We’ve done

- Growth of functions
- Asymptotic Notations ($O$, $o$, $\Omega$, $\omega$, $\Theta$)

Now

- Recurrence relations, solving them, Master theorem

Next

- Sorting
Recurrence Relations: Examples

FibA

\[ T(n) = T(n - 1) + T(n - 2) + \Theta(1) \]

Binary search

\[ T(n) \leq T([n/2]) + \Theta(1) \]

Merge sort

\[ T(n) = T([n/2]) + T([n/2]) + \Theta(n) \quad (1) \]

and tons of others

\[ T(n) = 4T(n/2) + n^2 \lg n \]
\[ T(n) = 3T(n/4) + \lg n \]
\[ T(n) = T(n/a) + T(a) \]

As we’ve done earlier, equations like (1) means:

“\( T(n) \) is \( T([n/2]) + T([n/2]) \) plus some function \( f(n) \) which is \( \Theta(n) \)”
Recurrent Relations: Methods

- Substitution method
- Recurrence tree
- Generating functions and others (take CSE 594 next sem)
- Master Theorem
The Substitution Method

Mostly to show $O$ and $\Omega$ relations

- Guess a solution
- Use induction to show it works

Recall the first lecture

$$T(n) = \begin{cases} 
  a & \text{if } n \leq 1 \\
  T(n-1) + T(n-2) + b & \text{if } n \geq 2 
\end{cases}$$

then we “guess”

$$T(n) = (a + b)F_{n+1} - b$$  \hfill (2)

where $F_n$ is the $n$th Fibonacci number

$$F_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n$$

$$= \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n = \Theta(\phi^n)$$

Hence (why?),

$$T(n) = \Theta(\phi^n)$$  \hfill (3)

We can show both (2) & (3) by induction.
More examples

- Given

\[
T(1) = \Theta(1)
\]
\[
T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + \Theta(n),
\]

which means

\[
T(1) = c_0
\]
\[
T(n) \geq T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + c_1n
\]
\[
T(n) \leq T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + c_2n
\]

- Guess: \( T(n) = \Theta(n \lg n). \)

- Need to show by induction that there are \( a, b > 0 \) s.t.

\[
an \lg n \leq T(n) \leq bn \lg n.
\]

Now try

- \( T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1. \)

- Guess \( T(n) = \Theta(n). \)
Misc. Notes

1. It is often safe to ignore integrality issues. For example:

\[ T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + n, \]

can be solved by saying

\[ T(n) = 2T(n/2) + n \]

which gives the exact same solution.

2. To see why integrality isn’t important, consider

\[ T(n) = 2T(\lfloor n/2 \rfloor + 17) + n. \]

It is still \( O(n \lg n) \). (Induction)

3. Avoid doing

\[ T(n) \leq 2c\lceil n/2 \rceil + n \]
\[ \leq cn + n \]
\[ = O(n) \]

which is dead wrong!!
Change of variable

Solve

\[ T(n) = 2T(\sqrt{n}) + 1 \]

Let \( m = \lg n \), then

\[ T(2^m) = 2T(2^{m/2}) + 1 \]

Let \( S(m) = T(2^m) \), then

\[ S(m) = 2S(m/2) + 1 \]

Hence,

\[ S(m) = O(m) \]

Thus,

\[ T(n) = S(\lg n) = O(\lg n) \]
How to guess?

Iterating the recurrence a few times

Like we did with the Fibonacci case.

Use recursion-tree method

Let’s try

\[ T(n) = 3T(\lfloor n/4 \rfloor) + \Theta(n^2). \]

Recursion tree suggests \( T(n) = O(n^2) \). Now rigorously shows it by induction.

Sick of induction already?

Now try

\[ T(n) = T(n/3) + T(2n/3) + O(n) \]
Master Theorem

Let $a \geq 1$, $b > 1$ be constants. Suppose

$$T(n) = aT(n/b) + f(n),$$

where $n/b$ could either be $\lceil n/b \rceil$ or $\lfloor n/b \rfloor$. Then

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some $\epsilon > 0$, then

   $$T(n) = \Theta(n^{\log_b a})$$

2. If $f(n) = \Theta(n^{\log_b a})$, then

   $$T(n) = \Theta(n^{\log_b a \lg n})$$

3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some $\epsilon > 0$, and if $af(n/b) \leq cf(n)$ for some constant $c < 1$ for all sufficiently large $n$, then

   $$T(n) = \Theta(f(n))$$
Examples

- \( T(n) = 8T(n/2) + n^5 \)
- \( T(n) = 8T(n/2) + n^4 \)
- \( T(n) = 8T(n/2) + n^3 \)
- \( T(n) = 3T(n/2) + n^2 \)
- \( T(n) = 3T(n/2) + n \)

Notes

- There is a gap between case 1 & case 2
- There is a gap between case 2 & case 3
- There is a gap within case 3