The Probabilistic Method

Techniques

- Union bound
- Argument from expectation
- Alterations
- The second moment method
- The (Lovasz) Local Lemma

And much more

- Alon and Spencer, “The Probabilistic Method”
- Bolobas, “Random Graphs”
Lovasz Local Lemma: Main Idea

- Recall the union bound technique:
  - want to prove $\text{Prob}[A] > 0$
  - $\overline{A} \Rightarrow$ (or $\iff$) some bad events $B_1 \cup \cdots \cup B_n$
  - done if $\text{Prob}[B_1 \cup \cdots \cup B_n] < 1$
- Could also have tried to show
  $$\text{Prob}[\overline{B_1} \cap \cdots \cap \overline{B_n}] > 0$$
- Would be much simpler if the $B_i$ were mutually independent, because
  $$\text{Prob}[\overline{B_1} \cap \cdots \cap \overline{B_n}] = \prod_{i=1}^{n} \text{Prob}[\overline{B_i}] > 0$$

Main Idea

Lovasz Local Lemma is a sort of generalization of this idea when the “bad” events are not mutually independent.
PTCF: Mutual Independence

Definition (Recall)
A set $B_1, \ldots, B_n$ of events are said to be mutually independent (or simply independent) if and only if, for any subset $S \subseteq [n],$

$$\text{Prob} \left[ \bigcap_{i \in S} B_i \right] = \prod_{i \in S} \text{Prob}[B_i]$$

Definition (New)
An event $B$ is mutually independent of events $B_1, \ldots, B_k$ if, for any subset $S \subseteq [k],$

$$\text{Prob} \left[ B \mid \bigcap_{i \in S} B_i \right] = \text{Prob}[B]$$

Question: can you find $B, B_1, B_2, B_3$ such that $B$ is mutually independent of $B_1$ and $B_2$ but not from all three?
Definition

Given a set of events $B_1, \cdots, B_n$, a directed graph $D = ([n], E)$ is called a dependency digraph for the events if every event $B_i$ is independent of all events $B_j$ for which $(i, j) \not\in E$.

- What’s a dependency digraph of a set of mutually independence events?
- Dependency digraph is **not unique**!
The Local Lemma

Lemma (General Case)

Let $B_1, \cdots, B_n$ be events in some probability space. Suppose $D = ([n], E)$ is a dependency digraph of these events, and suppose there are real numbers $x_1, \cdots, x_n$ such that

- $0 \leq x_i < 1$
- $\text{Prob}[B_i] \leq x_i \prod_{(i,j) \in E} (1 - x_j)$ for all $i \in [n]$

Then,

$$\text{Prob}\left[\bigcap_{i=1}^{n} \overline{B_i}\right] \geq \prod_{i=1}^{n} (1 - x_i)$$
Lemma (Symmetric Case)

Let $B_1, \cdots, B_n$ be events in some probability space. Suppose $D = ([n], E)$ is a dependency digraph of these events with maximum out-degree at most $\Delta$. If, for all $i$,

$$\Pr[B_i] \leq p \leq \frac{1}{e(\Delta + 1)}$$

then

$$\Pr\left[\bigcap_{i=1}^{n} \bar{B}_i\right] > 0.$$ 

The conclusion also holds if

$$\Pr[B_i] \leq p \leq \frac{1}{4\Delta}$$

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CSE 694 – A Fun Course
Example 1: Hypergraph Coloring

- \( G = (V, E) \) a hypergraph, each edge has \( \geq k \) vertices
- Each edge \( f \) intersects at most \( \Delta \) other edges
- Color each vertex randomly with red or blue
- \( B_f \): event that \( f \) is monochromatic

\[
\text{Prob}[B_f] = \frac{2}{2|f|} \leq \frac{1}{2^{k-1}}
\]

- There’s a dependency digraph for the \( B_f \) with max out-degree \( \leq \Delta \)

**Theorem**

\( G \) is 2-colorable if

\[
\frac{1}{2^{k-1}} \leq \frac{1}{e(\Delta + 1)}
\]
**Example 2: $k$-SAT**

**Theorem**

In a $k$-CNF formula $\varphi$, if no variable appears in more than $2^{k/e}$ clauses, then $\varphi$ is satisfiable.

Recently Moser (and Moser-Tardos) showed how to find such a truth assignment.
Example 3: Edge-Disjoint Paths

- \( \mathcal{N} \) a directed graph with \( n \) inputs and \( n \) outputs
- From input \( a_i \) to output \( b_i \) there is a set \( P_i \) of \( m \) paths
- In switching networks, we often want to find (or want to know if there exists) a set of edge-disjoint \((a_i \rightarrow b_i)\)-paths

Theorem

\[ 8nk \leq m \text{ and each path in } P_i \text{ shares an edge with at most } k \text{ paths in any } P_j, j \neq i. \text{ Then, there exists a set of edge-disjoint } (a_i \rightarrow b_i)-\text{paths.} \]