{j_2} \subseteq B(j', 3R_{j'})$. Otherwise, assume B_{j_2} is matched with B_{j_3} for some $j_3 \in \mathcal{C}'$. By the matching rule, $d(j_2, j_3) \leq d(j', j_2) = 2R'$. In this case, there is an open facility inside $B_{j_2} \cup B_{j_3} \subseteq B(j', 5R_{j'}$.

Now we prove the lemma for the solution $y^{(3)}$. If $y^{(2)}$ contains 2 fractional facilities, then to obtain $y^{(3)}$, we moved 1 fractional facility within some ball $B_{\tilde{j}}$, or moved 1 fractional facility from $B_{\tilde{j}}$ to $B_{\tilde{j}'}$ for some $(B_{\tilde{j}}, B_{\tilde{j}'}) \in \mathcal{M}$. It is easy to check that the argument for $y^{(2)}$ also works for $y^{(3)}$. If $y^{(2)}$ contains 1 fractional open facility, then after removing this facility, we still have that every full ball contains an open facility and every pair of matched partial balls contains an open facility. Thus, in the solution $y^{(3)}$, there is an open facility inside $B(j', 5R_{j'})$.

Proof (sketch of Theorem 3). We follow the same line of the algorithm for the knapsack median problem, except that we change the knapsack constraint (in LP (1) and LP (2)) to the constraints

$$\sum_{i \in S} y_i \leq r_{\mathcal{H}}(S), \forall S \subseteq \mathcal{F},$$

where $\mathcal{H}$ is the given matroid, and $r_{\mathcal{H}}(S)$ is the rank function of $\mathcal{H}$. We shall show that the basic solution $y^{(2)}$ for LP 2 is already an integral solution. This is true since the polytope of the LP is the intersection of a matroid polytope and a polytope given by a laminar system of constraints. By the same argument as in the proof of lemma 3.2 in [9], $y^{(2)}$ is an integral solution, i.e, we have $y^{(3)} = y^{(2)}$. Then, following the same proof of Lemma 7, we can show that $\mathsf{SOL}_j \leq 4\mathsf{LP}_j + 5\mathsf{LP}_j^{(2)}$ for every $j \in \mathcal{C}$, which immediately implies an 9-approximation for the matroid median problem.

B Handling the cardinality constraint

In this section, we show how to modify our algorithm so that it opens exactly k facilities. The main idea is to apply the technique of dependent rounding. For each pair of matched bundles, we have a 0-1 variable indicating whether there are 2 or 1 open facilities in the pair of bundles. For the unmatched bundle and each facility that is not in any bundle, we have a 0-1 variable indicating whether the facility or the bundle is open. In the algorithm described in section 2, these indicator variables are generated independently. The expected sum of these indicator variables is $k - |\mathcal{M}|$. To make sure that we open exactly k facilities, we need to guarantee that the sum is always $k - |\mathcal{M}|$. To achieve this, we can, for example, use the tree-based dependent rounding procedure introduced in [14]. After the indicator variables are determined, we sample separately each pair of bundles and the unmatched bundle .

Let us define some notations here. We are given n real numbers $x_1, x_2, \cdots, x_n$ with $0 \leq x_i \leq 1$ for every i and $m = \sum_{i=1}^{n} x_i$ is an integer. We use the random process in [14] to select exactly m elements from $[n]$. Let A_i denote the event that i is selected. For an event E, we use $\bar{E}$ to denote the negation of E. The random events $\{A_i : i \in [n]\}$ satisfy:

1. Marginal distribution : $\Pr[A_i] = x_i, \forall i \in [n]$;
2. Negative correlation : for any subset $S \subseteq [n]$, $\Pr\left[\bigwedge_{i\in S} A_i\right] \leq \prod_{i\in S} \Pr\left[A_i\right]$ and $\Pr\left[\bigwedge_{i\in S} \bar{A}_i\right] \leq \prod_{i\in S} \Pr\left[\bar{A}_i\right]$.

Then, for each i, we sample independently an event B_i using the following process for some $0 \leq p_i \leq q_i \leq 1$. If $\bar{A}_i$, with probability p_i, let B_i happen; if A_i, with probability q_i, let B_i happen. Thus, $\Pr[\bar{B}_i|\bar{A}_i] = 1 - p_i, \Pr[\bar{B}_i|A_i] = 1 - q_i$. We shall prove that B_i's are negatively correlated :

Lemma 8. *Let $U \subseteq [n]$, we have*

$$\Pr\left[\bigwedge_{i\in U} \bar{B}_i\right] \leq \prod_{i\in U} \Pr\left[\bar{B}_i\right]$$

Proof. Define $N_A(S) = \bigwedge_{i\in S} \bar{A}_i, M_A(S) = \bigwedge_{i\in S} A_i, N_B(S) = \bigwedge_{i\in S} \bar{B}_i$.

$$\begin{aligned}
\Pr\left[N_B(U)\right] &= \sum_{S\subseteq U} \Pr\left[N_A(S) \wedge M_A(U\setminus S)\right] \prod_{i\in S}(1-p_i) \prod_{i\in U\setminus S}(1-q_i) \\
&= \sum_{S\subseteq U} \sum_{T\supset S, T\subseteq U} (-1)^{|T\setminus S|} \Pr\left[N_A(T)\right] \prod_{i\in S}(1-p_i) \prod_{i\in U\setminus S}(1-q_i) \\
&= \sum_{T\subseteq U} \Pr\left[N_A(T)\right] \left(\prod_{i\in U\setminus T}(1-q_i)\right) \sum_{S\subseteq T} (-1)^{|T\setminus S|} \prod_{i\in S}(1-p_i) \prod_{i\in T\setminus S}(1-q_i) \\
&= \sum_{T\subseteq U} \Pr\left[N_A(T)\right] \left(\prod_{i\in U\setminus T}(1-q_i)\right) \prod_{i\in T}((1-p_i)-(1-q_i))
\end{aligned}$$

$$\begin{aligned}
\Pr\left[N_B(U)\right] &\leq \sum_{T\subseteq U} \left(\prod_{i\in T}(1-x_i)\right) \left(\prod_{i\in U\setminus T}(1-q_i)\right) \prod_{i\in T}(q_i-p_i) \\
&= \prod_{i\in U} \left((1-x_i)(q_i-p_i)+(1-q_i)\right) = \prod_{i\in U} \left((1-x_i)(1-p_i)+x_i(1-q_i)\right) \\
&= \prod_{i\in U} \Pr\left[\bar{B}_i\right]
\end{aligned}$$

The second equality used the inclusion-exclusion principle.

We are ready to apply the above lemma to our approximation algorithm. In the new algorithm, we sample the indicator variables using the dependence rounding procedure. So, we can guarantee that exactly k facilities are open. Then, we show that the expected connection cost for a client u does not increase :

Lemma 9. *The approximation ratio of the algorithm using the dependence rounding procedure is at most that of the original algorithm (i.e, the algorithm described in section 2).*

Proof. For the ease of description, we assume that every facility is in some bundle and all the bundles are matched. For a client u, we order the facilities in the ascending order of distances to u. Let $z_1, z_2, \cdots, z_t$ be the order. Then, u is connected to the first open facility in the order, and thue

$$\text{Expected connection cost of } u = \sum_{s=1}^{t} \Pr[\text{the first } s \text{ facilities are not open}] \left(d(u, z_{s+1}) - d(u, z_s)\right)$$

It suffices to show that for every s, the probability P_{new} that the first s facilities are not open in the new algorithm is at most the correspondent probability P_{old} in the old algorithm. For a pair i of bundles, let A_i be the variable indicating whether the pair i contains 2 or 1 open facility. Let $\bar{B}_i$ denote the event that the first s facilities that are in in bundle i are not open. Notice that in the new algorithm, B_i is independent of $\{A_j | j \neq i\} \cap \{B_j | j \neq i\}$ under the condition A_i ($\bar{A}_i$ as well). It's easy to see that $\Pr\left[\bar{B}_i | \bar{A}_i\right] \geq \Pr\left[\bar{B}_i | A_i\right]$.

Since $P_{new} = \Pr\left[\bigwedge_i \bar{B}_i\right]$ and $P_{old} = \prod_i \Pr[\bar{B}_i]$, we have $P_{new} \leq P_{old}$, by lemma 8. Thus, the expected connection cost of any facility in the new algorithm is at most its expected connection cost in the original algorithm.

Lemma 10. *The distribution of the k open facilities generated by our new algorithm satisfy the following negative correlation. Let T be a subset of facilities. We have*

$$\Pr\left[\bigwedge_{z \in T} z\right] \leq \prod_{z \in T} \Pr[z], \quad \Pr\left[\bigwedge_{z \in T} \bar{z}\right] \leq \prod_{z \in T} \Pr[\bar{z}]$$

where z also denotes the event that facility z is open, and $\bar{z}$ is the negation of z.

Proof. We only prove the second inequality; the proof for the first is symmetric. Again, for the ease of description, assume all facilities are in a pair of matched bundles. For a pair i of bundles, let $\bar{B}_i$ be the event that facilities in T that are in pair i are not open. Again, by lemma 8, we have $\Pr\left[\bigwedge_i \bar{B}_i\right] \leq \prod_i \Pr\left[\bar{B}_i\right]$. It's easy to see that $\Pr[\bar{B}_i] \leq \prod_z \Pr[\bar{z}]$, where the product is over all z's which are in T and are in pair i. Thus, we proved the lemma.

C Proof of the 3.25 approximation ratio

We now start the long journey of bounding the expected connection cost, denoted by E, of a client j with $d_{av}(j) = 1$. Before dispatching the analysis into the 3 cases, we outline our main techniques for the analysis. When analyzing each case, we will focus on a set of potential facilities $\mathcal{F}''$ and a set $\mathcal{C}''$ of clients. $\mathcal{C}''$ will be either $\{j, j_1, j_2, j_3\}$ (Section C.1 and C.3) or $\{j, j_1, j_2, j_3, j'\}$ (Section C.2). Each facility in $\mathcal{F}''$ serves at least 1 client in $\mathcal{C}''$. In the analysis, we only focus on the sub-metric induced by $\mathcal{F}'' \cup \mathcal{C}''$. Using similar argument as we proved lemma 10, we may assume the minimum dependence between the facilities in $\mathcal{F}''$. To be more specific, we assume

1. Facilities in $\mathcal{F}''$ that are not claimed by any facility in $\mathcal{C}''$ were sampled independently in the algorithm. We call these facilities *individual facilities*.
2. If $j' \in \mathcal{C}''$, $\mathcal{U}_{j'}$ and $\mathcal{U}_{j_1}$ are not matched in $\mathcal{M}$(We know $\mathcal{U}_{j_2}$ and $\mathcal{U}_{j_3}$ are matched).

With these conditions, we prove the following lemma.

Lemma 11. *Consider a set $\mathcal{G} \subseteq \mathcal{F}''$ of individual facilities. We have*

1. *The probability that $\mathcal{G}$ contains at least 1 open facility is at least $1 - e^{-vol(\mathcal{G})}$.*
2. *Under the condition that there is at least 1 open facility in $\mathcal{G}$, the expected distance between j and the closest open facility in $\mathcal{G}$ is at most $d(j, \mathcal{G})$.*

Proof. The probability that no facility in $\mathcal{G}$ is open is at most

$$\prod_{i \in \mathcal{G}} (1 - y_i) \leq \prod_{i \in \mathcal{G}} e^{-y_i} = e^{-\mathsf{vol}(\mathcal{G})}$$

This implies the first statement.

For the second statement, we sort all facilities in $\mathcal{G}$ according the non-decreasing distances to j. Let the order be $i_1, i_2, \cdots, i_m$. Then, the expected distance between j and the closest open facility in $\mathcal{G}$ is

$$\left(\sum_{t=1}^{m} \left(\prod_{s=1}^{t-1} (1 - y_{i_s}) \right) y_{i_t} d(j, i_t) \right) \Big/ \left(1 - \prod_{t=1}^{m} (1 - y_{i_t}) \right)$$

Compare the coefficients for $d(j, i_a)$ and $d(j, i_b)$ for some $a < b$ in the above quantity. The ratio of the two coefficients is $\dfrac{y_{i_a}}{y_{i_b}} / \displaystyle\prod_{t=a}^{b-1} (1 - y_{i_t}) \geq \dfrac{y_{i_a}}{y_{i_b}}$. Compared to $d(j, \mathcal{G}) = \sum_{t=1}^{m} y_{i_t} d(j, i_t) / \mathsf{vol}(\mathcal{G})$, the above quantity puts relatively more weights on smaller distances. Thus, the expected distance between j and the closest open facility in $\mathcal{G}$ is at most $d(j, \mathcal{G})$.

In the analysis for each case, the potential set $\mathcal{F}''$ is split into disjoint "atoms" $\mathcal{I}_1, \mathcal{I}_2, \cdots, \mathcal{I}_M$. Each atom is either a sub bundle (i.e a subset of a bundle) or a set of individual facilities. Typically, an atom is a region in the Venn diagram of some sets (for example, in subsection C.1, we consider the Venn diagram of sets $\mathcal{F}'_j, \mathcal{F}'_{j_1}, \mathcal{U}_{j_1}, \mathcal{U}_{j_2}, \mathcal{U}_{j_3}$). An atom contains facilities with the same "characterization", so that averaging the locations of the facilities in the same atom will still give a valid instance, from the view of $\mathcal{F}'' \cup \mathcal{C}''$.

Given a order o of the facilities in $\mathcal{F}''$, the value of o, denoted by $\mathsf{val}(o)$, is defined as the connection cost from j to the first open facility in o. It is easy to see that $E = \mathbb{E}(\mathsf{val}(o))$, if o is the order where facilities are sorted according to the increasing distances to j. Also, for every order o, $E \leq \mathbb{E}(\mathsf{val}(o))$. In our analysis, we only consider the orders where the facilities in the same atom are consecutive and are sorted optimally. Thus, each atom is equivalent to a single facility at the weighted averaging location of the facilities in the atom, by the first statement of lemma 11(if the atom is a sub bundle, we can also average the locations). The equivalent facility has y value $\mathsf{vol}(\mathcal{I}_t)$ if $\mathcal{I}_t$ is a sub-bundle and $1 - e^{-\mathsf{vol}(\mathcal{I}_t)}$ if $\mathcal{I}_t$ is an atom of individual facilities, by the second statement of 11.

At each step, we maintain a *partial order* p. A partial order p is an order $(\mathcal{S}_1, \mathcal{S}_2, \cdots \mathcal{S}_m)$, where $\{\mathcal{S}_1, \mathcal{S}_2, \cdots, \mathcal{S}_m\}$ forms a partitioning for the atoms $\{\mathcal{I}_1, \mathcal{I}_2, \cdots, \mathcal{I}_M\}$. The *value* of p, denoted by $\mathsf{val}(p)$, is the distance between j and the closest open atom (notice that each atom is already replaced by a facility) in $\mathcal{S}_t$, where $\mathcal{S}_t$ is the first set in p containing an open atom. Initially, the partial order p contains only 1 set $\{\mathcal{I}_1, \mathcal{I}_2, \cdots, \mathcal{I}_M\}$ and $\mathbb{E}(\mathsf{val}(p)) = E$. In each step, we may *refine* the partial order p by splitting some set $\mathcal{S}_i$ into smaller subsets and replace $\mathcal{S}_i$ with some order of these subsets; or we may *merge* some atoms in some set $\mathcal{S}_t$ into 1 atom. Both operations can only increase $\mathbb{E}(\mathsf{val}(p))$.

Now, we shall consider the 3 cases one by one. Let's recall some notations here. $d_1 = d(j, j_1), d_2 = d(j, j_2), d_3 = d(j, j_3), 2R = 2R_{j_1} = d(j_1, j_2), 2R' = 2R_{j_2} = d(j_2, j_3), \mathcal{F}'_j = \mathcal{F}_j \cap B(j, d_1), \mathcal{F}'_{j_1} = \mathcal{F}_{j_1} \cap B(j_1, 1.5R)$.

C.1 The case $2 \le d_1 \le 4, \mathcal{F}'_j \cap \mathcal{F}'_{j_1} \subseteq \mathcal{U}_{j_1}$

In this subsection, we consider the case $2 \le d_1 \le 4, \mathcal{F}'_j \cap \mathcal{F}'_{j_1} \subseteq \mathcal{U}_{j_1}$. The potential set to be considered is $\mathcal{F}'' = \mathcal{F}'_j \cup \mathcal{F}'_{j_1} \cup \mathcal{U}_{j_2} \cup \mathcal{U}_{j_3}$.

Essentially, we would like to show that figure C.1 is the worst instance in this case. Notice that we are not using the facilities in $\mathcal{F}_{j_2} \backslash \mathcal{U}_{j_2}$. The expected connection cost of j in this case is

$$e^{-\frac{3}{4}} \left(\frac{3}{4} \cdot 4 + \frac{1}{4} \left(1 - e^{-\frac{1}{4}}\right) \times 8 + \frac{1}{4} e^{-\frac{1}{4}} \cdot \frac{1}{2} \cdot 12 + \frac{1}{4} e^{-\frac{1}{4}} \cdot \frac{1}{2} \cdot 22 \right) = e^{-\frac{3}{4}} \left(5 + 2.25 e^{-\frac{1}{4}}\right) < 3.19$$

At some point, we relaxed the objective function. $e^{-3/4}\left(5 + 2.25e^{-3/16}\right) < 3.243$ is the upper bound we can prove.

The idea is showing that W.L.O.G, we can assume $\mathcal{F}'_j \cup \mathcal{F}'_{j_1}$ and $\mathcal{U}_{j_2} \cup \mathcal{U}_{j_3}$ are disjoint. Then, under the condition that no facility is open in $\mathcal{F}'_j \cup \mathcal{F}'_{j_1}$, the expected connection cost of j is at most $(1/2)(d_1 + 2R) + (1/2)(d_1 + 2.5R) = d_1 + 3.25R$. Then, by using $d_1 + 3.25R$ as a "backup" value, we can ignore j_2 and j_3.

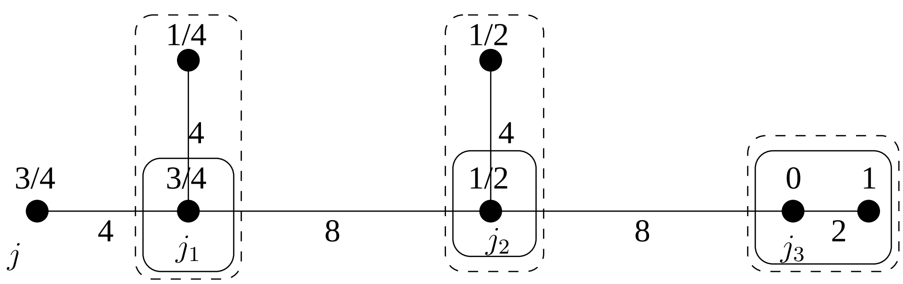

Fig. C.1. The worst case for $2 \le d_1 \le 4, \mathcal{F}'_j \cap \mathcal{F}'_{j_1} \subseteq \mathcal{U}_{j_1}$. The 3 solid rectangles represent the facilities claimed by j_1, j_2 and j_3; the 3 dash rectangles represent the facilities serving j_1, j_2 and j_3. $1/4$ fractional facility at j_1 is serving j.

Initially, an atom is a region in the Venn diagram of the following 5 sets : $\mathcal{F}'_j, \mathcal{F}'_{j_1}, \mathcal{U}_{j_1}, \mathcal{U}_{j_2}, \mathcal{U}_{j_3}$. We first consider the partial order $p = \left(\mathcal{F}'_j \cup \mathcal{F}'_{j_1}, \mathcal{U}_{j_2} \backslash (\mathcal{F}'_j \cup \mathcal{F}'_{j_1}), \mathcal{U}_{j_3} \backslash (\mathcal{F}'_j \cup \mathcal{F}'_{j_1})\right)$.

Let E_3 be the expected value of p under the condition that no atom in $\mathcal{F}'_j \cup \mathcal{F}'_{j_1} \cup \mathcal{U}_{j_2}$ is open. We prove

Lemma 12. *If $d_2 + 0.5R' \ge d_1$, $E_3 \le d_2 + 2.5R'$.*

Proof. Under the condition that no atom in $\mathcal{F}'_j \cup \mathcal{F}'_{j_1} \cup \mathcal{U}_{j_2}$ is open, there is exactly 1 open atom in $\mathcal{U}_{j_3} \backslash (\mathcal{F}'_j \cup \mathcal{F}'_{j_1})$. Thus, $E_3 = d(j, \mathcal{U}_{j_3} \backslash (\mathcal{F}'_j \cup \mathcal{F}'_{j_1}))$. Since $d(j_3, \mathcal{F}_{j_3}) \le R_{j_3}/2$ and $d(j_3, \mathcal{F}_{j_3} \backslash \mathcal{U}_{j_3}) \ge R_{j_3}$, we have $d(j_3, \mathcal{U}_{j_3}) \le R_{j_3}/2 \le R'/2$.

$\mathcal{U}_{j_3} \cap (\mathcal{F}'_j \cup \mathcal{F}'_{j_1})$ contains 2 components : $\mathcal{U}_{j_3} \cap \mathcal{F}'_j, \mathcal{U}_{j_3} \cap (\mathcal{F}'_{j_1} \backslash \mathcal{F}'_j)$. If the average distance from j_3 to each of the 2 components is at least $R'/2$, then $d(j_3, \mathcal{U}_{j_3} \backslash (\mathcal{F}'_j \cup \mathcal{F}'_{j_1}))$ is at most $R'/2$ and thus E_3 is at most $d_2 + 2R' + R'/2 \le d_2 + 2.5R'$. So, it suffices to consider the case where at least 1 component has average distance strictly smaller than $R'/2$ to j_3.

If $d(j_3, \mathcal{U}_{j_3} \cap \mathcal{F}'_j) < R'/2$, then since $d(j, \mathcal{U}_{j_3} \cap \mathcal{F}'_j) \le d_1$(recall that $\mathcal{F}'_j = \mathcal{F}_j \cap B(j, d_1)$), we have $d(j, j_3) \le d_1 + R'/2$. Since $d(j_3, \mathcal{U}_{j_3} \backslash \mathcal{F}'_j) \le 1.5R'$, we have $E_3 \le d_1 + R'/2 + 1.5R' \le d_2 + 2.5R'$.

If $d(j_3, \mathcal{U}_{j_3} \cap (\mathcal{F}'_{j_1} \backslash \mathcal{F}_j)) < R'/2$, then since $d(j_1, \mathcal{U}_{j_3} \cap (\mathcal{F}'_{j_1} \backslash \mathcal{F}_j)) \leq 1.5R$, we have $d(j_1, j_3) < 1.5R + R'/2 \leq 2R$, which contradicts the fact that the closest neighbor of j_1 in $\mathcal{C}' \backslash \{j_1\}$ has distance $2R$.

Let $\beta_2 = \mathsf{vol}(\mathcal{U}_{j_2} \backslash (\mathcal{F}'_j \cup \mathcal{F}'_{j_1})), b_2 = d(j_2, \mathcal{U}_{j_2} \backslash (\mathcal{F}'_j \cup \mathcal{F}'_{j_1}))$, and E_2 be the expected value of p under the condition that no atom in $\mathcal{F}'_j \cup \mathcal{F}'_{j_1}$ is open.

Lemma 13. *$E_2 \leq d_1 + 3.25R$.*

Proof. Under the condition that no atom in $\mathcal{F}'_j \cup \mathcal{F}'_{j_1}$ is open, there is at least 1 open atom in $(\mathcal{U}_{j_2} \cup \mathcal{U}_{j_3}) \backslash (\mathcal{F}'_j \cup \mathcal{F}'_{j_1})$. We first try to connect j to $\mathcal{U}_{j_2} \backslash (\mathcal{F}'_j \cup \mathcal{F}'_{j_1})$; if this fails, we connect j to $\mathcal{U}_{j_3} \backslash (\mathcal{F}'_j \cup \mathcal{F}'_{j_1})$. E_2 is the expected connection cost.

If $d_2 + 0.5R' \leq d_1$, then $d(j, \mathcal{U}_{j_2} \backslash (\mathcal{F}'_j \cup \mathcal{F}'_{j_1})) \leq d_2 + 1.5R' \leq d_1 + R$ and $d(j, \mathcal{U}_{j_3} \backslash (\mathcal{F}'_j \cup \mathcal{F}'_{j_1})) \leq d_2 + 2R' + 1.5R' \leq d_1 + 3R$. Clearly in this case, $E_2 \leq \max\{d_1 + R, d_1 + 3R\} = d_1 + 3R$. So, we assume $d_2 + 0.5R' > d_1$. Thus, by lemma 12, $E_3 \leq d_2 + 2.5R'$.

Let $\theta = \mathsf{vol}(\mathcal{U}_{j_2} \cap (\mathcal{F}'_j \cup \mathcal{F}'_{j_1})), t = d(j_2, \mathcal{U}_{j_2} \cap (\mathcal{F}'_j \cup \mathcal{F}'_{j_1}))$. Then

$$
\begin{aligned}
E_2 &\leq \frac{\beta_2}{1-\theta}(d_2 + b_2) + \frac{1-\theta-\beta}{1-\theta} E_3 \\
&\leq \frac{\beta_2}{1-\theta}\left(d_2 + \frac{R'/2 - \theta t - (1-\theta-\beta_2)R'}{\beta_2}\right) + \frac{1-\theta-\beta_2}{1-\theta}(d_2 + 2.5R') \\
&\leq d_2 + \frac{R'/2 - \theta t}{1-\theta} + \frac{1-\theta-\beta_2}{1-\theta} 1.5R' \leq d_2 + \frac{R'/2 - \theta t}{1-\theta} + \frac{(R'/2 - \theta t)/R'}{1-\theta} 1.5R' \\
&= d_2 + 2.5\frac{R'/2 - \theta t}{1-\theta}
\end{aligned}
$$

If $t \geq R'/2$, then $E_2 \leq d_2 + 2.5R'/2 \leq d_1 + 2R + 1.25R' \leq d_1 + 3.25R$.

Thus, we can assume that $t < R'/2$. This implies either $d(j_2, \mathcal{U}_{j_2} \cap \mathcal{F}'_j) < R'/2$ or $d(j_2, \mathcal{U}_{j_2} \cap (\mathcal{F}'_{j_1} \backslash \mathcal{F}'_j)) < R'/2$. $d(j_2, \mathcal{U}_{j_2} \cap (\mathcal{F}'_{j_1} \backslash \mathcal{F}'_j)) < R'/2$ implies

$$
d(j_1, j_2) \leq d(j_2, \mathcal{U}_{j_2} \cap (\mathcal{F}'_{j_1} \backslash \mathcal{F}'_j)) + d(j_1, \mathcal{U}_{j_2} \cap (\mathcal{F}'_{j_1} \backslash \mathcal{F}'_j)) < R'/2 + 1.5R \leq 2R,
$$

contradicting $d(j_1, j_2) = 2R$. So, we only need to consider the case $d(j_2, \mathcal{U}_{j_2} \cap \mathcal{F}'_j) \leq R'/2$. In this case, $d_2 \leq d(j, \mathcal{U}_{j_2} \cap \mathcal{F}'_j) + d(j_2, \mathcal{U}_{j_2} \cap \mathcal{F}'_j) \leq d_1 + R'/2$. Then, $d(j, \mathcal{U}_{j_2} \backslash (\mathcal{F}'_j \cup \mathcal{F}'_{j_1})) \leq d_1 + R'/2 + 1.5R' \leq d_1 + 2R'$ and $d(j, \mathcal{U}_{j_3} \backslash (\mathcal{F}'_j \cup \mathcal{F}'_{j_1})) = E_3 \leq d_2 + 2.5R' \leq d_1 + 3R'$. Thus, we have $E_2 \leq \max\{d_1 + 2R', d_1 + 3R'\} \leq d_1 + 3.25R$.

By lemma 13, we can replace the order p by $\left(\mathcal{F}'_j \cup \mathcal{F}'_{j_1}, d_1 + 3.25R\right)$. That is, connect j to the closest open atom in $\mathcal{F}'_j \cup \mathcal{F}'_{j_1}$, if it exists; otherwise use $d_1 + 3.25R$ as the value of p. By merging many atoms into one, we can redefine atoms as regions in the Venn diagram of the sets $\mathcal{F}'_j, \mathcal{F}'_{j_1}, \mathcal{U}_{j_1}$. This means that we can ignore j_2 and j_3 from now on.

Now, we refine the order p to $\left(\mathcal{F}'_j \backslash \mathcal{F}'_{j_1}, \mathcal{F}'_j \cap \mathcal{F}'_{j_1}, \mathcal{U}_{j_1} \backslash (\mathcal{F}'_j \cup \mathcal{F}'_{j_1}), \mathcal{F}'_{j_1} \backslash \mathcal{U}_{j_1}, d_1 + 3.25R\right)$.

Define $\alpha = \mathsf{vol}(\mathcal{F}'_j \backslash \mathcal{U}_{j_1}), a = d(j, \mathcal{F}'_j \backslash \mathcal{U}_{j_1}), \alpha_1 = \mathsf{vol}(\mathcal{F}'_j \cap \mathcal{F}'_{j_1}), a_1 = d(j_1, \mathcal{F}'_j \cap \mathcal{F}'_{j_1}), a'_1 = d(j, \mathcal{F}'_j \cap \mathcal{F}'_{j_1}), \beta_1 = \mathsf{vol}(\mathcal{U}_{j_1} \backslash \mathcal{F}'_j)$. Define $b_1 = d(j_1, \mathcal{U}_{j_1} \backslash \mathcal{F}'_j), \gamma = \mathsf{vol}(\mathcal{F}'_{j_1} \backslash \mathcal{U}_{j_1}), c = d(j_1, \mathcal{F}'_{j_1} \backslash \mathcal{U}_{j_1})$. Define $s = d_{av}(j_1)$. See figure C.2 for illustration.

Let E_1 be the expected value of p under the condition that the atom $\mathcal{F}'_j \backslash \mathcal{F}'_{j_1}$ is not open.

$$
\begin{aligned}
E_1 &= \alpha_1 a'_1 + \beta_1(d_1 + b_1) + (1-\alpha_1-\beta_1)(1-e^{-\gamma})(d_1 + c) + (1-\alpha_1-\beta_1)e^{-\gamma}(d_1 + 3.25R) \\
&= \alpha_1 a'_1 + (1-\alpha_1)d_1 + \beta_1 b_1 + (1-\alpha_1-\beta_1)\left((1-e^{-\gamma})c + 3.25e^{-\gamma}R\right)
\end{aligned}
$$

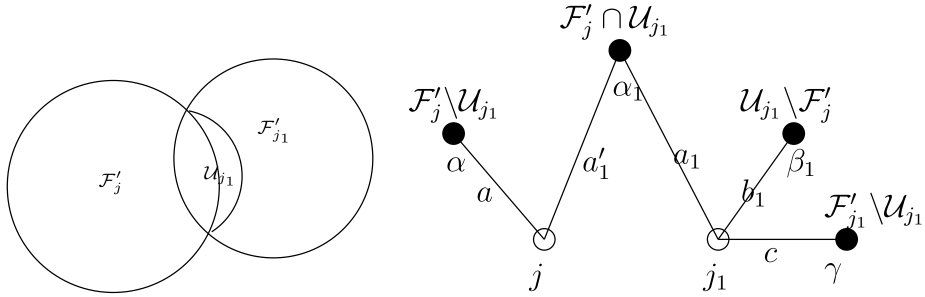

Fig. C.2. The definition of variables

Let $E = \mathbb{E}(\mathsf{val}(p))$, then $E = (1 - e^{-\alpha})a + e^{-\alpha}E_1$.

Consider the following optimization problem :

PROBLEM(1) max $E = (1 - e^{-\alpha})a + e^{-\alpha}E_1$, where

$$E_1 = \alpha_1 a'_1 + (1 - \alpha_1)d_1 + \beta_1 b_1 + (1 - \alpha_1 - \beta_1)\left((1 - e^{-\gamma})c + 3.25e^{-\gamma}R\right)$$

s.t

$$\begin{array}{rccc} \alpha a + \alpha_1 a'_1 + (1 - \alpha - \alpha_1)d_1 \leq 1 & a'_1 \leq d_1 & 2 \leq d_1 \leq 4 \\ \alpha + \alpha_1 \leq 1 & s \leq 1 & a_1 + a'_1 \geq d_1 \\ \alpha_1 a_1 + \beta_1 b_1 + \gamma c + (1 - \alpha_1 - \beta_1 - \gamma)1.5R \leq s & b_1 \leq 1.5R & R \leq c \leq 1.5R \\ \alpha_1 + \beta_1 + \gamma \leq 1 & R \leq 16/3 & s \leq R/2 \end{array}$$

We need to mention constraint $R \leq 16/3$. To see this, define $\theta = \mathsf{vol}(\mathcal{F}_j \cap \mathcal{F}_{j_1})$, $l = d(j, \mathcal{F}_j \backslash \mathcal{F}_{j_1})$. Then $R \leq d_1 + l$(assuming $\mathsf{vol}(\mathcal{U}_{j_1}) < 1$). If $l > d_1/(d_1 - 1)$, then $\theta \geq 1 - 1/l = 1/d_1$ and

$$\begin{aligned} d_{av}(j_1) \geq\ & \theta(d_1 - d(j, \mathcal{F}_j \cap \mathcal{F}_{j_1})) = \theta\left(d_1 - \frac{1 - (1 - \theta)l}{\theta}\right) > \theta d_1 - 1 + (1 - \theta)d_1/(d_1 - 1) \\ \geq\ & (1/d_1)d_1 - 1 + (1 - 1/d_1)d_1/(d_1 - 1) = 1 \end{aligned}$$

which leads to a contradiction. The last inequality above used that $d_1 \geq d_1/(d_1 - 1)$. Since $2 \leq d_1 \leq 4$, $R \leq d_1 + d_1/(d_1 - 1) \leq 4 + 4/3 = 16/3$.

In subsection D.1, we prove that value of optimization problem 1 is at most 3.243.

C.2 The case $2 \leq d_1 \leq 4, \mathcal{F}'_j \cap \mathcal{F}'_{j_1} \not\subseteq \mathcal{U}_{j_1}$

We only sketch the analysis for this case $\mathcal{F}'_j \cap \mathcal{F}'_{j_1} \not\subseteq \mathcal{U}_{j_1}$, since the techniques are exactly the same as the previous case. As we already showed, there is a client $j' \in \mathcal{C}', j' \neq j_1$ with $d(j, j') \leq d_1 + 1.5R$. There are two cases.

1. $j' \in \{j_2, j_3\}$.
   In this case, we know j_2(or j_3) is close to j, compared to the previous case, where $d(j, j_2)$ could be $d_1 + 2R$. With this gain, we can consider a smaller potential set : $\mathcal{F}'_j \cup \mathcal{U}_{j_1} \cup \mathcal{U}_{j_2} \cup \mathcal{U}_{j_3}$. Using similar argument, we can first show that $E_2 \leq d_1 + 1.5R + 2.5R/2 = d_1 + 2.75R$ and then prove $E \leq e^{-3/4}(4 + 2.75) < 3.189$.

2. $j' \notin \{j_2, j_3\}$.
   In this case, we know that there is a client $j' \in \mathcal{C}'$ other than j_1, j_2, j_3 that is close to j. We can use $\mathcal{F}'_j \cup \mathcal{U}_{j_1} \cup \mathcal{U}_{j'} \cup \mathcal{U}_{j_2} \cup \mathcal{U}_{j_3}$ as the potential set. We consider the order $\mathcal{F}'_j, \mathcal{U}_{j_1} \backslash \mathcal{F}'_j, \mathcal{U}_{j'} \backslash \mathcal{F}'_j, \mathcal{U}_{j_2} \backslash \mathcal{F}'_j, \mathcal{U}_{j_3} \backslash \mathcal{F}'_j$. Similarly, let E_2 be the expected connection cost under the condition that no facilities in $\mathcal{F}'_j \cup \mathcal{U}_{j_1}$ is open. We can first show that $E_2 \leq d_1 + (1/2)1.5R + (1/2)3.25R = d_1 + 2.375R$, i.e the maximum value is achieved when $\mathcal{F}'_j \cup \mathcal{U}_{j_1}$ and $\mathcal{U}_{j'} \cup \mathcal{U}_{j_2} \cup \mathcal{U}_{j_3}$ are disjoint. Then, we can prove $E \leq e^{-3/4}(4 + 2.375) = 6.375e^{-3/4} < 3.02$.

C.3 The case $0 \leq d_1 \leq 2$

In this case, we use the potential set $\mathcal{F}'' = \mathcal{F}_j \cup \mathcal{U}_{j_1} \cup \mathcal{U}_{j_2} \cup \mathcal{U}_{j_3}$.

We show that the worst instance in this case is give by figure C.3. Notice that we are not using the facilities in $\mathcal{F}_{F_2} \backslash \mathcal{F}_{U_2}$. The expected connection cost for j is upper bounded by

$$\left(1 - \frac{1}{R}\right) \cdot 0 + \frac{1}{R}(1 - e^{-1/R}) \cdot R + \frac{1}{R}e^{-1/R} \cdot \frac{1}{2} \cdot 2R + \frac{1}{R}e^{-1/R} \cdot \frac{1}{2} \cdot 4.5R = 1 + 2.25e^{-1/R}$$

As R tends to ∞, the above upper bound tends to 3.25.

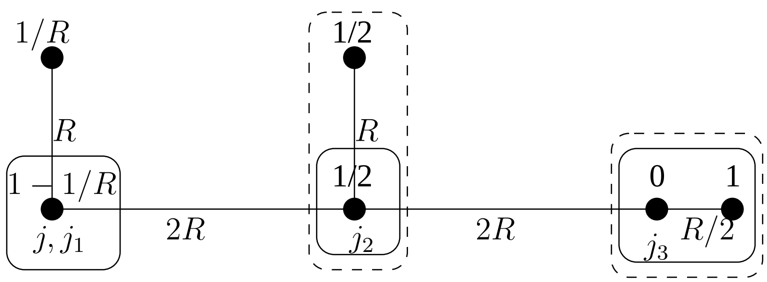

Fig. C.3. The worst instance for $0 \leq d_1 \leq 2$.

The proof is similar to the proof in subsection C.1. We first show that in the worst case, $\mathcal{F}_j \cup \mathcal{U}_{j_1}$ and $\mathcal{U}_{j_2} \cup \mathcal{U}_{j_3}$ are disjoint. Then, under the condition that no facilities are open in $\mathcal{F}_j \cup \mathcal{U}_{j_1}$, the expected connection cost of j is at most $d_1 + 3.25R$.

An atom is a region in the Venn diagram of sets $\mathcal{F}_j, \mathcal{U}_{j_1}, \mathcal{U}_{j_2}, \mathcal{U}_{j_3}$. The partial order p we first consider is $(\mathcal{F}_j, \mathcal{U}_{j_1} \backslash \mathcal{F}_j, \mathcal{U}_{j_2} \backslash \mathcal{F}_j, \mathcal{U}_{j_3} \backslash \mathcal{F}_j)$. Similar to the first case, define E_3 to be the expected value of p under the condition that there is no open atom in $\mathcal{F}_j \cup \mathcal{U}_{j_1} \cup \mathcal{U}_{j_2}$ is open.

Define $\alpha_3 = \mathsf{vol}(\mathcal{F}_j \cap \mathcal{U}_{j_3}), a_3 = d(j_3, \mathcal{F}_j \cap \mathcal{U}_{j_3}), a'_3 = d(j, \mathcal{F}_j \cap \mathcal{U}_{j_3}), \beta_3 = \mathsf{vol}(\mathcal{U}_{j_3} \backslash \mathcal{F}_j), b_3 = d(j_3, \mathcal{U}_{j_3} \backslash \mathcal{F}_j)$.

Lemma 14. *If $\alpha_3 \leq 1/2$ and $a'_3 \leq d_2 + 1.5R'$, then $E_3 \leq d_2 + 2.5R$.*

Proof. Under the condition that there is no open atom in $\mathcal{F}_j \cup \mathcal{U}_{j_1} \cup \mathcal{U}_{j_2}$, the atom $\mathcal{U}_{j_3} \backslash \mathcal{F}_j$ is open. Thus, $E_3 = d(j, \mathcal{U}_{j_3} \backslash \mathcal{F}_j)$.

Let $R'' = R_{j_3}$ and thus $R'' \leq R', d_{av}(j_3) \leq R''/2$, and

$$b_3 \leq \frac{R''/2 - \alpha_3 a_3 - (1-\alpha_3-\beta_3)R''}{\beta_3} \leq \frac{R''/2 - \alpha_3 a_3 - (1-\alpha_3-\beta_3)R'' + (1-\alpha_3-\beta_3)R''}{\beta_3 + (1-\alpha_3-\beta_3)}$$
$$= \frac{R''/2 - \alpha_3 a_3}{1-\alpha_3} \leq \frac{R'/2 - \alpha_3 a_3}{1-\alpha_3}$$

where the second inequality used the fact that $\frac{R''/2-\alpha_3 a_3}{1-\alpha_3} \leq \frac{R''/2}{1-1/2} = R''$

Since $1 - \frac{\alpha_3}{1-\alpha_3} \geq 0, d_3 \leq d_2 + 2R'$ and $d_3 - a_3 \leq a'_3$,

$$E_3 \leq d_3 + \frac{R'/2 - \alpha_3 a_3}{1-\alpha_3} \leq d_3 + (d_2 + 2R' - d_3) + \frac{R'/2 - \alpha_3(a_3 + d_2 + 2R' - d_3)}{1-\alpha_3}$$
$$\leq d_2 + 2R' + \frac{R'/2 - \alpha_3(d_2 + 2R' - a'_3)}{1-\alpha_3} \leq d_2 + 2R' + \frac{R'/2 - \alpha_3 R'/2}{1-\alpha_3} = d_2 + 2.5R'$$

Define $\alpha_2 = \mathsf{vol}(\mathcal{F}_j \cap \mathcal{U}_{j_2}), a_2 = d(j_2, \mathcal{F}_j \cap \mathcal{U}_{j_2}), a'_2 = d(j, \mathcal{F}_j \cap \mathcal{U}_{j_2}), \beta_2 = \mathsf{vol}(\mathcal{U}_{j_2} \backslash \mathcal{F}_j), b_2 = d(j_2, \mathcal{U}_{j_2} \backslash \mathcal{F}_j)$. Let E_2 be the expected value of p under the condition that no atom in $\mathcal{F}_j \cup \mathcal{U}_{j_1}$ is open. We show

Lemma 15. *If $\alpha_3 \leq 1/2, a'_2 \leq d_1 + 1.5R$ and $\alpha_2 \leq 2/7$, then $E_2 \leq d_1 + 3.25R$.*

Proof. We first consider the case $a'_3 > d_2 + 1.5R'$(assuming $\alpha_3 > 0$). Consider the balls $B(j_1, R)$ and $B(j_2, R')$. The two balls are disjoint and each has volume at least $1/2$ and thus the union of the two balls has volume at least 1. Since there are facilities in $\mathcal{F}_j$ that are a'_3 away from j, we have $a'_3 \leq \max\{d_1 + R, d_2 + R'\}$. According to the assumption $a'_3 > d_2 + 1.5R'$, we get $d_2 + 1.5R' < d_1 + R$. In this case $E_2 \leq \max\{d_2 + 1.5R', d_2 + 2R' + 1.5R'\} = d_2 + 3.5R' < d_1 + R + 2R' \leq d_1 + 3R \leq d_1 + 3.25R$.

Now we can assume $a'_3 \leq d_2 + 1.5R'$. By lemma 14, $E_3 \leq d_2 + 2.5R'$. So,

$$E_2 \leq \frac{\beta_2}{1-\alpha_2}\left(d_2 + \frac{R'/2 - \alpha_2 a_2 - (1-\alpha_2-\beta_2)R'}{\beta_2}\right) + \frac{1-\alpha_2-\beta_2}{1-\alpha_2}(d_2 + 2.5R')$$
$$= d_2 + \frac{R'/2 - \alpha_2 a_2}{1-\alpha_2} + \frac{1-\alpha_2-\beta_2}{1-\alpha_2}1.5R' \leq d_2 + \frac{R'/2 - \alpha_2 a_2}{1-\alpha_2} + \frac{(R'/2 - \alpha_2 a_2)/R'}{1-\alpha_2}1.5R'$$
$$= d_2 + 2.5\frac{R'/2 - \alpha_2 a_2}{1-\alpha_2} \leq d_2 + 2.5\frac{R/2 - \alpha_2 a_2}{1-\alpha_2}$$

Since $d_2 \leq d_1 + 2R, 1 - \frac{2.5\alpha_2}{1-\alpha_2} \geq 0$ and $d_2 - a_2 \leq a'_2$, we have

$$E_2 \leq d_2 + (d_1 + 2R - d_2) + 2.5\frac{R/2 - \alpha_2(a_2 + d_1 + 2R - d_2)}{1-\alpha_2}$$
$$\leq d_1 + 2R + 2.5\frac{R/2 - \alpha_2(d_1 + 2R - a'_2)}{1-\alpha_2}$$
$$\leq d_1 + 2R + 2.5R/2 = d_1 + 3.25R$$

At this time, we assume $\alpha_3 \leq 1/2, a'_2 \leq d_1 + 1.5R$ and $\alpha_2 \leq 2/7$. We shall consider the other missing cases later. By lemma 15, we can change p to $(\mathcal{F}_j, \mathcal{U}_{j_1} \backslash \mathcal{F}_j, d_1 + 3.25R)$. We redefine atoms as regions in the Venn diagram of the 2 sets $\mathcal{F}_j, \mathcal{U}_{j_1}$. From now on, we can forget about j_2 and j_3. Let E be the expected value of p.

Define $\alpha = \mathsf{vol}(\mathcal{F}_j \backslash \mathcal{U}_{j_1})$ and $a = d(j, \mathcal{F}_j \backslash \mathcal{U}_{j_1}), \alpha_1 = \mathsf{vol}(\mathcal{F}_j \cap \mathcal{U}_{j_1}), a'_1 = d(j, \mathcal{F}_j \cap \mathcal{U}_{j_1}), a_1 = d(j_1, \mathcal{F}_j \cap \mathcal{U}_{j_1})$. Define $\beta_1 = \mathsf{vol}(\mathcal{U}_{j_1} \backslash \mathcal{F}_j), b_1 = d(j_1, \mathcal{U}_{j_1} \backslash \mathcal{F}_j)$. Define $s = d_{av}(j_1)$. These definitions are the same as the definitions in subsection C.1(see figure C.2).

We prove

Lemma 16. *If* $\alpha_3 \leq 1/2, a'_2 \leq d_1 + 1.5R$ *and* $\alpha_2 \leq 2/7$, $E \leq 3.25$.

Proof. By lemma 15, we can consider the following maximization problem :

PROBLEM(2)

$$\max \quad E = \begin{cases} (1 - e^{-\alpha})a + e^{-\alpha}\alpha_1 a'_1 + T & a \leq a'_1 \\ \alpha_1 a'_1 + (1 - e^{-\alpha})(1 - \alpha_1)a + T & a'_1 \leq a \end{cases}$$

$$T = e^{-\alpha}\beta_1(d_1 + b_1) + e^{-\alpha}(1 - \alpha_1 - \beta_1)(d_1 + 3.25R) = e^{-\alpha}(1 - \alpha_1)d_1 + e^{-\alpha}\left(\beta_1 b_1 + 3.25(1 - \alpha_1 - \beta_1)R\right)$$

s.t.

$$\begin{array}{cccc} \alpha + \alpha_1 = 1 & \alpha a + \alpha_1 a'_1 = 1 & \alpha_1 + \beta_1 \leq 1 & \alpha_1 a_1 + \beta_1 b_1 + (1 - \alpha_1 - \beta_1)R \leq 1 \\ d_1 \leq 2 & a'_1 + a_1 \geq d_1 & b_1 \leq 1.5R & d_1 + a \geq R \end{array}$$

It is not hard to see that $E = \mathbb{E}(\mathsf{val}(p))$. All the constraints are straight forward except the last one : $d_1 + a \geq R$. This is true if we assume facilities in $B(j_1, R)$ are all claimed by j_1 (since otherwise $\mathsf{vol}(\mathcal{U}_{j_1}) = 1$ and we can prove that the expected connection cost of j is at most $d_1 + 1 \leq 3$) and $\mathcal{F}_j \backslash \mathcal{U}_{j_1}$ is not empty (otherwise, we also have $\mathsf{vol}(\mathcal{U}_{j_1}) = 1$).

We shall prove in subsection D.2 that the above optimization problem has value 3.25.

Now we remove the 3 assumptions : $\alpha_3 \leq 1/2, a'_2 \leq d_1 + 1.5R$ and $\alpha_2 \leq 7/2$. We consider the 3 conditions separately.

1. $\alpha_3 > 1/2$. Notice that all we need for lemma 16 is that $E_2 \leq d_1 + 3.25R$. So, we can assume $E_2 > d_1 + 3.25R$. By $\alpha_3 > 1/2$, we get $\alpha_2 < 1/2$ and $d(j, \mathcal{U}_{j_2} \backslash \mathcal{F}_j) \leq d_1 + 2R + \frac{R/2}{1-\alpha_2} \leq d_1 + 3R$. Then since $d_1 + 3.25R < E_2 \leq \max\{d(j, \mathcal{U}_{j_2}), E_3\} \leq \max\{d_1 + 3R, E_3\}$, we have $E_3 > d_1 + 3.25R$. We also have $R \geq 2d_{av}(j_1) \geq 2(1 - d_1)$.
   It is not hard to prove that
   $$E_3 \leq \begin{cases} \frac{1}{\alpha_3} + \frac{R/2}{1-\alpha_3} & 1/2 < \alpha_3 \leq 2/3 \\ \frac{1}{\alpha_3} + 3R - \frac{R}{\alpha_3} & 2/3 < \alpha_3 \leq 1 \end{cases}$$
   For a fixed R, $E_3 \leq \max\{2 + R, 1.5 + 1.5R, 1 + 2R\} = \max\{2 + R, 1 + 2R\}$. If $R \geq 1$, then $E_3 \leq 1 + 2R \leq 3.25R$, contradicting the assumption that $E_3 > d_1 + 3.25R$. So, $R < 1$ and thus $d_1 + 3.25R < 2 + R < 3$. Notice that if we consider the order $(\mathcal{U}_{j_1}, \mathcal{U}_{j_2}, \mathcal{U}_{j_3})$, we can get an upper bound $d_1 + 3.25d_{av}(j_1) \leq d_1 + 1.625R$ for the expected connection cost of j. Now $d_1 + 1.625R \leq d_1 + 3.25R < 3$. So, the expected connection cost of j is at most 3 in this case.
2. $a'_2 \geq d_1 + 1.5R$. In this case, we have $d_{\min}(j_1, \mathcal{F}_{j_1} \backslash \mathcal{F}_j) \geq d_1 + 1.5R - d_1 = 1.5R$, since otherwise some facilities in $\mathcal{F}_{j_1} \backslash \mathcal{F}_j$ should be claimed by j, leading to a contradiction. Thus, $\mathcal{U}_{j_1} \subseteq \mathcal{F}_j$, and $\alpha_1 a_1 + (1 - \alpha_1)1.5R \leq s$, where $s = d_{av}(j_1)$. Since $R \geq 2s$, we have $\alpha_1 \geq 2/3, \alpha \leq 1/3$.
   Since $E_2 \leq \max\{d_1 + 2R + 1.5R, E_3\} \leq d_1 + 4.5R$. We can consider the order $(\mathcal{F}_j, d_1 + 4.5R)$. Then,

If $a \le a'_1$, then

$$\begin{aligned} E &= (1-e^{-\alpha})a + e^{-\alpha}\alpha_1 a'_1 + e^{-\alpha}(1-\alpha_1)(d_1+4.5R) \\ &\le (1-e^{-\alpha})a + e^{-\alpha}\alpha_1 a'_1 + e^{-\alpha}(1-\alpha_1)d_1 + 1.5e^{-\alpha}R \\ &:= (1-e^{-\alpha})a + e^{-\alpha}\alpha_1 a'_1 + T \end{aligned}$$

If $a > a'_1$, then

$$\begin{aligned} E &\le \alpha_1 a'_1 + (1-e^{-\alpha})(1-\alpha_1)a + e^{-\alpha}(1-\alpha_1)d_1 + 1.5e^{-\alpha}R \\ &= \alpha_1 a'_1 + (1-e^{-\alpha})(1-\alpha_1)a + T \end{aligned}$$

We have the following constraints :

$$\begin{array}{cccc} \alpha+\alpha_1 = 1 & \alpha a + \alpha_1 a'_1 = 1 & \alpha_1 a_1 + (1-\alpha_1)1.5R \le 1 & d_1 \le 2 \\ a'_1 + a_1 \ge d_1 & \alpha \le 1/3 & d_1 + a \ge R & \end{array}$$

Let's compare the above maximization problem with problem (2) with restriction $\beta_1 = b_1 = 0$. We can see that the two sets of constraints are almost the same, except that the constraint $\alpha_1 a_1 + (1-\alpha_1)1.5R \le 1$, which is stronger than the correspondent constraint $\alpha_1 a_1 + (1-\alpha_1)R \le 1$ in problem (2), and the above optimization problem has one more constraint $\alpha \le 1/3$. Thus the constraints of the above problem are stronger than that of problem (2).
Let's compare the two objective functions. They are only different in the definition of T. In the above problem, the value of T is $e^{-\alpha}(1-\alpha_1)d_1 + 1.5R$, which is at most the correspondent value $e^{-\alpha}(1-\alpha_1)d_1 + 3.25e^{-\alpha}R$. Thus the above problem has value at most that of problem (2), which is 3.25.

3. $\alpha_2 \ge 2/7, a'_2 < d_1 + 1.5R$. Again, we can assume $E_2 > d_1 + 3.25R$.
If $2/7 \le \alpha_2 \le 1/2$, it's not hard to see that $E_2 \le a'_2 + \frac{1/2}{1-\alpha_2}2.5R$. If $\alpha_1 \ge 0.1$, we consider the order $\left(\mathcal{F}_j \cap \mathcal{U}_{j_1}, \mathcal{U}_{j_1} \backslash \mathcal{F}_j, \mathcal{F}_j \cap \mathcal{U}_{j_2}, a'_2 + \frac{1/2}{1-\alpha_2}2.5R\right)$. Notice that we do not use the facilities in $\mathcal{F}_j \backslash (\mathcal{U}_{j_1} \cup \mathcal{U}_{j_2})$. Define $\beta_1 = \mathsf{vol}(\mathcal{U}_{j_1} \backslash \mathcal{F}_j), b_1 = d(j_1, \mathcal{U}_{j_1} \backslash \mathcal{F}_j)$, then

$$\begin{aligned} E \le &\quad \alpha_1 a'_1 + \beta_1(d_1+b_1) + (1-\alpha_1-\beta_1)\alpha_2 a'_2 + (1-\alpha_1-\beta_1)(1-\alpha_2)\left(a'_2 + \frac{1/2}{1-\alpha_2}2.5R\right) \\ = &\quad \alpha_1 a'_1 + \beta_1(d_1+b_1) + (1-\alpha_1-\beta_1)(a'_2 + 1.25R) \end{aligned}$$

Consider the following optimization problem :

PROBLEM(3) $\quad \max E = \alpha_1 a'_1 + \beta_1(d_1+b_1) + (1-\alpha_1-\beta_1)(a'_2+1.25R) \quad$ s.t.

$$\begin{array}{ccc} \alpha a + \alpha_1 a'_1 + \alpha_2 a'_2 \le 1 & \alpha_1 a_1 + \beta_1 b_1 + (1-\alpha_1-\beta_1)R \le s & a'_2 + \dfrac{1/2}{1-\alpha_2}2.5R \ge d_1 + 3.25R \\ 2/7 \le \alpha_2 \le 1/2 & \alpha+\alpha_1+\alpha_2 = 1 & s \le \min\{R/2, 1\} \end{array}$$

In subsection D.3, we show that $E \le 3.5$, and $E \le 3.21$ if we have $\alpha_1 \ge 0.1$. If $\alpha_1 \le 0.1$, we consider the order $\left(\mathcal{F}_j \backslash (\mathcal{U}_{j_1} \cup \mathcal{U}_{j_2}), \mathcal{F}_j \cap \mathcal{U}_{j_1}, \mathcal{U}_{j_1} \backslash \mathcal{F}_j, \mathcal{F}_j \cap \mathcal{U}_{j_2}, a'_2 + \frac{1/2}{1-\alpha_2}2.5R\right)$. Then,

$$E \le (1-e^{-\alpha})a + e^{-\alpha}\left(\alpha_1 a'_1 + \beta_1(d_1+b_1) + (1-\alpha_1-\beta_1)(a'_2+1.25R)\right) \le (1-e^{-\alpha})a + 3.5e^{-\alpha}$$

We have $\alpha = 1 - \alpha_1 - \alpha_2 \ge 1 - 1/2 - 0.1 = 0.4$ and $a \le 1/\alpha$. So,

$$E \le (1-e^{-\alpha})(1/\alpha) + 3.5e^{-\alpha} \le (1-e^{-0.4})/0.4 + 3.5e^{-0.4} \le 3.18.$$

If $1/2 \le \alpha_2 \le 1$, we have $d_1 + 3.25R < E_2 \le a_2' + \dfrac{(\alpha_2 - 1/2)R}{\alpha_2} + 2.5R$. Thus $d_1 + b_1 \le d_1 + 1.5R \le a_2' + 1.25R$.
Consider the order $p = \left(\mathcal{F}_j \cap \mathcal{U}_{j_1}, \mathcal{U}_{j_1} \backslash \mathcal{F}_j, \mathcal{U}_{j_2} \backslash \mathcal{F}_j, a_2' + \frac{(\alpha_2 - 1/2)R}{\alpha_2} + 2.5R\right)$, we have

$$
\begin{aligned}
E &= \alpha_1 a_1' + \beta_1(d_1 + b_1) + (1 - \alpha_1 - \beta_1)\alpha_2 a_2' + (1 - \alpha_1 - \beta_1)(1 - \alpha_2)\left(a_2' + \frac{(\alpha_2 - 1/2)R}{\alpha_2} + 2.5R\right) \\
&= \alpha_1 a_1' + \beta_1(d_1 + b_1) + (1 - \alpha_1 - \beta_1)\left(a_2' + (1 - \alpha_2)\left(3.5 - \frac{0.5}{\alpha_2}\right)R\right) \\
&\le \alpha_1 a_1' + \beta_1(d_1 + b_1) + (1 - \alpha_1 - \beta_1)\left(a_2' + 1.25R\right) \\
&\le \alpha_1 a_1' + \left(1 - \alpha_1 - \frac{s - \alpha_1 a_1}{R}\right)d_1 + \frac{s - \alpha_1 a_1}{R}(a_2' + 1.25R) \\
&= \alpha_1 a_1' + (1 - \alpha_1)d_1 + \frac{s - \alpha_1 a_1}{R}(a_2' - d_1) + 1.25(s - \alpha_1 a_1) \\
&\le \alpha_1 a_1' + (1/2 - \alpha_1)d_1 + \frac{1}{2}a_2' + 1.25
\end{aligned}
$$

The third inequality used $d_1 + b_1 \le a_2' + 1.25R$ and $\alpha_1 a_1 + \beta_1 b_1 + (1 - \alpha_1 - \beta_1)R \le s$. The right hand side is maximized only if $b_1 = 0$ and $\beta_1 = 1 - \alpha_1 - (s - \alpha_1 a_1)/R$.
Since $\alpha_1 a_1' + \frac{1}{2}a_2' \le \alpha_1 a_1' + \alpha_2 a_2' \le 1$, $(1/2 - \alpha_1)d_1 \le 1$, we have $E \le 1 + 1 + 1.25 \le 3.25$.

D Solving optimization problems

In this section, we solve the optimization problems mentioned before. The domain over which we need to find a maximum point is a closed body Ω, and the function f over Ω is continuous.

The technique we shall use is finding all local maxima using local adjustments. That is, for every $x \in \Omega$ which does not satisfy some condition, we can change it locally to $x' \in \Omega$ such that x' satisfies the condition and $f(x') \ge f(x)$, then we only need to consider the points in Ω that satisfy the condition.

In some cases, we may break the closed body Ω into two bodies according to a given function g : one body Ω_1 with $g(x) \ge 0$ and the other body Ω_2 with $g(x) < 0$. We will find local maxima in Ω_1 and Ω_2 separately. Ω_2 is not closed; it has an open boundary $g(x) < 0$. Since the function f is continuous and the boundary $g(x) = 0$ is already considered in Ω_1, we do not need to consider the local maxima at the boundary $g(x) = 0$ when dealing with Ω_2. If we can apply some local adjustment to move the points arbitrarily close to the boundary $g(x) = 0$, then we can ignore these points. In this case, we say that the local adjustment "hits the open boundary".

All the variables in the following optimization problem are non-negative real numbers.

D.1 Optimization problem (1)

PROBLEM (1) max $E = (1 - e^{-\alpha})a + e^{-\alpha}E_1$, where

$$E_1 = \alpha_1 a_1' + (1 - \alpha_1)d_1 + \beta_1 b_1 + (1 - \alpha_1 - \beta_1)\left((1 - e^{-\gamma})c + 3.25e^{-\gamma}R\right)$$

s.t

$$
\begin{aligned}
\alpha a + \alpha_1 a_1' + (1 - \alpha - \alpha_1)d_1 \le 1 \quad & a_1' \le d_1 \qquad 2 \le d_1 \le 4 \\
\alpha + \alpha_1 \le 1 \quad & s \le R/2 \quad\; a_1 + a_1' \ge d_1 \\
\alpha_1 a_1 + \beta_1 b_1 + \gamma c + (1 - \alpha_1 - \beta_1 - \gamma)1.5R \le s \quad & b_1 \le 1.5R \qquad R \le c \le 1.5R \\
\alpha_1 + \beta_1 + \gamma \le 1 \quad & R \le 16/3 \qquad\quad s \le 1
\end{aligned}
$$

We prove that the above maximization problem has value at most 2.423.

Notice that decreasing a_1 does not change E, since it is independent of a_1. We can decrease a_1 until $d_1 = a_1 + a_1'$, which comes before $a_1 = 0$ due to the constraint $a_1' \leq d_1$. Thus, we can assume $a_1' + a_1 = d_1$. We can also assume $\alpha_1 a_1 + \beta_1 b_1 + \gamma c + (1 - \alpha_1 - \beta_1 - \gamma)1.5R = s$.

We first prove

Lemma 17. *For fixed* $\alpha, \alpha_1, a, a_1, d_1$,

$$E_1 \leq d_1 - \alpha_1 a_1 + (1 - \alpha_1 a_1)\left(1 + 2.25e^{-3(1-\alpha_1 a_1)/16}\right)$$

Proof. We fix $\alpha, \alpha_1, a, a_1, d_1$ and apply local adjustments on the other variables.

If $c > R$, we can decrease c and γ so that $\gamma c + (1 - \alpha - \beta_1 - \gamma)1.5R$ does not change. i.e, we decrease c by ϵ and decrease γ by $\frac{\gamma}{1.5R-c}\epsilon$. The increment of E is $e^{-\alpha}(1 - \alpha_1 - \beta_1)$ times

$$-(1 - e^{-\gamma})\epsilon + e^{-\gamma}\frac{\gamma\epsilon}{1.5R - c}((d_1 + 3.25R) - (d_1 + c)) \geq (e^{-\gamma} - 1 + 4.5\gamma e^{-\gamma})\epsilon \geq 0$$

So, we can decease c to R, which comes before γ becomes 0. Thus,

$$E_1 = d_1 - \alpha_1 a_1 + \beta_1 b_1 + (1 - \alpha_1 - \beta_1)(1 + 2.25e^{-\gamma})R$$

If $b_1 > 0$ and $b \neq R$, we decrease b_1 by ϵ, decrease β_1 by $\beta_1\epsilon/(R - b_1)$ and increase γ by $\beta\epsilon/(R - b_1)$ so that $\beta_1 + \gamma$ and $\beta_1 b_1 + \gamma c$ do not change. (If $b_1 > R$, we increase β_1 and decrease γ). The increment of E_1 will be

$$\begin{aligned}
\mathbf{d}E_1 &= -\beta_1\epsilon - \frac{\beta_1\epsilon}{R - b_1}b_1 + \frac{\beta_1\epsilon}{R - b_1}(1 + 2.25e^{-\gamma})R - \frac{\beta_1\epsilon}{R - b_1}(1 - \alpha_1 - \beta_1)e^{-\gamma}2.25R \\
&= \frac{\beta\epsilon}{R - b_1}\left(-(R - b_1) - b_1 + (1 + 2.25e^{-\gamma})R - 2.25(1 - \alpha_1 - \beta_1)e^{-\gamma}R\right) \\
&= \frac{\beta\epsilon}{R - b_1}2.25(\alpha_1 + \beta_1)e^{-\gamma}R \geq 0
\end{aligned}$$

So, we can decrease b_1 to 0 or R. If $b_1 = R$, we can change decrease β_1 and increase γ so that E_1 will only increase, until $\beta_1 = 0$. In this case, we can assume $b_1 = 0$. So, we always have $b_1 = 0$.

$$E_1 = d_1 - \alpha_1 a_1 + (1 - \alpha_1 - \beta_1)(1 + 2.25e^{-\gamma})R$$

Now, if $\alpha_1 + \beta_1 + \gamma < 1$, decrease β_1 by ϵ, increase γ by 3ϵ, so that $\gamma R + (1 - \alpha_1 - \beta_1 - \gamma)1.5R$ does not change. The increment of E_1 will be

$$\mathbf{d}E_1 = \left(1 + 2.25e^{-\gamma}\right)R\epsilon - (1 - \alpha_1 - \beta_1)(2.25e^{-\gamma}R \times 3\epsilon) = (1 + 2.25e^{-\gamma} - 6.75(1 - \alpha_1 - \beta_1)e^{-\gamma})R\epsilon$$

Since $\alpha_1 a_1 + \gamma R + (1 - \alpha_1 - \beta_1 - \gamma)1.5R = s \leq R/2$, we have $\alpha_1 + \beta_1 \geq 1/2, \alpha_1 + \beta_1 + \gamma \geq 2/3$. Denoting $z = \alpha_1 + \beta_1$, then $1/2 \leq z \leq 1$ and $\gamma \geq 2/3 - z$.

$$\mathbf{d}E_1 = \left(1 + 2.25e^{-\gamma} - 6.75(1 - z)e^{-\gamma}\right)R\epsilon = (1 - 4.5e^{-\gamma} + 6.75ze^{-\gamma})$$

If $z \geq 2/3$, then $\mathbf{d}E_1 \geq 0$; otherwise,

$$\begin{aligned}
\mathbf{d}E_1 &\geq (1 - 2.25(2 - 3z)e^{-(2/3-z)})R\epsilon \geq \left(1 - 2.25 \times (2 - 3 \times 1/2)e^{1/2-2/3}\right)R\epsilon \\
&= (1 - 1.125e^{-1/6})R\epsilon \geq 0
\end{aligned}$$

Thus, the above operation can only increase E_1, even if we don't have $\beta \geq 0$. We now remove the condition that $\beta \geq 0$ and apply the operation until $\alpha_1 + \beta_1 + \gamma = 1$. So, we have $\alpha_1 + \beta_1 + \gamma = 1$ and $\alpha_1 a_1 + \gamma R = s, s \leq 1$. Then, $\gamma = (s - \alpha_1 a_1)/R$ and

$$E_1 = d_1 - \alpha_1 a_1 - \gamma R(1 + 2.25e^{-(s-\alpha_1 a_1)/R}) \leq d_1 - \alpha_1 a_1 + (s - \alpha_1 a_1)(1 + 2.25e^{-3(s-\alpha_1 a_1)/16})$$

Let $z = s - \alpha_1 a_1$ and thus $0 \leq z \leq 1$. $z(1 + 2.25e^{-3z/16})$ is an increasing function of z; indeed, the derivative of the function to z is $(1 + 2.25e^{-3z/16}) - 2.25ze^{-3z/16} \times 3/16 \geq 0$. Thus

$$E_1 \leq d_1 - \alpha_1 a_1 + (1 - \alpha_1 a_1)(1 + 2.25e^{-3(1-\alpha_1 a_1)/16})$$

This concludes the proof.

It suffices to solve the following optimization problem :

$$\max E = (1 - e^{-\alpha})a + e^{-\alpha}\left(d_1 - \alpha_1 a_1 + (1 - \alpha_1 a_1)\left(1 + 2.25e^{-3(1-\alpha_1 a_1)/16}\right)\right) \quad \text{s.t.}$$

$$\alpha a - \alpha_1 a_1 + (1-\alpha)d_1 \leq 1 \qquad 2 \leq d_1 \leq 4 \qquad \alpha + \alpha_1 \leq 1 \qquad a_1 \leq d_1$$

where the first constraint is from $\alpha a + \alpha_1 a_1' + (1 - \alpha - \alpha_1)d_1 \leq 1$ and $a_1' = d_1 - a_1$.

We can increase a until $\alpha a - \alpha_1 a_1 + (1-\alpha)d_1 = 1$.

If we decrease a by $(1-\alpha)\epsilon$ and increase d by $\alpha\epsilon$, the increment of E is

$$\mathbf{d}E = -(1 - e^{-\alpha})(1-\alpha)\epsilon + e^{-\alpha}\alpha\epsilon = (e^{-\alpha} + \alpha - 1)\epsilon \geq 0$$

We can apply the operation until $a = 0$ or $d_1 = 4$. We have two cases :

1. $a = 0$.
   In this case, $a_1 = ((1-\alpha)d_1 - 1)/\alpha_1$ and thus $1 - \alpha_1 a_1 = 2 - (1-\alpha)d_1$. So,

   $$E = e^{-\alpha}(1 + \alpha d_1) + e^{-\alpha}(2 - (1-\alpha)d_1)(1 + 2.25e^{-3(2-(1-\alpha)d_1)/16})$$

   subject to $(1-\alpha)d_1 \geq 1, 2 \leq d_1 \leq 4, \alpha \leq 1, (1 - \alpha - \alpha_1)d_1 \leq 1$.
   If we increase α by ϵ, the increment of E will be

   $$\begin{aligned}
   \mathbf{d}E &= -e^{-\alpha}\epsilon\left(1 + \alpha d_1 + (2 - (1-\alpha)d_1)\left(1 + 2.25e^{-3(2-(1-\alpha)d_1)/16}\right)\right) + e^{-\alpha}d_1\epsilon \\
   &\quad + e^{-\alpha}\left(d_1\epsilon\left(1 + 2.25e^{-3(2-(1-\alpha)d_1)/16}\right) - (2 - (1-\alpha)d_1)(27/64)e^{-3(2-(1-\alpha)d_1)/16}\epsilon\right) \\
   &\geq -e^{-\alpha}\epsilon(1 + \alpha d_1 + 1 + 2.25e^{-3/16}) + e^{-\alpha}d_1\epsilon + e^{-\alpha}(d_1\epsilon(1 + 2.25e^{-3/16}) - (27/64)e^{-3/16}\epsilon) \\
   &= e^{-\alpha}\epsilon\left(-2 - 2.25e^{-3/16} + (2 + 2.25e^{-3/16} - \alpha)d_1 - (27/64)e^{-3/16}\right) \\
   &\geq e^{-\alpha}\epsilon\left(-2 - 2.25e^{-3/16} + (2 + 2.25e^{-3/16} - 1) \times 2 - (27/64)e^{-3/16}\right) \geq 1.8e^{-3/16}e^{-\alpha}\epsilon \geq 0
   \end{aligned}$$

   So, we can increase α until $(1-\alpha)d_1 = 1$, i.e $d_1 = 1/(1-\alpha), 1/2 \leq \alpha \leq 3/4$. So,

   $$E = e^{-\alpha}\left(1 + \frac{\alpha}{1-\alpha}\right) + e^{-\alpha}(1 + 2.25e^{-3/16}) = e^{-\alpha}\left(\frac{1}{1-\alpha} + 1 + 2.25e^{-3/16}\right)$$

   $$\frac{\mathbf{d}E}{\mathbf{d}\alpha} = e^{-\alpha}\left(\frac{1}{(1-\alpha)^2} - \frac{1}{1-\alpha} - 1 - 2.25e^{-3/16}\right)$$

   There is a $\alpha \in [1/2, 3/4]$ such that the differential is 0, but it corresponds to a local minimum. So,

   $$E \leq \max\left\{e^{-1/2}\left(\frac{1}{1-1/2} + 1 + 2.25e^{-3/16}\right), e^{-3/4}\left(\frac{1}{1-3/4} + 1 + 2.25e^{-3/16}\right)\right\} < 3.243$$

2. $d_1 = 4$.
   In this case, $a = (1 - 4(1-\alpha) + \alpha_1 a_1)/\alpha$.

$$E = \frac{1-e^{-\alpha}}{\alpha}(1 - 4(1-\alpha) + \alpha_1 a_1) + (4 - \alpha_1 a_1)e^{-\alpha} + e^{-\alpha}(1-\alpha_1 a_1)(1 + 2.25e^{-3(1-\alpha_1 a_1)/16})$$

   subject to $a_1 \le 4, \alpha_1 a_1 \ge (3 - 4\alpha), \alpha + \alpha_1 \le 1$.
   Notice that $a_1\alpha_1$ appears as a whole in the right side hand. Decrease $a_1\alpha_1$ by ϵ, the increment of E will be

$$\begin{aligned}\mathbf{d}E &= -\frac{1-e^{-\alpha}}{\alpha}\epsilon + e^{-\alpha}\epsilon + e^{-\alpha}(1 + 2.25e^{-3(1-\alpha_1 a_1)/16})\epsilon - e^{-\alpha}(1-\alpha_1 a_1)(27/64)e^{-3(1-\alpha_1 a_1)/16} \\ &\ge -\frac{1-e^{-\alpha}}{\alpha}\epsilon + e^{-\alpha}\left(2 + 1.8e^{-3(1-\alpha_1 a_1)/16}\right)\epsilon \ge -\frac{1-e^{-\alpha}}{\alpha}\epsilon + 3e^{-\alpha}\epsilon \\ &= \frac{\epsilon}{\alpha}(3\alpha e^{-\alpha} + e^{-\alpha} - 1) \ge \frac{\epsilon}{\alpha}\min\left\{3 \times 0 \times e^{-0} + e^{-0} - 1, 3e^{-1} + e^{-1} - 1\right\} \ge 0\end{aligned}$$

   So, we can assume $\alpha_1 a_1 = \max\{0, 3 - 4\alpha\}$.
   If $\alpha \ge 3/4$, then $\alpha_1 a_1 = 0$,

$$\begin{aligned}E &= \frac{1-e^{-\alpha}}{\alpha}(4\alpha - 3) + e^{-\alpha}(4 + 1 + 2.25e^{-3/16}) = 4 - 3\frac{1-e^{-\alpha}}{\alpha} + e^{-\alpha}(1 + 2.25e^{-3/16}) \\ &\le 4 - 3\frac{1-e^{-3/4}}{3/4} + e^{-3/4}(1 + 2.25e^{-3/16}) = e^{-3/4}(5 + 2.25e^{-3/16}) < 3.243\end{aligned}$$

   If $\alpha \le 3/4$, then $\alpha_1 a_1 = 3 - 4\alpha$.

$$\begin{aligned}E &= (1+4\alpha)e^{-\alpha} + e^{-\alpha}(4\alpha - 2)(1 + 2.25e^{-3(4\alpha-2)/16}) \\ &\le (1 + 4 \times 3/4)e^{-3/4} + e^{-3/4}(4 \times 3/4 - 2)(1 + 2.25e^{-3(4\times 3/4-2)/16}) = e^{-3/4}(5 + 2.25e^{-3/16}) \\ &< 3.243\end{aligned}$$

D.2 Optimization problem (2)

PROBLEM (2)

$$\max \quad E = \begin{cases} (1-e^{-\alpha})a + e^{-\alpha}\alpha_1 a_1' + T & a \le a_1' \\ \alpha_1 a_1' + (1-e^{-\alpha})(1-\alpha_1)a + T & a_1' \le a \end{cases}$$

$$T = e^{-\alpha}(1-\alpha_1)d_1 + e^{-\alpha}\left(\beta_1 b_1 + 3.25(1 - \alpha_1 - \beta_1)R\right)$$

s.t.

$$\begin{array}{cccc} \alpha + \alpha_1 = 1 & \alpha a + \alpha_1 a_1' = 1 & \alpha_1 + \beta_1 \le 1 & \alpha_1 a_1 + \beta_1 b_1 + (1-\alpha_1-\beta_1)R \le 1 \\ d_1 \le 2 & a_1' + a_1 \ge d_1 & b_1 \le 1.5R & d_1 + a \ge R \end{array}$$

We prove that the above optimization problem has value 3.25.
Notice that

$$E = \min\left\{(1 - e^{-\alpha}a + e^{-\alpha}\alpha_1 a_1', \alpha_1 a_1' + (1-e^{-\alpha})(1-\alpha_1)a\right\} + T$$

That is, when $a \le a_1'$, the first quantity is smaller and when $a > a_1'$, the second quantity is smaller.

W.L.O.G, we can assume $b_1 = 0$. Indeed, if $b_1 \geq R$, we can decrease β to 0 and since $b_1 \leq 1.5R \leq 3.25R$, the value of the objective function can only increase. So, assume $b_1 \leq R$. In this case, we can decrease b_1 by ϵ and β_1 by $\beta_1\epsilon/(R-b_1)$, the increment of E is $e^{-\alpha}$ times

$$-\beta_1\epsilon - \frac{\beta_1\epsilon}{R-b_1}b_1 + \frac{\beta_1\epsilon}{R-b_1}3.25R = \frac{\beta_1\epsilon}{R-b_1}\left(-R+b_1-b_1+3.25R\right) = 2.25\frac{\beta_1\epsilon}{R-b_1}R \geq 0$$

Thus, we can assume $b_1 = 0$. $T = e^{-\alpha}(1-\alpha_1)d_1 + 3.25e^{-\alpha}(1-\alpha_1-\beta_1)R$.

Since $(1-\alpha_1-\beta_1)R$ appears as a whole, we can decrease β_1 and decrease R so that $(1-\alpha_1-\beta_1)$ does not change. The operation does not make any constraint invalid. Thus, we can assume $\beta_1 = 0$.

Up to now, the problem becomes

$$\max \quad E = \begin{cases} (1-e^{-\alpha})a + e^{-\alpha}\alpha_1 a_1' + T & a \leq a_1' \\ \alpha_1 a_1' + (1-e^{-\alpha})(1-\alpha_1)a + T & a_1' \leq a \end{cases}$$

$$T = e^{-\alpha}(1-\alpha_1)d_1 + 3.25e^{-\alpha}(1-\alpha_1)R, \quad \text{s.t.}$$

$$\begin{array}{ccc} \alpha + \alpha_1 = 1 & \alpha a + \alpha_1 a_1' = 1 & \alpha_1 a_1 + (1-\alpha_1)R \leq 1 \\ d_1 \leq 2 & a_1' + a_1 \geq d_1 & d_1 + a \geq R \end{array}$$

We first solve the problem for $d_1 = 2$.

Lemma 18. *If $d_1 = 2$, $E \leq 3.185$.*

Proof. For the above optimization problem, we remove the condition $d_1 + a \geq R$ and relax the objective function to $E = (1-e^{-\alpha})a + e^{-\alpha}\alpha_1 a_1' + T$. The two operations can only increase the value of E, (recall that E was $\min\{(1-e^{-\alpha})a + e^{-\alpha}\alpha_1 a_1', \alpha_1 a_1' + (1-e^{-\alpha})(1-\alpha_1 a_1)a\} + T$). We can increase R until $\alpha_1 a_1 + (1-\alpha_1)R = 1$. Thus $T = 2e^{-\alpha}(1-\alpha_1) + 3.25e^{-\alpha}(1-\alpha_1 a_1)$. We decrease a_1 until $a_1 = \max\{0, 2-a_1'\}$.

1. If $a_1 = 2 - a_1'$.

$$\begin{aligned} E &= (1-e^{-\alpha})a + e^{-\alpha}\alpha_1 a_1' + 2e^{-\alpha}(1-\alpha_1) + 3.25e^{-\alpha}(1-\alpha_1 a_1) \\ &= (1-e^{-\alpha})a + 2e^{-\alpha} - e^{-\alpha}\alpha_1 a_1 + 3.25e^{-\alpha}(1-\alpha_1 a_1) \end{aligned}$$

subject to

$$\alpha + \alpha_1 = 1, \quad \alpha a + \alpha_1(2-a_1) = 1, \quad \alpha_1 a_1 \leq 1, \quad a_1 \leq 2$$

The maximum point falls into 1 of the following 3 cases :

(a) $a = 1/\alpha, a_1 = 2$

$$\begin{aligned} E &= (1-e^{-\alpha})/\alpha + 2e^{-\alpha} - 2\alpha_1 e^{-\alpha} + 3.25e^{-\alpha}(1-2\alpha_1) \\ &= \frac{1-e^{-\alpha}}{\alpha} + e^{-\alpha}(8.5\alpha - 3.25) \leq 1 + 4.25e^{-1} < 2.57 \end{aligned}$$

(b) $a = 0, a_1 = 2 - 1/\alpha_1, \alpha_1 \geq 1/2$

$$E = 2e^{-\alpha} - 2\alpha_1 e^{-\alpha} + e^{-\alpha} + 3.25e^{-\alpha}(2-2\alpha_1) = e^{-\alpha}(1+8.5\alpha) \leq e^{-1/2}(1+8.5\times 1/2) < 3.185$$

(c) $a = (1-2\alpha_1)/\alpha, a_1 = 0, \alpha_1 \leq 1/2$

$$\begin{aligned} E &= (1-e^{-\alpha})(2-1/\alpha) + 2e^{-\alpha} + 3.25e^{-\alpha} = 2 - (1-e^{-\alpha})/\alpha + 3.25e^{-\alpha} \\ &\leq 2 - (1-e^{-1/2})/(1/2) + 3.25e^{-1/2} < 5.25e^{-1/2} < 3.185 \end{aligned}$$

2. $a_1 = 0$ and $a_1' > 2$.

$$E = (1-e^{-\alpha})a + e^{-\alpha}\alpha_1 a_1' + 2e^{-\alpha}(1-\alpha_1) + 3.25e^{-\alpha}$$

subject to

$$\alpha + \alpha_1 = 1, \quad \alpha a + \alpha_1 a_1' = 1, \quad a_1' > 2$$

Increase a by $\alpha_1\epsilon$ and decrease a_1' by $\alpha\epsilon$, the increment of E will be

$$\mathbf{d}E = (1-e^{-\alpha})\alpha_1\epsilon - e^{-\alpha}\alpha_1\alpha\epsilon = \alpha_1\epsilon(1-e^{-\alpha}-e^{-\alpha}\alpha) \geq 0$$

Thus, we can increase a and decrease a_1' until we hit the open boundary $a_1' > 2$.

So, we already considered the case $d_1 = 2$ and we can change the condition $d_1 \leq 2$ to $d_1 < 2$.
The optimization problem now is :

$$\max \quad E = \begin{cases} (1-e^{-\alpha})a + e^{-\alpha}\alpha_1 a_1' + T & a \leq a_1' \\ \alpha_1 a_1' + (1-e^{-\alpha})(1-\alpha_1)a + T & a_1' \leq a \end{cases}$$

$$T = e^{-\alpha}(1-\alpha_1)d_1 + 3.25e^{-\alpha}(1-\alpha_1)R, \quad \text{s.t.}$$

$$\begin{array}{ccc} \alpha + \alpha_1 = 1 & \alpha a + \alpha_1 a_1' = 1 & \alpha_1 a_1 + (1-\alpha_1)R \leq 1 \\ d_1 < 2 & a_1' + a_1 \geq d_1 & d_1 + a \geq R \end{array}$$

We can decrease a_1 until $a_1 = \max\{0, d_1 - a_1\}$. This will not violate any constraint above, and does not change the objective function, since it is independent of a_1. If $a_1 = 0$ and $a_1' > d_1$, we can increase d_1 until $a_1' = d_1$ or we hit the open boundary $d_1 = 2$.

Thus, we can assume we have $a_1 + a_1' = d_1$.

Up to now, the objective function becomes

$$\max \quad E = \begin{cases} (1-e^{-\alpha})a + e^{-\alpha}\alpha_1(d_1 - a_1) + T & a \leq d_1 - a_1 \\ \alpha_1(d_1 - a_1) + (1-e^{-\alpha})(1-\alpha_1)a + T & d_1 - a_1 < a \end{cases}$$

where $T = e^{-\alpha}(1-\alpha_1)d_1 + 3.25e^{-\alpha}(1-\alpha_1)R$, s.t.

$$\alpha + \alpha_1 = 1 \quad \alpha a + \alpha_1(d_1 - a_1) = 1 \quad a_1 \leq d_1 < 2 \quad \alpha_1 a_1 + (1-\alpha_1)R \leq 1 \quad d_1 + a \geq R$$

If $\alpha_1 a_1 + (1-\alpha_1)R < 1$, we can increase d_1, a_1 and R at the same rate. It is easy to see that this can only increase E. We can do this until $\alpha_1 a_1 + (1-\alpha_1)R = 1$ or we hit the open boundary $d = 2$.

So, $R = (1-\alpha_1 a_1)/(1-\alpha_1)$ and $T = e^{-\alpha}(1-\alpha_1)d_1 + 3.25e^{-\alpha}(1-\alpha_1 a_1)$.

Depending the whether $a \leq d_1 - a_1$, we have 2 cases.

1. $a \le d_1 - a_1$. In this case

$$\begin{aligned} E &= (1-e^{-\alpha})a + e^{-\alpha}\alpha_1(d_1 - a_1) + e^{-\alpha}(1-\alpha_1)d_1 + 3.25e^{-\alpha}(1-\alpha_1 a_1) \\ &= (1-e^{-\alpha})a + e^{-\alpha}(d_1 - \alpha_1 a_1) + 3.25e^{-\alpha}(1-\alpha_1 a_1) \end{aligned}$$

Constraints are

$$\alpha + \alpha_1 = 1 \quad \alpha a + \alpha_1(d_1 - a_1) = 1 \quad a + a_1 \le d_1 < 2, \quad \alpha_1 a_1 \le 1, \qquad \frac{1-\alpha_1 a_1}{1-\alpha_1} \le d_1 + a$$

If we decrease d_1 and a_1 by ϵ, the increment of E will be

$$\mathbf{d}E = e^{-\alpha}(-1 + \alpha_1 + 3.25\alpha_1) = e^{-\alpha}(4.25\alpha_1 - 1) = e^{-\alpha}(3.25 - 4.25\alpha)$$

We have two cases :

(a) $\alpha \le 3.25/4.25$. We can decrease d_1 and a_1 at the same rate until $a_1 = 0$ or $\frac{1-\alpha_1 a_1}{1-\alpha_1} = d_1 + a$.

i. $a_1 = 0$. In this case,

$$E = (1-e^{-\alpha})a + e^{-\alpha}d_1 + 3.25e^{-\alpha}$$

subject to

$$\alpha \le 3.25/4.25, \quad \alpha a + (1-\alpha)d_1 = 1, \quad a \le d_1 < 2, \quad 1 \le (d_1 + a)\alpha,$$

The second and the fourth constraints imply $d_1 = 0$ or $\alpha \ge 1/2$. If $d_1 = 0$ then $a = 0$, which contradicts $\alpha a + (1-\alpha)d_1 = 1$. Thus, $\alpha \ge 1/2$.

For a fixed α, E is a linear function of a and d. Since $e^{-\alpha}/(1-\alpha) \ge (1-e^{-\alpha})/\alpha$, we can decrease a and increase d_1 so that $\alpha a + (1-\alpha)d_1$ does not change E only increases. E is maximized when $a = 0$ and $d = 1/(1-\alpha)$. In this case,

$$E = e^{-\alpha}/(1-\alpha) + 3.25e^{-\alpha} \le e^{-1/2}/(1-1/2) + 3.25e^{-1/2} = 5.25e^{-1/2} < 3.185$$

ii. $\frac{1-\alpha_1 a_1}{1-\alpha_1} = d_1 + a, a_1 > 0$. In this case,

$$a = \frac{2-d_1}{2\alpha}, \quad a_1 = d_1 - \frac{d_1}{2(1-\alpha)}, \quad \frac{2-d_1}{2} \le \alpha < 1/2, \quad d_1 < 2$$

$$E = (1-e^{-\alpha})\frac{2-d_1}{2\alpha} + e^{-\alpha}(d_1/2 + \alpha d_1) + 3.25e^{-\alpha}\left(1 - (1/2-\alpha)d_1\right)$$

For a fixed α, we should either maximize d_1 or minimize d_1. Thus, either we hit the open boundary $d_1 = 2$ or $d_1 = 2-2\alpha$. So, we only consider $d_1 = 2-2\alpha$. In this case, $0 < \alpha \le 1/2$, and

$$\begin{aligned} E &= (1-e^{-\alpha}) + e^{-\alpha}(1 - \alpha + 2\alpha - 2\alpha^2) + 3.25e^{-\alpha}(1 - (1/2-\alpha)(2-2\alpha)) \\ &= 1 + e^{-\alpha}\alpha(10.75 - 8.5\alpha) \end{aligned}$$

The maximum is achieved at $\alpha = 1/2$ and $E \le 1 + 3.25e^{-1/2} < 2.972$.

(b) $\alpha \ge 3.25/4.25$. We can increase d_1 to a_1 until we hit the open boundary $d_1 = 2$.

2. $a > d_1 - a_1$

$$E = \alpha_1(d_1 - a_1) + (1 - e^{-\alpha})(1 - \alpha_1)a + e^{-\alpha}(1 - \alpha_1)d_1 + 3.25e^{-\alpha}(1 - \alpha_1 a_1)$$

The constraints are

$$\begin{array}{ccc} \alpha + \alpha_1 = 1, & \alpha a + \alpha_1(d_1 - a_1) = 1, & a_1 \leq d_1 2, \\ \alpha_1 a_1 \leq 1, & \dfrac{1 - \alpha_1 a_1}{1 - \alpha_1} \leq d_1 + a, & a + a_1 > d_1 \end{array}$$

Again, depending on the value of α, we may choose to decrease or increase d_1 and a_1.

(a) $\alpha \leq 3.25/4.25$. We can decrease d_1 and a_1 at the same rate until $a_1 = 0$ or $(1 - \alpha_1 a_1) = (d_1 + a)(1 - \alpha_1)$.

i. $a_1 = 0$. $E = (1 - \alpha)d_1 + (1 - e^{-\alpha})\alpha a + e^{-\alpha}\alpha d_1 + 3.25e^{-\alpha}$ subject to

$$\alpha a + (1 - \alpha)d_1 = 1, \quad d_1 < 2, \quad 1 \leq (d_1 + a)\alpha, \qquad a > d_1, \quad \alpha \leq 3.25/4.25$$

By the first and the third constraint, either $d_1 = 0$ or $\alpha \geq 1/2$.
If $d_1 = 0$, $a = 1/\alpha$, $E = (1 - e^{-\alpha}) + 3.25e^{-\alpha} = 1 + 2.25e^{-\alpha} \leq 3.25$.
Now, consider the case $\alpha \geq 1/2$. The third constraint is always true. For a fixed α, E is a linear function of a and d_1. We should either maximize d or maximize a. If we need to maximize d_1, we will hit the open boundary $a = d_1$. So, suppose we need to maximize a. Thus, we have $d_1 = 0$ and $a = 1/\alpha, 0 \leq \alpha \leq 3.25/4.25$ and $E = 1 - e^{-\alpha} + 3.25e^{-\alpha} = 1 + 2.25e^{-\alpha} \leq 3.25$.

ii. $1 - \alpha_1 a_1 = (d_1 + a)(1 - \alpha_1), a_1 > 0$. In this case,

$$E = \alpha_1(d_1 - a_1) + (1 - e^{-\alpha})(1 - \alpha_1)a + e^{-\alpha}(1 - \alpha_1)d_1 + 3.25e^{-\alpha}(1 - \alpha_1 a_1)$$

The constraints are

$$\begin{array}{ccc} \alpha + \alpha_1 = 1, & \alpha a + \alpha_1(d_1 - a_1) = 1, & 0 < a_1 \leq d_1 < 2 \\ \alpha_1 a_1 \leq 1, & \dfrac{1 - \alpha_1 a_1}{1 - \alpha_1} = d_1 + a, & a + a_1 > d_1 \end{array}$$

So, we have

$$a = \frac{2 - d_1}{2\alpha}, \quad a_1 = d_1 - \frac{d_1}{2(1 - \alpha)}, \quad \alpha < \min\left\{1/2, \frac{2 - d_1}{2}\right\}, \quad d_1 < 2$$

$$E = \frac{d_1}{2} + (1 - e^{-\alpha})\frac{2 - d_1}{2} + e^{-\alpha}\alpha d_1 + 3.25e^{-\alpha}\left(1 - (1/2 - \alpha)d_1\right)$$

For a fixed α, we should either maximize d_1 or minimize d_1. If we want to maximize d_1, we will hit the open boundary $\alpha = (2 - d_1)/2$. If we want to minimize d_1, we will have $d_1 = 0$, in which case $E = 1 - e^{-\alpha} + 3.25e^{-\alpha} = 1 + 2.25e^{-\alpha} \leq 3.25$.

(b) $\alpha > 3.25/4.25$. We can increase d_1 and a_1 until we hit the open boundary $d_1 = 2$.

D.3 Optimization problem 3

PROBLEM (3) max $E = \alpha_1 a_1' + \beta_1(d_1 + b_1) + (1 - \alpha_1 - \beta_1)(a_2' + 1.25R)$ s.t.

$$\alpha a + \alpha_1 a_1' + \alpha_2 a_2' \leq 1 \qquad \alpha_1 a_1 + \beta_1 b_1 + (1-\alpha_1-\beta_1)R \leq s \quad a_2' + \frac{1/2}{1-\alpha_2}2.5R \geq d_1 + 3.25R$$
$$2/7 \leq \alpha_2 \leq 1/2 \qquad \alpha + \alpha_1 + \alpha_2 = 1 \qquad s \leq \min\{R/2, 1\}$$
$$b_1 \leq 1.5R$$

We want to show that $E \leq 3.5$, and $E \leq 3.21$ if $\alpha_1 \geq 0.1$.

$a_2' + 1.25R = a_2' + 2.5R - 1.25R \geq a_2' + \frac{1/2}{1-\alpha_2}2.5R - 1.25R \geq d_1 + 3.25R - 1.25R = d_1 + 2R.$

If $b_1 \geq R$, we can decrease β until $\beta = 0$. If $b_1 < R$, we can decrease b_1 by ϵ and decrease β by $\beta_1\epsilon/(R - b_1)$ until $b_1 = 0$.

Thus, we can assume $b_1 = 0, \beta_1 \geq 1 - \alpha_1 - (s - \alpha_1 a_1)/R$. Then

$$\begin{aligned}
E &\leq \alpha_1 a_1' + (1 - \alpha_1 - (s - \alpha_1 a_1)/R)d_1 + \frac{s - \alpha_1 a_1}{R}(a_2' + 1.25R) \\
&= \alpha_1 a_1' + (1-\alpha_1)d_1 + \frac{s - \alpha_1 a_1}{R}(a_2' - d_1) + 1.25(s - \alpha_1 a_1) \\
&\leq \alpha_1 a_1' + (1-\alpha_1)d_1 + \frac{1}{2}(a_2' - d_1) + 1.25s
\end{aligned}$$

subject to

$$\alpha a + \alpha_1 a_1' + \alpha_2 a_2' \leq 1 \quad a_2' + \frac{1/2}{1-\alpha_2}5s \geq d_1 + 6.5s \quad 2/7 \leq \alpha_2 \leq 1/2 \quad \alpha + \alpha_1 + \alpha_2 = 1 \quad s \leq 1$$

If $\alpha_1 \geq 1/2$, then decrease d_1 to 0,

$$E \leq \alpha_1 a_1' + \frac{1}{2}a_2' + 1.25s \leq \frac{\alpha_1 a_1' + (1/2)a_2'}{\alpha_1 a_1' + \alpha_2 a_2'} + 1.25 \leq \frac{1/2}{2/7} + 1.25 = 3$$

If $\alpha_1 \leq 1/2$, increase d_1 until $a_2' - d_1 = \left(6.5 - \frac{5/2}{1-\alpha_2}\right)s$.

$$E \leq \alpha_1 a_1' + (1-\alpha_1)d_1 + \left(4.5 - \frac{5/4}{1-\alpha_2}\right)s$$

subject to $\alpha_1 a_1' + \alpha_2 d_1 + \alpha_2\left(6.5 - \frac{5/2}{1-\alpha_2}\right)s \leq 1.$

Comparing the ratio of the coefficients, we have

$$\frac{\alpha_1}{\alpha_1} = 1; \frac{1-\alpha_1}{\alpha_2} \leq 3.5; \frac{4.5 - (5/4)/(1-\alpha_2)}{\alpha_2(6.5 - (5/2)(1-\alpha_2))} \leq \frac{2.75}{6/7} < 3.21$$

So $E \leq 3.5$. If $\alpha_1 \geq 0.1$, $\frac{1-\alpha_1}{\alpha_2} \leq 3.15$ and thus $E \leq 3.21$.