TRAFFIC CONGESTION IN EXPANDERS AND (p, δ)–HYPERBOLIC SPACES

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ABSTRACT. In this paper we define the notion of (p, δ)–Gromov hyperbolic space where we relax Gromov *slimness* condition to allow that not all but a positive fraction of all the triangles are δ–slim. Furthermore, we study their traffic congestion under geodesic routing. We also construct a constant degree family of expanders with congestion $\Theta(n^2)$ in contrast to random regular graphs that have congestion $O(n \log^3(n))$ (as shown in [17]).

1. INTRODUCTION

The purpose of this work is to continue the study of traffic congestion under geodesic routing. By geodesic routing we mean that the path chosen to route the traffic between the nodes is the shortest path. We assume there is a pre-defined consistent way to break ties so that the shortest path between any two nodes is uniquely defined. Our set up throughout the paper is the following. Let $\{G_n\}_{n=1}^{\infty}$ be a family of connected graphs where G_n has n nodes. For each pair of nodes in G_n, consider a unit flow of traffic that travels through the shortest path between nodes as we previously discussed. Hence, the total traffic flow in G_n is equal to $n(n-1)/2$. Given a node $v \in G_n$ we define $\mathcal{L}_n(v)$ as the total flow passing through the node v. Let M_n be the maximum vertex flow across the network

$$M_n := \max \Big\{ \mathcal{L}_n(v) \, : \, v \in G_n \Big\}.$$

It is easy to see that for any graph $n - 1 \le M_n \le n(n-1)/2$.

It was observed experimentally in [16], and proved formally in [1, 2], that if the family is Gromov hyperbolic then the maximum vertex congestion scales as $\Theta(n^2)$. More precisely, let $\{G_n\}_{n=1}^{\infty}$ be an increasing sequence of finite simple graphs such that $|G_n| = n$ and consider a traffic flowing in this graphs that is uniform and geodesically routed between all pairs of nodes. If this sequence is uniformly Gromov δ–hyperbolic, for some non–negative δ, then there is a sequence of nodes in $\{x_n\}_{n=1}^{\infty}$ with $x_n \in G_n$ such that the total traffic passing through x_n is greater than cn^2 for some positive constant c independent on n. These highly congested nodes are called the *core* of the graph.

In this work we extend this analysis and study what happens with the traffic congestion if we relax the slimness condition so that not all but a fraction of all the triangles is δ–slim. More precisely, we say that a metric (X, d) is (p, δ)–hyperbolic if for at least a p fraction of the 3–tuples $(u, v, w) \in X \times X \times X$ the geodesic triangle $\triangle_{uvw}$ is δ–slim. The case p equals 1 corresponds to the classic Gromov δ–hyperbolic space. We show that the congestion in these graphs scales as $\Omega(p^2 n^2 / D_n^2)$ where D_n is the diameter of G_n.

Another important family of graphs are expanders. In graph theory, an expander graph is a sparse graph that has strong connectivity properties. Expander constructions have spawned research in pure and applied mathematics, with several applications to complexity theory, design of robust computer networks, and the theory of error-correcting codes. It is well known that random regular graphs are a large family of expanders. We proved in [17] that random d–regular graphs have maximum vertex congestion scaling as $O(n\log^3_{d-1}(n))$. Therefore, it is a natural question to ask if expanders have always low congestion under geodesic routing. In Section 5, we show that this is not the case. More precisely, we construct a family of expanders $\{G_n\}_{n=1}^{\infty}$ with maximum vertex congestion $\Theta(n^2)$.

2. Preliminaries

2.1. **Hyperbolic Metric Spaces.** In this Section we review the notion of Gromov δ–hyperbolic space. There are many equivalent definitions of Gromov hyperbolicity but the one we take as our definition is the property that triangles are *slim.*

Definition 2.1. *Let $\delta > 0$. A geodesic triangle in a metric space X is said to be δ–slim if each of its sides is contained in the δ–neighbourhood of the union of the other two sides. A geodesic space X is said to be δ–hyperbolic if every triangle in X is δ–slim.*

It is easy to see that any tree is 0-hyperbolic. Other examples of hyperbolic spaces include, any finite graph, the fundamental group of a surface of genus greater or equal than 2, the classical hyperbolic space, and any regular tessellation of the hyperbolic space (i.e. infinite planar graphs with uniform degree q and p–gons as faces with $(p-2)(q-2) > 4$).

2.2. **Expanders.** We say that a family of graphs $\{G_n\}_{n=1}^{\infty}$ is a c–expander family if the edge expansion (also isoperimetric number or Cheeger constant) $h(G_n) \geq c$ where

$$h(G_n) = \min\left\{\frac{|\partial S|}{|S|} \;:\; S \subset G_n \text{ with } 1 \leq |S| \leq |G_n|/2\right\}$$

and ∂S is the edge boundary of S, i.e., the set of edges with exactly one endpoint in S.

3. Congestion on (p,δ)–Hyperbolic Graphs

In this Section we generalize the definition of Gromov hyperbolic spaces to include spaces where not all but a fixed proportion of the triangles in the metric space are *slim.* Furthermore we study what are the traffic characteristics in these graphs.

Definition 3.1 ((p,δ)-hyperbolic)**.** *We say that a metric (X,d) is (p,δ)–hyperbolic if for at least a p fraction of the 3–tuples $(u,v,w) \in X \times X \times X$ the geodesic triangle $\triangle_{uvw}$ is δ–slim.*

A classical δ–hyperbolic spaces corresponds to p equals one.

Theorem 3.1. *Let (X,d) be a (p,δ)-hyperbolic metric of size n. Let D be the diameter of the metric and $M = \max\{|B(u,\delta)| : u \in X\}$ be the maximum number of points in a ball of radius δ. Then there exists a point $a \in X$ such that the traffic passing through a is at least $p^2n^2/(D^2M^3)$.*

Before proving the theorem, we prove the following useful lemma.

Lemma 3.2. *Let $G = (U, V, E)$ be a bipartite graph such that $|U| = |V| = n$ and $|E| \geq pn^2$. The edges of G are colored in such a way that every vertex in $U \cup V$ is incident to at most t colors (u is incident to a color c if u is incident to an edge with color c). Then, there must be a color that is used by at least $(pn/t)^2$ edges in E.*

Proof. Define three random variables A, B and C as follows. We randomly select an edge $(u, v) \in E$ and let $A = u, B = v$ and C be the color of the edge (u, v).

Since $h(A) \leq \log n$, $h(B) \leq \log n$ and $h(A, B) \geq \log(pn^2) = 2 \log n - \log(1/p)$, we have $I(A; B) = h(A) + h(B) - h(A, B) \leq \log(1/p)$. ($h$ is the entropy function and $I(A, B)$ is the mutual information between A and B.) Moreover, if we know $A = u$, then there can be at most t possible colors for C. Thus, we have $h(C|A) \leq \log t$. Similarly, $h(C|B) \leq \log t$. Hence,

$$\begin{aligned} h(C|B) &\geq I(C; A|B) = h(A|B) - h(A|C, B) = h(A) - I(A; B) - h(A|C, B) \\ &\geq h(A) - I(A; B) - h(A|C) = I(A; C) - I(A; B). \end{aligned}$$

Thus $h(C) = h(C|A) + I(A; C) \leq h(C|A) + h(C|B) + I(A; B) \leq \log(t^2/p)$.

Notice that $|E| \geq pn^2$. The inequality implies that there must be a color that is used by at least $pn^2/(t^2/p) = (pn/t)^2$ edges. □

Now, we proceed to prove the theorem. For any 3 points $u, v, w \in X$, let c_{uvw} be the barycenter of the triangle $\triangle_{uvw}$. By a simple counting argument, there must be a point $w \in X$ such that for at least p fraction of the ordered pairs $(u, v) \in X \times X$, $\triangle_{uvw}$ is δ–slim. We fix such a point w from now on. Define $G = (U = X, V = X, E)$ as follows. For any two vertices $u \in U, v \in V$, $(u, v) \in E$ iff $\triangle_{uvw}$ is δ–slim. The color of $(u, v) \in E$ is c_{uvw}. Then, $|E| \geq pn^2$. Moreover, if c is the color of (u, v), then c is in the δ-neighbourhood of $[uw]$. Thus, any vertex u can be incident to at most DM colors. By Lemma 3.2, there must be a color c that is used by at least $p^2n^2/(DM)^2$ edges in E. Notice that for each such edge $(u, v) \in E$, c is in the δ-neighbourhood of $[uv]$. Thus, $[uv]$ will contain a vertex in the ball $B(c, \delta)$. Since $|B(c, \delta)| \leq M$, for some vertex $c' \in B(c, \delta)$ and $p^2n^2/(D^2M^3)$ different ordered pairs $(u, v) \in X \times X$, $[uv]$ contains c'.

4. Traffic on Expanders

In this section we construct a constant degree family of expanders with $\Theta(n^2)$ congestion. This result is in contrast to random regular graphs that have congestion $O(n \log^3(n))$ (as shown in [17]).

Theorem 4.1. *There exists a family $\{G_n\}_{n=1}^{\infty}$ of constant–degree expanders with congestion $\Theta(n^2)$.*

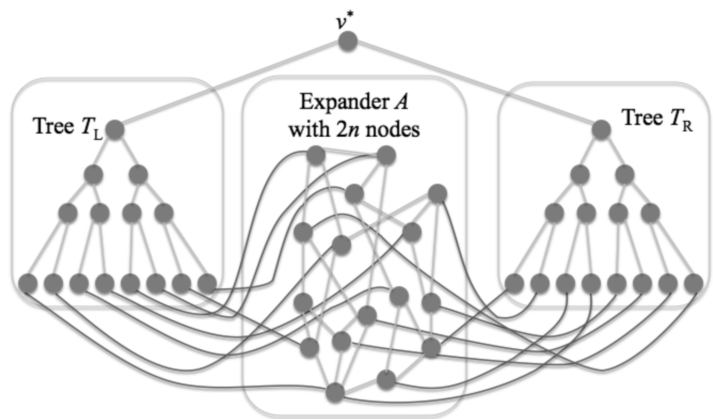

FIGURE 1. Construction of the expander G. There are two trees T_{L} and T_{R}, each with n leaves, a root node v^* and an expander A with $2n$ nodes. We connect v^* to the two roots of T_{L} and T_{R}. We create a random matching between the $2n$ leaves of T_{L} and T_{R} and the $2n$ vertices of A; there is a path of length c connecting each pair in the matching. We also replace edges of A with paths of length c.

4.1. **Construction of the Expander Graph.** For an even integer h, let T be a tree of depth h defined as follows. Each node in depth 0 to $h/2-1$ of T has 3 children, and each node in depth $h/2$ to $h-1$ has 2 children (root has depth 0 and leaves have depth h). Thus, T has exactly $n := 3^{h/2} \times 2^{h/2} = \sqrt{6}^h$ leaves. We use $L(T)$ to denote the set of leaves of T. Define $\lambda(d)$ to be the number of leaves in a sub–tree of T rooted at some vertex of depth $h-d$. Then, we have

$$\lambda(d) = \begin{cases} 2^d & 0 \le d \le h/2 \\ 2^{h/2}3^{d-h/2} & h/2 < d \le h \end{cases}.$$

Construct a graph G as follows. (See Figure 1.) Let A be a degree–3 expander of size $2n$. Let T_{L} and T_{R} be two copies of the tree T. We create a random matching between the $2n$ leaves of T_{L} and T_{R} and the vertices of A. We add a *matching edge* between each matched pair of vertices. We also add an vertex v^* that is connected to the roots of T_{L} and T_{R}. Finally, we scale matching edges and expander edges by a factor of c (i.e., replace those edges with paths of length c) for some constant even number c to be determined later.

Proposition 4.1. *The graph G is an expander of degree at most 4 and size $\Theta(n)$.*

Proof. Since replacing edges with paths of length c only decrease the expansion by a factor of c, we only need to consider the graph obtained before the scaling operation. Let G' be that graph. For the simplicity of the notation, let T' be the tree rooted at v^* with 2 sub–trees T_{L} and T_{R} and $L(T') = L(T_{\mathsf{L}}) \cup L(T_{\mathsf{R}})$. We also use $I(T')$ to denote the set of inner vertices of T'.

Assume that G' is not an 0.01α-expander. Then, let $S \subseteq V(G')$ be a set of size at most $|V(G')|/2$ such that $E_{G'}(S, V(G') \setminus S) < 0.01\alpha\,|S|$.

We notice the following two facts :

(1) If $|S \cap I(T')| \geq |S \cap L(T')| + s$ then there must be at least s edges between $S \cap V(T')$ and $V(T') \setminus S$.
(2) If $\big| |S \cap L(T')| - |S \cap V(A)| \big| \geq s$ then there must be at least s matching edges between S and $V(G') \setminus S$.

By the first fact, we can assume $|S \cap I(T')| \leq 0.6|S|$. Then

$$\left|S \cap L(T')\right| + |S \cap V(A)| \geq 0.4|S|.$$

By the second fact, we have $|S \cap V(A)| \geq 0.15|S|$. If $|S \cap V(A)| \leq |V(A)|/2$ then $E_{G'}(S, V(G') \setminus S) \geq 0.15\alpha|S|$. Thus, we have $|S \cap V(A)| \geq |V(A)|/2$. If $|S \cap V(A)| \leq 0.9\,|V(A)|$ then

$$E_{G'}(S, V(G') \setminus S) \geq \alpha\,|V(A) \setminus S| \geq \frac{0.1}{0.9}\alpha\,|S \cap V(A)| \geq \alpha|S|/60.$$

Thus, we have $|S \cap V(A)| \geq 0.9\,|V(A)|$, which implies $|S \cap L(T')| \geq 0.9\,|V(A)| - 0.1|S|$ by the second fact. Then $|S| \geq 1.8\,|V(A)| - 0.1|S|$, implying that

$$|S| \geq 1.6\,|V(A)| \geq \frac{1.6}{3}\left|V(G')\right| > 0.5\left|V(G')\right|,$$

a contradiction. □

4.2. **Proof of the High Congestion in G.** Consider the set $L(T_{\mathsf{L}}) \times L(T_{\mathsf{R}})$ of n^2 pairs. We shall show that for a constant fraction of pairs (u, v) in this set, the shortest path connecting u and v will contain v^*. It is easy to see that for every $(u, v) \in L(T_{\mathsf{L}}) \times L(T_{\mathsf{R}})$, there is a path of length $2h + 2$ connecting u and v that goes through v^*.

Focus on the graph $G \setminus v^*$. We are interested in the number of pairs $(u, v) \in L(T_{\mathsf{L}}) \times L(T_{\mathsf{R}})$ such that $d_{G \setminus v^*}(u, v) \leq 2h + 2$. Fix a vertex $u \in L(T_{\mathsf{L}})$ from now on. Consider the set $\mathcal{P}$ of simple paths starting at u and ending at $L(T_{\mathsf{R}})$. We say a path $P \in \mathcal{P}$ has rank r if it enters and leaves the expander r times. Notice that we always have $r \geq 1$, since we must use the expander A from u to $L(T_{\mathsf{R}})$.

For a path P of rank r, we define the *pattern* of P, denoted by $\mathsf{ptn}(P)$, as a sequence $t = (t_1, t_2, \cdots, t_{2r+1})$ of $2r + 1$ non-negative integers as follows. For $1 \leq i \leq r$, ct_{2i} is the length of the sub-path of P correspondent to the i-th traversal of P in the expander A. (Recall that we replaced each expander edge with a path of length c.) The path P will contain $r + 1$ sub-paths in the two copies of T (the first and/or the last sub-path might have length 0). Let $2t_{2i-1}$ be the length of the i-th sub-path in the tree. Notice that P can only enter and leave the trees through leaves and thus the lengths of those sub-paths must be even. If some path $P \in \mathcal{P}$ has $\mathsf{ptn}(P) = (t_1, t_2, \cdots, t_{2r+1})$, then the length of P is exactly $\mathsf{len}(t_{[2r+1]}) := 2\sum_{i=0}^{r} t_{2i+1} + c\sum_{i=1}^{r}(t_{2i} + 2)$, where $t_{[2r+1]}$ denotes the sequence $(t_1, t_2, \cdots, t_{2r+1})$. Notice that the $c(t_{2i} + 2)$ term in the definition of len includes the $2c$ edges replaced by the 2 matching edges through which the path enters and leaves the expander.

We call a sequence $(t_1, t_2, \cdots, t_{2r+1})$ of non-negative integers *a valid pattern* of rank r if $\mathsf{len}(t_{[2r+1]}) \leq 2h + 2$.

Lemma 4.2. *The number of paths $P \in \mathcal{P}$ with $\mathsf{ptn}(P) = t_{[2r+1]}$ is at most*

$$\left(\frac{3}{2}\right)^r \left[\prod_{i=1}^{r} \lambda(t_{2i-1}) 2^{t_{2i}}\right] \lambda(t_{2r+1}).$$

Proof. For any leaf v in T, we have at most $\lambda(t)$ possible simple paths of length $2t$ in T that start at v and end at $L(T)$. For a degree-3 graph(in particular, a degree-3 expander), we have at most $\frac{3}{2} \times 2^t$ simple paths that start at any fixed vertex. From a leaf in T_{L} or T_{R}, we only have one way to enter the expander. Similarly, we only have one way to leave the expander from a vertex in the expander. Thus, the total number of simple paths of pattern $t_{[2r+1]}$ is at most

$$\lambda(t_1)\left(\frac{3}{2}2^{t_2}\right)\lambda(t_3)\left(\frac{3}{2}2^{t_4}\right)\cdots\lambda(t_{2r+1}) = \left(\frac{3}{2}\right)^r \left[\prod_{i=1}^{r} \lambda(t_{2i-1}) 2^{t_{2i}}\right] \lambda(t_{2r+1}).$$

□

We fix the rank $r \geq 1$ from now on. For some integer $\ell \in [0, r]$, suppose $t_{[2\ell]}$ is a prefix of some valid pattern of rank r. Define $W(t_{[2\ell]})$ to be the maximum $t_{2\ell+1}$ such that $t_{[2\ell+1]}$ is a prefix of some valid pattern of rank r. That is,

$$W\left(t_{[2\ell]}\right) = \left\lfloor \frac{2h + 2 - 2\sum_{i=1}^{\ell} t_{2i-1} - c\sum_{i=1}^{\ell}(t_{2i}+2) - 2c(r-\ell)}{2} \right\rfloor,$$

Similarly, we define

$$W\left(t_{[2\ell-1]}\right) = \left\lfloor \frac{2h + 2 - 2\sum_{i=1}^{\ell} t_{2i-1} - c\sum_{i=1}^{\ell-1}(t_{2i}+2) - 2c(r-\ell)}{c} \right\rfloor - 2.$$

For some prefix $t_{[2\ell]}$ of some valid pattern of rank r, let $\mathcal{P}_{t_{[2\ell]}}$ be the set of paths $P \in \mathcal{P}$ of rank r such that $t_{[2\ell]}$ is a prefix of $\mathsf{ptn}(P)$. We prove that

Lemma 4.3. *For any $0 \leq \ell \leq r$, we have*

$$\left|\mathcal{P}_{t_{[2\ell]}}\right| \leq 2C^{r-\ell}\left(\frac{3}{2}\right)^{\ell} \prod_{i=1}^{\ell}\left[\lambda(t_{2i-1})2^{t_{2i}}\right] \times \sqrt{6}^{W\left(t_{[2\ell]}\right)}$$

for some large enough constant C.

Proof. For $\ell = r$, Lemma 5.2 implies

$$\begin{aligned}\left|\mathcal{P}_{t_{[2r]}}\right| &\le \sum_{t_{2r+1}=0}^{W\left(t_{[2r]}\right)} \left(\frac{3}{2}\right)^r \prod_{i=1}^{r} \left[\lambda(t_{2i-1})2^{t_{2i}}\right] \lambda(t_{2r+1}) \\ &= \left(\frac{3}{2}\right)^r \prod_{i=1}^{r} \left[\lambda(t_{2i-1})2^{t_{2i}}\right] \sum_{t_{2r+1}=0}^{W\left(t_{[2r]}\right)} \lambda(t_{2r+1}) \\ &\le 2\left(\frac{3}{2}\right)^r \prod_{i=1}^{r} \left[\lambda(t_{2i-1})2^{t_{2i}}\right] \times \sqrt{6}^{W\left(t_{[2r]}\right)}.\end{aligned}$$

The last inequality holds since $\sum_{t_{2r+1}=0}^{W\left(t_{[2r]}\right)} \lambda(t_{2r+1}) \le 2\lambda\left(W\left(t_{[2r]}\right)\right) \le 2\sqrt{6}^{W\left(t_{[2r]}\right)}$.

Now, suppose the lemma holds for some $1 \le \ell \le r$ and we shall prove that it holds for $\ell - 1$. By the induction hypothesis, we have

$$\begin{aligned}\left|\mathcal{P}_{t_{[2\ell-2]}}\right| &\le \sum_{t_{2\ell-1}=0}^{W\left(t_{[2\ell-2]}\right)} \sum_{t_{2\ell}=0}^{W\left(t_{[2\ell-1]}\right)} 2C^{r-\ell}\left(\frac{3}{2}\right)^\ell \prod_{i=1}^{\ell} \left[\lambda(t_{2i-1})2^{t_{2i}}\right] \times \sqrt{6}^{W\left(t_{[2\ell]}\right)} \\ &= 2C^{r-\ell}\left(\frac{3}{2}\right)^\ell \prod_{i=1}^{\ell-1} \left[\lambda(t_{2i-1})2^{t_{2i}}\right] \sum_{t_{2\ell-1}=0}^{W\left(t_{[2\ell-2]}\right)} \lambda(t_{2\ell-1}) \sum_{t_{2\ell}=0}^{W\left(t_{[2\ell-1]}\right)} 2^{t_{2\ell}}\sqrt{6}^{W\left(t_{[2\ell]}\right)}.\end{aligned}$$

It is sufficient to prove that $\sum_{t_{2\ell-1}=0}^{W\left(t_{[2\ell-2]}\right)} \lambda(t_{2\ell-1}) \sum_{t_{2\ell}=0}^{W\left(t_{[2\ell-1]}\right)} 2^{t_{2\ell}}\sqrt{6}^{W\left(t_{[2\ell]}\right)} \le \frac{2C}{3}\sqrt{6}^{W\left(t_{[2\ell-2]}\right)}$.

$$\text{LHS} \le \frac{1}{1-2/\sqrt{6}^{c/2}} \sum_{t_{2\ell-1}=0}^{W\left(t_{[2\ell-2]}\right)} \lambda(t_{2\ell-1})\sqrt{6}^{W\left(t_{[2\ell-1]}\bowtie(0)\right)} \tag{4.1}$$

$$\le \frac{1}{1-2/\sqrt{6}^{c/2}} \left(\frac{\sqrt{6}^{W\left(t_{[2\ell-2]}\bowtie(0,0)\right)}}{1-\sqrt{2/3}} + \frac{\lambda\left(W\left(t_{[2\ell-2]}\right)\right)}{1-\sqrt{2/3}}\right) \tag{4.2}$$

$$\le \frac{1}{1-2/\sqrt{6}^{c/2}} \frac{2}{1-\sqrt{2/3}}\sqrt{6}^{W\left(t_{[2\ell-2]}\right)}, \tag{4.3}$$

where $t_{[2\ell-1]} \bowtie (1)$ denotes the sequence obtained by concatenating the two sequences $t_{[2\ell-1]}$ and (1).

We explain Inequalities (5.1),(5.2) and (5.3) one by one. Focus on the term $Q = 2^{t_{2\ell}}\sqrt{6}^{W\left(t_{[2\ell]}\right)}$. If we increase $t_{2\ell}$ by 1, then $W\left(t_{[2\ell]}\right)$ will decrease by exactly $c/2$, by the definition of W. (We assumed c is an even number.) Thus, Q will decrease by a factor of $\sqrt{6}^{c/2}/2$. By the rule of the geometric sum,

$$\sum_{t_{2\ell}=0}^{W\left(t_{[2\ell-1]}\right)} Q \le \frac{Q|_{t_{2\ell}=0}}{1-2/\sqrt{6}^{c/2}} = \frac{\sqrt{6}^{W\left(t_{[2\ell-1]}\bowtie(0)\right)}}{1-2/\sqrt{6}^{c/2}},$$

implying Inequality (5.1).

Now focus on the term $Q = \lambda(t_{2\ell-1})\sqrt{6}^{W\left(t_{[2\ell-1]}\bowtie(0)\right)}$. If we increase $t_{2\ell-1}$ by 1, then $W\left(t_{[2\ell-1]} \bowtie (0)\right)$ will decrease by 1. Then, Q will either decrease by a factor of $\sqrt{6}/2 = \sqrt{3/2}$, or increase by a factor of $3/\sqrt{6} = \sqrt{3/2}$, depending on whether $t_{2\ell-1} \leq h/2$. We can split the sum $\sum_{t_{2\ell-1}=0}^{W(t_{[2\ell-2]})} Q$ into 2 sums at the point $h/2$ if necessary. Again, using the geometric sum, we have

$$\sum_{t_{2\ell-1}=0}^{W(t_{[2\ell-2]})} Q \leq \frac{Q|_{t_{2\ell-1}=0}}{1-\sqrt{2/3}} + \frac{Q|_{t_{2\ell-1}=W(t_{[2\ell-2]})}}{1-\sqrt{2/3}},$$

implying Inequality (5.2).

Inequality (5.3) follows from the fact that $W\left(t_{[2\ell-2]} \bowtie (0,0)\right) = W\left(t_{[2\ell-2]}\right)$ and $\lambda(t) \leq \sqrt{6}^t$ for any $t \in [0,h]$.

This finishes the proof if we let $C = \dfrac{3}{\left(1-2/\sqrt{6}^{c/2}\right)\left(1-\sqrt{2/3}\right)}$. □

Notice that $W(\emptyset) = \left\lfloor \frac{2h+2-2rc}{2} \right\rfloor = h+1-rc$ for an even integer c. Thus,

$$|\mathcal{P}_\emptyset| \leq 2C^r\sqrt{6}^{h+1-rc} = 2\sqrt{6}\left(\frac{C}{\sqrt{6}^c}\right)^r \sqrt{6}^h = 2\sqrt{6}\left(\frac{C}{\sqrt{6}^c}\right)^r n.$$

$\mathcal{P}_\emptyset$ is essentially the set of paths in $\mathcal{P}$ with rank r. For a large enough c, we have $C < \sqrt{6}^c$. Summing up over all $r \geq 1$, we have that the total number of paths in $\mathcal{P}$ is at most

$$\frac{2\sqrt{6}C/\sqrt{6}^c}{1-C/\sqrt{6}^c}n.$$

For a large enough constant c (say, $c = 6$), the number will be at most $n/2$.

Then, consider the graph G. We know there is a path of length $2h+2$ from u to v via v^* for any vertex $v \in L(T_\mathsf{R})$. Thus, for a fixed vertex $u \in L(T_\mathsf{L})$, there are at least $n/2$ vertices $v \in L(T_\mathsf{R})$ such that the minimum length between u and v contains v^*. Therefore, there are $n^2/2$ pairs (u,v) such that the shortest path between u and v contain v^*.

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