This Week’s Agenda

- Introduction to Queueing Theory
  - Exponential Distribution, Poisson Process, Discrete Time Markov Chain
  - Continuous Time Markov Chain, Birth and Death Process
  - Queueing Networks
Exponential Distribution

$T$ exponentially distributed with rate $\lambda$ if

$$f_T(t) = \begin{cases} 
\lambda e^{-\lambda t} & t \geq 0 \\
0 & t < 0 
\end{cases}$$  \hspace{1cm} (1)$$

$f_T(t)$ is the density function of $T$. We also write

$$T = \text{exponential}(\lambda).$$

The cdf of $T$ is then

$$F_T(t) = \Pr[T \leq t] = \int_{-\infty}^{t} f_T(x) \, dx = 1 - e^{-\lambda t}. \hspace{1cm} (2)$$

Equivalently,

$$F_T(t) = \Pr[T > t] = e^{-\lambda t}. \hspace{1cm} (3)$$
Memoryless Random Variables

$T$ is said to be *memoryless* if

$$\Pr[T > t_1 + t_2 \mid T > t_1] = \Pr[T > t_2]$$

**Theorem 1.** A continuous random variable $X$ is memoryless if and only if it has an exponential distribution

**Some facts**

Let $T = \text{exponential} (\lambda)$. Then,

$$E[e^{xT}] = \int_{-\infty}^{\infty} e^{xt} f_T(t) \, dt = \cdots = \sum_{n=0}^{\infty} \frac{n!}{\lambda^n} \frac{x^n}{n!}$$

Hence,

$$E[T] = \frac{1}{\lambda}$$

$$\text{Var}[T] = \frac{1}{\lambda^2}$$
Example: Exponential Race Problem

- \( k \) lines carrying incoming packet streams are connected to a router.
- The interarrival times \( T_1, \ldots, T_k \) are independent
- \( T_i = \text{exponential}(\lambda_i) \)

Questions

- What’s the probability that the first packet comes from line 1?
- What’s the distribution of the first arrival time \( \min\{T_i\} \)?
- What’s the distribution of the last arrival time \( \max\{T_i\} \)?
Example: Exponential Race Problem

\[
\Pr[\text{first arrival is from line 1}] = \Pr[T_1 = \min\{T_1, \cdots, T_k\}] = \int_0^\infty \Pr[T_2 > t] \cdots \Pr[T_k > t] f_{T_1}(t) \, dt = \int_0^\infty e^{-(\lambda_1+\cdots+\lambda_k)t} \lambda_1 e^{-\lambda_1 t} \, dt = \frac{\lambda_1}{\lambda_1 + \lambda_2 + \cdots + \lambda_k}
\]

Let \(Z = \min\{T_1, \cdots, T_k\}\). The cdf of \(Z\) is

\[
F_Z(t) = P[Z \leq t] = 1 - P[Z > t] = 1 - \left(1 - e^{-(\lambda_1+\cdots+\lambda_k)t}\right). \tag{1}
\]

Similarly, \(W = \max\{T_1, \cdots, T_k\}\), and

\[
F_W(t) = P[W \leq t] = \prod_{i=1}^k (1 - e^{-\lambda_i t}). \tag{2}
\]

Intuitively, why is it that \(Z\) is exponential but \(W\) is not?
The \( n \)th packet arrival time

- Packets are arriving at a server.
- Inter-arrival time is exponential(\( \lambda \)).
- \( T_i \) is the time that the \( i \)th packet arrives.
- \( S_n = T_1 + \cdots + T_n \).

Question

Compute the cdf of \( S_n \)
The $n$th packet arrival time

The cdf of $S_n$ can be computed as follows.

$$F_{S_n}(t) = \Pr[S_n \leq t]$$
$$= \int_0^t (\Pr[S_n - T_1 \leq t - x]) f_{T_1}(x) \, dx$$

Inductively, we get $F_{S_n}(t)$ and

$$f_{S_n}(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}$$

which is Gamma$(n, \lambda)$.
Stochastic Processes

- A *stochastic process* is a collection of random variables indexed by some set $T$:

  $$\{X(t), t \in T\}$$

- Elements of $T$ are often thought of as points in time

- The set of all possible values of the $X(t)$ are called the *state space* of the process

- When $T$ is countable the process is said to be *discrete-time*

- When $T$ is an interval of the real line, then the process is called a *continuous-time* process
Example: Bernoulli Process

A Bernoulli process is a sequence \( \{X_1, X_2, \ldots, \} \) of independent Bernoulli random variables with parameter \( p \), i.e.

\[
\begin{align*}
\Pr[X_i = 1] &= p \\
\Pr[X_i = 0] &= 1 - p
\end{align*}
\]

We are interested in the following quantities

\[
S_n = X_1 + \cdots + X_n \\
T_n = \text{number of slots from the } (n - 1)\text{th } 1 \text{ to the } n\text{th } 1 \\
Y_n = T_1 + \cdots + T_n
\]

Questions

- Compute the probability mass functions of \( S_n, T_n, Y_n \)
- Compute the expectations and variances of \( S_n, T_n, Y_n \)
Example: Bernoulli Process

\[ p_{S_n}(k) = \binom{n}{k} p^k (1 - p)^{n-k}, \quad 0 \leq k \leq n \]
\[ p_{T_n}(k) = (1 - p)^{k-1} p \]
\[ p_{Y_n}(k) = \binom{k-1}{n-1} p^n (1 - p)^{k-n}, \quad k \geq n. \]

\[ E[S_n] = np \]
\[ \text{Var}[S_n] = np(1 - p) \]

\[ E[T_n] = \frac{1}{p} \]
\[ \text{Var}[T_n] = \frac{1 - p}{p^2} \]

\[ E[Y_n] = \frac{n}{p} \]
\[ \text{Var}[Y_n] = \frac{n(1 - p)}{p^2} \]
Poisson Distribution

$X$ has Poisson distribution with parameter $\mu$, written as

$X = \text{Poisson}(\mu)$ if

$$\Pr[X = n] = e^{-\mu} \frac{\mu^n}{n!}$$

We have $E[X] = \mu$ and $\text{Var}[X] = \mu$.

**Theorem 2.** If $X = \text{Poisson}(\lambda)$ and $Y = \text{Poisson}(\mu)$, then

$X + Y = \text{Poisson}(\lambda + \mu)$, given that $X$ and $Y$ are independent.
Poisson Process

- There are several equivalent definitions. We give the most intuitive here.

- Let \( T_1, \ldots, T_n, \ldots \) be i.i.d. random variables which are all exponential(\( \lambda \))

- Think of \( T_i \) as the inter-arrival time between the \((i - 1)\)th event and the \(i\)th event

- Let \( S_n = T_1 + \cdots + T_n \)

- Define the random process \( \{N(t), t \geq 0\} \) by

\[
N(t) = \max\{n : S_n \leq t\}.
\]

Then, the process is called a Poisson process with rate \( \lambda \)

It is easy to see that

\[
\Pr[N(t) = n] = \Pr[S_n \leq t, S_{n+1} > t] = \int_0^t (P[T_{n+1} > t - x]) f_{S_n}(x) \, dx = \ldots = e^{-\lambda t} \cdot \frac{(\lambda t)^n}{n!} = \text{Poisson}(\lambda t)
\]
Merging Independent Poisson Processes

Let $N_1(t), \ldots, N_k(t)$ be independent Poisson processes.

Then, $N(t) = N_1(t) + \ldots + N_k(t)$ is called the *merging*, or the *superposition* of Poisson processes.

**Proposition 3.** The merging, or superposition of independent Poisson processes $N_1(t), N_2(t), \cdots, N_k(t)$ with rates $\lambda_1, \lambda_2, \cdots, \lambda_k$ is a new Poisson process $N(t)$ with rate

$$\lambda = \sum_{i=1}^{k} \lambda_i.$$
Splitting Independent Poisson Processes

We can split $N(t)$ with rate $\lambda$ into $N_i(t)$ with probability $p_i$, where $1 \leq i \leq k$ and $p_1 + \cdots + p_k = 1$.

This act is called splitting or thinning the Poisson process.

**Theorem 4.** $N_i(t)$ is a Poisson process with rate $\lambda p_i$.

**Proof.**

$$\Pr[N_1(t) = n_1, \ldots, N_k(t) = n_k] = \prod_{j=1}^{k} e^{-\lambda p_j t} \frac{(\lambda p_j t)^{n_j}}{n_j!}$$

Thus,

$$\Pr[N_1(t) = n] = \sum_{n_2, \ldots, n_k} \Pr[N_1(t) = n, N_2(t) = n_2, \ldots, N_k(t) = n_k]$$

$$= \cdots$$

$$= e^{-\lambda p_1 t} \frac{(\lambda p_1 t)^n}{n!} \prod_{j=2}^{k} \sum_{n_j=0}^{\infty} e^{-\lambda p_j t} \frac{(\lambda p_j t)^{n_j}}{n_j!}$$

$$= e^{-\lambda p_1 t} \frac{(\lambda p_1 t)^n}{n!}$$