On writing proofs about asymptotic relations

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In this document, I formally write a few things discussed in previous lectures. Most problems are first analyzed in a draft form, indicating how I think about the solution to the problem. Then, formal proofs are presented. When writing homework solutions, only formal proofs are required.

**Problem 1.** Given two functions \( f, g : \mathbb{N}^+ \to \mathbb{R}^+ \) such that both \( f(n) \) and \( g(n) \) tend to \( \infty \) as \( n \to \infty \), is it true that \( \lg(f(n)) = O(\lg(g(n))) \) implies \( f(n) = O(g(n)) \)?

*Informal analysis.* The value \( \lg(f(n)) \) roughly is the “power part” of the function \( f(n) \). If \( f(n) = n^3 \), then \( \lg(f(n)) = 3 \lg n \). The relation \( \lg(f(n)) = O(\lg(g(n))) \) says that the power-part of \( f(n) \) is upper bounded by some constant times the power-part of \( g(n) \). Hence, it is possible that \( \lg(f(n)) \) is greater than \( \lg(g(n)) \) by a constant factor, yet the relation \( \lg(f(n)) = O(\lg(g(n))) \) still holds. For instance, \( f(n) = n^{100} \), \( g(n) = n^1 \), i.e. \( \lg(f(n)) = 100 \lg(g(n)) \), yet \( \lg(f(n)) = O(\lg(g(n))) \). However, clearly \( n^{100} \neq O(n) \). This is a perfect counter example to the claim! □

*Formal proof.* NOT TRUE. Take, for instance, \( f(n) = n^{100}, g(n) = n \). Then, \( \lg(f(n)) = 100 \lg n = O(\lg(n)) \), yet \( n^{100} \neq O(n) \). □

Note again: a formal proof is all we need for homework problems. Do not go at length explaining your thinking!

**Problem 2.** Given two functions \( f, g : \mathbb{N}^+ \to \mathbb{R}^+ \) such that both \( f(n) \) and \( g(n) \) tend to \( \infty \) as \( n \to \infty \), is it true that \( \lg(f(n)) = o(\lg(g(n))) \) implies \( f(n) = o(g(n)) \)?

*Informal analysis.* In Problem 1, the assertion was not true because the \( O \) relation is not very strong: \( f \) could be \( O(g) \) even though \( f \) is a constant factor greater than \( g \). The \( o \) relation, however, indicates that the power-part of \( g \) grows infinitely faster than the power-part of \( f \). It only makes sense then, that \( g \) grows infinitely faster than \( f \).

How are we going to prove something like this? Let’s start from the definitions.

**What we know** is: \( \lg(f(n)) = o(\lg(g(n))) \), which, by definition, means that for all \( c > 0 \), \( \lg(f(n)) \leq c \lg(g(n)) \) for large enough \( n \) (say \( n \geq n_0 \), for some \( n_0 \)).

**What we want** is: for all \( \bar{c} > 0 \), \( f(n) \leq \bar{c} g(n) \) when \( n \geq n_1 \), for some \( n_1 \).

Let’s start from what we want to prove, to see what it is equivalent to, at the same time try to make a connection to what we know. Consider any constant \( \bar{c} > 0 \).

\[
f(n) \leq \bar{c} g(n) \iff \lg(f(n)) \leq \lg(g(n)) + \lg(\bar{c}).
\]

The reason we want to take \( \lg \) is clear: what we know involves the \( \lg \) of the two functions!

Now, for **any** constant \( c \),

\[
\lg(f(n)) \leq c \lg(g(n)), \quad \text{for } n \geq n_c.
\]

*Please let me know of any mistakes/typos as soon as you find them*
How do we use this to show
\[ \lg(f(n)) \leq \lg(g(n)) + \lg(\bar{c}), \quad \text{for large enough } n. \]
(2)

It is only natural to pick \( c > 0 \), so that
\[ c \lg(g(n)) \leq \lg(g(n)) + \lg(\bar{c}), \]
in which case (3) and (1) imply (2)!

When \( \lg(\bar{c}) \geq 0 \), we can pick \( c = 1 \) and (3) would definitely hold.

When \( \lg(\bar{c}) < 0 \), (3) is equivalent to
\[ c \lg(g(n)) \leq \lg(g(n)) + \lg(\bar{c}) - \lg(\bar{c}) \leq (1 - c) \lg(g(n)) \]

We thus have to choose \( c \) so that \( 1 - c > 0 \), in which case the last inequality is the same as
\[ \frac{-\lg(\bar{c})}{1 - c} \leq \lg(g(n)), \]
or
\[ 2^{-\frac{\lg(\bar{c})}{1 - c}} \leq g(n). \]
This is definitely true since the left hand side is a constant (for a fixed \( \bar{c} \) and a constant \( c < 1 \) we have chosen), while \( g(n) \) was assumed to go to \( \infty \). (This is true for \( n \) is large enough!)

**Formal proof.** We want to show that, for any \( \bar{c} > 0 \), there is some constant \( n_0 \) such that \( f(n) \leq \bar{c}g(n) \) when \( n \geq n_0 \).

Consider any \( \bar{c} > 0 \).

**Case 1:** \( \bar{c} \geq 1 \), or \( \lg(\bar{c}) \geq 0 \).

Since \( \lg(f(n)) = o(\lg(g(n))) \), by definition there is some \( n_1 \) such that
\[ \lg(f(n)) \leq 1 \cdot \lg(g(n)) \quad \text{for all } n \geq n_1. \]

Thus,
\[ \lg(f(n)) \leq \lg(g(n)) + \lg(\bar{c}), \quad \forall n \geq n_1, \]
which is equivalent to
\[ f(n) \leq \bar{c}g(n), \quad \forall n \geq n_1. \]
Hence, when \( \bar{c} \geq 1 \), we can pick \( n_0 = n_1 \), and our assertion is proved.

**Case 2:** \( 0 < \bar{c} < 1 \), or \( \lg(\bar{c}) < 0 \).

Since \( \lg(f(n)) = o(\lg(g(n))) \), by definition there is some \( n_1 \) such that
\[ \lg(f(n)) \leq \frac{1}{2} \cdot \lg(g(n)) \quad \forall n \geq n_1. \]

Since \( \lim_{n \to \infty} g(n) = \infty \), there is some \( n_2 \) such that
\[ 2^{-\frac{\lg(\bar{c})}{1/2}} \leq g(n), \quad \forall n \geq n_2. \]
Now, pick \( n_0 = \max\{n_1, n_2\} \), we have, for all \( n \geq n_0 \),
\[ 2^{-\frac{\lg(\bar{c})}{1/2}} \leq g(n) \]
\[ \iff -\lg(\bar{c}) \leq \frac{1}{2} \lg(g(n)) \]
\[ \iff \frac{1}{2} \lg(g(n)) \leq \lg(\bar{c}) + \lg(g(n)) = \lg(\bar{c} \cdot g(n)). \]
Consequently, for all \( n \geq n_0 \), we have

\[
\log(f(n)) \leq \frac{1}{2} \cdot \log(g(n)) \leq \log(\bar{c} \cdot g(n)),
\]

which is the same as

\[
f(n) \leq \bar{c} \cdot g(n), \, \forall n \geq n_0,
\]

as desired. \( \square \)