Week 5, Spr 2009

Bypass operation on node j:

\[
\begin{array}{c}
\text{from} \\
1 \xrightarrow{r_1} j \xrightarrow{r_2} k \\
\end{array}
\]

\[
\begin{array}{c}
\text{to} \\
0 \xrightarrow{r^*_2} k \\
\end{array}
\]

For each arc into or out of \( j \):

\[
\begin{array}{c}
j = 1 \\
1 \text{ to } n \text{ do} \\
\text{for every pair of states } i, k \in G, \\
\text{if there is an arrow from } i \text{ to } j \text{ and there is an arrow from } j \text{ to } k, \\
\text{perform a bypass operation to remove state } j. \\
\end{array}
\]

Then, perform a union operation.

When we are done, we have \( S \xrightarrow{r} F \).

Thus, every regular language is described by a regular expression.
Pumping lemma: If \( A \) is a regular language, then there is a number \( p \) (pumping length) so that for every string \( s \in A \) of length \( \geq p \), \( s \) is a concatenation of strings of the form 
\[
s = xyz
\]
such that
1. for each \( i \geq 0 \) \( xy^i z \in A \),
2. \( |y| > 0 \), and
3. \( |xy| < p \).

If a regular language has a long string \( s \), then it has infinitely many long strings \( xy^i z \).
Application
Let $B = \{0^n1^n | n \geq 0\}$

We claim $B$ is not regular. Suppose $B$ is regular. Then, the pumping lemma applies. Let $p$ be the pumping length. Select

$s = 0^p1^p$.

$s \in B$ and $|s| \geq p$. So

$s = xyz$,

where

$|y| \neq 0$ and $xy^2z \notin B$.

We show that the result of the pumping lemma is a contradiction. Since $|xy| \leq p$, $xy$ consists only of 0's. Thus, the string $xy^2z$ has more 0's than 1's. This is a contradiction.
In general, think of applying the pumping lemma as an "adversary argument."

1. You wish to prove that $L$ is not regular. Adversary claims $L$ is regular.

2. Adversary selects $p$. You may not use $p$ to your convenience.

3. You choose a string $s$ in $L$. Your choice depends on $p$.

4. Adversary selects $x, y, z$, and $p$ such that $s = xyz$ and subject to the constraints $|xy| \leq p$ and $|y| \geq 1$.

5. You achieve a contradiction to the pumping lemma by showing for $x, y, z$ picked by the adversary that some word $xy^2z \notin L$. 
Example: 
\[ L = \{ 1^2, 1121 \} \] is not regular.

Assume \( L \) is regular. 
Let \( p \) be constant in pumping lemma. 
Select 
\[ s = 1p^2. \]
Let \( s = xy^i z \), where \( 1 \leq i \leq p \) and \( 1 \leq |y| \leq 1. \)
Let \( i = 2. \)
\[ 1p^2 = xy^2 z \leq p^2. \]
Thus, \( 1p^2 \) is not perfect square. 
So, \( xy^2 z \notin L. \)
Thus, \( L \) is not regular.
Proof of the pumping lemma:
(P.S. This one you need to learn!)

Let \( M = (Q, \Sigma, \delta, q_0, F) \) be a DFA that recognizes \( A \).
Let \( p = \|Q\| \) be the number of states.
Let \( s = a_1 \ldots a_n \) be a string in \( A \) of length \( n \), \( n \geq p \).

\[ q_i = \delta(q_0, a_1 \ldots a_i) \]

It is not possible for all of the \( n+1 \) states
\[ q_0, q_1, \ldots, q_n \]
to be distinct because there are only \( p < n+1 \) states. Thus, there are two integers \( j \) and \( k \), \( 0 \leq j < k \leq p \) such that \( q_j = q_k \).

Since \( j < k \), the string \( a_{j+1} \ldots a_k \) has length
at least one. Let \( y = a_{j+1} \ldots a_k \).

Let \( x = a_1 \ldots a_j \). Since \( k \leq p \),
\[ |xy| \leq p. \]
Since $s \in L$, $q_n \in F$.

Let $z = a_{i+1} \ldots a_n$.

Observe that the path without going through the loop accepts $xz$.

Go around the loop $i$ times to accept $x y^i z$, with $i \geq 1$. 
Closure properties are useful.

**Example.** Let

\[ L = \{ w \in \{0,1\}^* \mid w \text{ has an} \]

\[ \text{equal number of 0's and 1's} \]

If \( L \) is regular, then

\[ L \cap \{0^n1^n \mid n \geq 0\} \]

is regular.

But \( \{0^n1^n \mid n \geq 0\} \) is not regular.

So, \( L \) is not regular.