Transmission Fault-Tolerance of Iterated Line Digraphs

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Abstract

Many interconnection networks can be constructed with line digraph iterations. In this paper, we will establish a general result on super line-connectivity based on the line digraph iteration which improves and generalizes several existing results in the literature.

Key Words: line digraph iterations, super line-connectivity, interconnection networks.

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1 Introduction

When an interconnection network contains possible link-fault there are two fault-tolerance measures in the literature.

The deterministic measure is the maximum number of faulty links which, in any case, cannot disconnect the network. This measure is called the line-connectivity.

The probabilistic measure is the probability of the network being connected when links fail with certain probabilistic distribution. Let $F$ be the family of all line-cuts of a digraph $G$. By the exclusion-inclusion principle,

$$\text{Prob}(G \text{ connected}) = 1 - \text{Prob}(G \text{ disconnected})$$

$$= 1 - \sum_{c \in F} \text{Prob}(c) + \sum_{c_1, c_2 \in F, c_1 \neq c_2} \text{Prob}(c_1 \cup c_2) - \ldots$$

When all links are independent, $\text{Prob}(c)$ (respectively $\text{Prob}(c_1 \cup c_2)$) is a product of failure probabilities of links in $c$ (respectively in $c_1 \cup c_2$). Therefore, if every link has the same fault probability of a small number, then $\text{Prob}(G \text{ connected})$ depends mainly on the number of the minimum line-cuts.

Consider a digraph $G$ with line-connectivity $c$. If a vertex of $G$ has $c$ in-links (or $c$ out-links) other than loops, then those $c$ in-links (out-links) form a line-cut of size $c$. Those line-cuts are called natural line-cuts. A digraph $G$ is said to have super line-connectivity $c$ if its line-connectivity is $c$ and every line-cut of size $c$ is natural. Clearly, the super line-connected digraph, in some sense, reaches maximum fault-tolerance.

Given a degree bound $d$, many constructions have been found in the literature to achieve the maximum connectivity $d$ and near-minimum diameter [16, 5], including Kautz digraphs, cyclically-modified de Bruijn digraphs, generalized cycles, etc. Do they also have super line-connectivity? This is an interesting question. Indeed, several related research works have been published in the literature [17, 3].

In this paper, we study the super line-connectivity with line digraph iterations. In fact, many interconnection networks can be constructed with line digraph iterations, such as de Bruijn digraphs [2], Kautz digraphs [12], some of generalized de Bruijn digraphs [5, 13], Imase-Itoh digraphs [10, 11, 9], large bipartite digraphs[15], and large generalized cycles[7]. We will show that the super line-connectivity can be generally established through line digraph iterations.
2 Main Results

Consider a \(d\)-regular digraph \(G\), that is, every vertex of \(G\) has in-degree \(d\) and out-degree \(d\). Suppose each vertex of \(G\) has at most one loop. A vertex with a loop is called a loop-vertex. A cyclic modification of \(G\) is a digraph obtained from \(G\) by deleting all loops and connecting all loop-vertices into a cycle.

**Lemma 2.1.** Let \(d \geq 2\). Suppose \(G\) is a \(d\)-regular digraph that each vertex has at most one loop. Then every cyclic modification of \(G\) has super line-connectivity \(d\) if and only if \(G\) satisfies the following conditions:

(a) \(G\) has super line-connectivity at least \(d - 1\), and

(b) every line-cut of size \(d\) breaks the vertex set of \(G\) into two parts \(A\) and \(B\) such that either every part contains a loop-vertex or one of \(A\) and \(B\) is a singleton.

**Proof.** For sufficiency, assume \(G\) has properties (a) and (b). Consider a line-cut \(C\) of size \(d\) in a cyclic modification \(G^*\) of \(G\). Suppose \(C\) breaks the vertex set of \(G^*\) into two parts \(A\) and \(B\) such that every link from \(A\) to \(B\) belongs to \(C\). By (b), we have two cases.

**Case 1.** Both \(A\) and \(B\) contain at least one loop-vertex. Then, \(C\) must contain a link \(e\) from the cycle \(G^* \setminus G\). Then, \(C - \{e\}\) forms a line-cut of \(G\). By (a), \(C - \{e\}\) must be natural. Thus, either \(A\) or \(B\) contains only one vertex, so \(C\) must be natural in \(G^*\).

**Case 2.** Either \(A\) or \(B\) contains only one vertex. Thus, \(C\) must be natural in \(G^*\).

For necessity, we first assume that \(G\) does not have property (b). This means that there exists a line-cut \(C\) of size \(d\) which breaks the vertex set of \(G\) into two parts \(A\) and \(B\) such that \(|A| \geq 2\), \(|B| \geq 2\), and either \(A\) or \(B\) contains no loop-vertex. Clearly, \(C\) is also an evidence to witness that \(G^*\) has no super line-connectivity \(d\).

Now, we assume that \(G\) does not have property (a). Suppose \(C\) is a line-cut of size \(d - 1\) which breaks the vertex set of \(G\) into two parts \(A\) and \(B\) such that \(|A| \geq 2\), \(|B| \geq 2\), and all links from \(A\) to \(B\) belong to \(C\). We connect all loop-vertices in \(A\) into a path \(P_A\) and all loop-vertices in \(B\) into a path \(P_B\), and then connect two path into a cycle \(Q\). With this cycle, we can obtain a cyclic modification \(G^*\) of \(G\) such that \(C\) together with the link in the cycle \(Q\) from \(A\) to \(B\) form a line-cut of size \(d\) for \(G^*\), which witnesses that \(G^*\) has no super line-connectivity \(d\).  

We should be careful with the case \(d = 1\). In fact, Lemma 2.1 does not hold for \(d = 1\). For a counterexample, consider a digraph \(G\) consisting of disjoint union of two loops and a cycle of
size three. The cyclic modification of $G$ is not connected. In fact, when $d - 1 = 0$, the condition (a) is vague.

It is worth mentioning that if $G$ has no loop, then conditions (a) and (b) are equivalent to the fact that $G$ has super line-connectivity $d$. In fact, it follows from (a) that $G$ has line-connectivity $d$. It then follows from (b) that every line-cut of size $d$ is natural. Hence, $G$ has super line-connectivity $d$. Conversely, if $G$ has super line-connectivity $d$, then (a) and (b) hold trivially.

For any digraph $G = (V, E)$, we denote by $L(G)$ the line digraph of $G$ defined as follows: The vertex set of $L(G)$ is $E$. For $(a, b), (c, d) \in E$, there exists a link in $L(G)$ from $(a, b)$ to $(c, d)$ if and only if $b = c$. For any natural number $k \geq 1$, recursively define $L^k(G) = L(L^{k-1}(G))$, where $L^0(G) = G$.

**Theorem 2.2.** Let $G$ be a $d$-regular digraph where each vertex has at most one loop. If every cyclic modification of $G$ has super line-connectivity $d$, then for $k \geq 1$, every cyclic modification of $L^k(G)$ also has super line-connectivity $d$ unless $d = 2$ and $G$ contains a loop.

**Proof.** For $d = 1$, since $G$ is $d$-regular, $G$ consists of disjoint union of cycles. If $G$ has no loop, then $G$ is a cycle since $G$ has super line-connectivity 1. Thus, for every $k \geq 1$, $L^k(G)$ is a cycle and hence has super line-connectivity $d$. If $G$ has a loop, then every cycle in $G$ is a loop because every cyclic modification of $G$ has super line-connectivity 1. This means that every vertex of $G$ has a loop and so does every vertex of $L^k(G)$ for $k \geq 1$. Hence, every cyclic modification of $L^k(G)$ has super line-connectivity 1.

Next, we assume $d \geq 2$. By Lemma 2.1, it suffices to show that if $G$ has properties (a) and (b), then $L(G)$ has properties (a) and (b). The fact that $L^k(G)$ satisfies (a) and (b) then follows by induction.

To do so, consider a minimum line-cut $C$ of $L(G)$. Since $G$ is $d$-regular, $L(G)$ is also $d$-regular. Hence, the line-connectivity of $L(G)$ is at most $d$, i.e., $|C| \leq d$. Suppose $C$ breaks the vertex set of $L(G)$ into two parts $A$ and $B$ such that no link other than those in $C$ is from $A$ to $B$. Let

$$
U = \{(u, v) \mid ((u, v), (v, w)) \in C, \text{ for some vertex } w \text{ of } G \}\n$$

$$
W = \{(v, w) \mid ((u, v), (v, w)) \in C, \text{ for some vertex } u \text{ of } G \}\n$$

$$
V = \{v \mid ((u, v), (v, w)) \in C, \text{ for some vertices } u, w \text{ of } G \}\n$$

Our plan is to first show that $C$ is a natural line-cut of $L(G)$ of size at least $d - 1$, namely $L(G)$ satisfies condition (a). We show several claims as follows.
Claim 1. If $|V| \geq 2$, then $|A| > |C|$ and $|B| > |C|$.

Proof. Note that for each $v \in V$, there are $d$ out-links and $d$ in-links at $v$. Each of the out-links belongs to either $W$ or $A$ and each of the in-links belongs to either $U$ or $B$. Moreover, for each $v \in V$, there exists at least one in-link in $A$ and at least one out-link in $B$. Therefore, when $|V| \geq 2$ and no loop-vertex exists in $V$, we have $|A| \geq 2(d+1) - |C| > |C|$ and $|B| \geq 2(d+1) - |C| > |C|$. When there exists loop-vertex in $V$, $L(G)$ must have a loop-vertex. Therefore, $|C| \leq d - 1$. Therefore, $|A| \geq 2d - |C| > |C|$ and $|B| \geq 2d - |C| > |C|$.

Claim 2. $|V| = 1$.

Proof. For contradiction, suppose $|V| \geq 2$. By Claim 1, $|A| > |C|$ and $|B| > |C|$. Since $|A| > |C|$, $U$ is a vertex-cut of $L(G)$ and hence a line-cut of $G$. If $G$ has no loop, then $G$ has super line-connectivity $d$. Thus, $|U| \geq d$. Note that $|U| \leq |C| \leq d$. Therefore, $|U| = |C| = d$ and hence $U$ is a natural line-cut of $G$. If $G$ has a loop, then $L(G)$ has a loop. Hence, $|U| \leq |C| \leq d - 1$. However, in this case, $G$ has super line-connectivity $d - 1$. Therefore, $|U| = |C| = d - 1$ and $U$ is a natural line-cut of $G$. Similarly, we can show that $W$ is natural and $|W| = d$ if $G$ has no loop and $d - 1$ if $G$ has a loop. Hence, we have $|C| = |U| = |W|$. It follows that any two links in $U$ cannot share the same ending vertex (recall the assumption that $|V| \geq 2$). Therefore, $U$ must consist of out-links at a vertex $x$ and $W$ must consist of in-links at a vertex $y$ (Fig. 1). It also follows that $|V| = |C|$. We next show that any $v$ in $V$ is not a loop-vertex. In fact, for otherwise, suppose some $v \in V$ has a loop. Then the loop being in $A$ would introduce a link $((v, v), (v, y))$ from $A$ to $B$, but not in $C$, and the loop being in $B$ would introduce a link $((x, v), (v, v))$ from $A$ to $B$, but not in $C$, a contradiction.
Note that at each $v \in V$, every in-link other than $(x, v)$ belongs to $B$ and every out-link other than $(v, y)$ belongs to $A$. Those links induce $(d - 1)^2$ links in $L(G)$ from $B$ to $A$. Therefore, there exist at least $|C|(d - 1)^2$ links in $L(G)$ from $B$ to $A$. However, as $L(G)$ is $d$-regular, every vertex in $L(G)$ has the same in-degree and out-degree. It follows that in $L(G)$ the number of links from $B$ to $A$ equals the number of links from $A$ to $B$. Therefore, $|C|(d - 1)^2 \leq |C|$. It follows that $d = 2$ (Fig. 2). Since $|C| = |V| \geq 2$, we must have $|C| = 2$. Thus, we may write $V = \{v_1, v_2\}$. Note that every path from $v_1$ to $y$ in $G$, not containing link $(v_1, y)$, must pass through vertex $x$ and hence must contain link $(x, v_2)$. To see this, suppose $P$ is a path from $v_1$ to $y$ not going through $x$. It is clear that $P$ must go through $v_2$. The first link in $P$ is an out-link of $v_1$ and thus it is in $A$. The last link in $P$ is $(v_2, y)$ which is in $B$. Hence, there must be a transition from $A$ to $B$ along the way. Thus, $P$ induces a link of $L(G)$ connecting $A$ to $B$ which is not in $C$. This means that $(v_1, y)$ and $(x, v_2)$ form a line-cut of size $|C| = d = 2$, which is not natural. However, $|C| = d$ implies that $L(G)$ has no loop and hence $G$ has no loop. It follows that $G$ has super line-connectivity $d$, contradicting the existence of an un-natural line-cut of size $d$.

**Claim 3.** $|U| = 1$ or $|W| = 1$.

**Proof.** For contradiction, suppose $|U| \geq 2$ and $|W| \geq 2$. By Claim 2, $|V| = 1$, i.e., $V = \{v\}$. This means that for any $(u, v) \in U$ and $(v, w) \in W$, $((u, v), (v, w)) \in C$. It follows that $|U| \cdot |W| = |C|$. Since $|U| \geq 2$ and $|W| \geq 2$, we have $|C| - |W| = (|U| - 1)|W| \geq |W|$ and $|C| - |U| = |U||(W| - 1) \geq |U|$. Therefore, $(|C| - |U|)(|C| - |W|) \geq |U| \cdot |W| = |C|$. Note that $v$ has at least $|C| - |U|$ in-links not in $U$, which must belong to $B$, and at least $|C| - |W|$ out-links.

![Figure 2: $d = 2$.](image-url)
not in $W$, which must belong to $A$. Those links at $v$ induce at least $(|C| - |U|)(|C| - |W|)$ links in $L(G)$ from $B$ to $A$. However, the number of links from $B$ to $A$ equals the number of links from $A$ to $B$, which equals $|C|$. Therefore, $(|C| - |U|)(|C| - |W|) \leq |C|$. This means that $(|C| - |U|)(|C| - |W|) = |C|$ and all links from $B$ to $A$ in $L(G)$ are also located at $v$ in $G$ (Fig. 3). (An link of $L(G)$ is said to be located at a vertex of $G$ if the link is in the form $((u, v), (v, w))$.) Now, consider a link $(u, v) \notin U$, which is not a loop at $v$. Such a link exists because $|C| - |U| \geq |U| \geq 2$. Then $(u, v)$ must belong to $B$ and all in-links at $u$ must also belong to $B$. Note that the number of in-links at $v$ other than those in $U$ is $d - |U| \leq d - 2$ and $G$ is at least $(d - 1)$-line-connected. Therefore, after deleting all in-links at $v$ which are not in $U$, the remaining digraph is still connected. Hence, there exists a path from $u$ to $v$ passing through a link in $U$. This path would induce a link in $L(G)$ from $B$ to $A$, not located at $v$, a contradiction.

Claim 4. $C$ is a natural line-cut of $L(G)$.

Proof. By Claim 3, $|U| = 1$ or $|W| = 1$. It follows that if $|C| = d$, then $C$ is natural. Next, we assume $|C| \leq d - 1$. First, we consider the case that $|U| = 1$. Note that $|W| \leq |C| \leq d - 1$. Let $V = \{v\}$. There exists at least one out-link (possibly a loop) at $v$ not in $W$. Suppose $(v, w)$ is an out-link at $v$ not in $W$. Then $(v, w)$ must belong to $A$.

If $(v, w)$ is not a loop, then $(v, w) \notin U$, i.e., $A - U \neq \emptyset$. Define

$$X = \{x \mid (u, x) \in A - U \text{ for some vertex } u \text{ of } G\}$$

$$Y = \{y \mid (y, w) \in B \text{ for some vertex } w \text{ of } G\}.$$ 

Then any vertex $z$ not in $X \cup Y$ must satisfy property that all in-links at $z$ belong to $B$ and all out-links at $z$ belong to $A$. Thus, the existence of such a vertex $z$ induces $d^2$ links from $B$ to $A$ in $L(G)$. Since the number of links from $A$ to $B$ equals the number of links from $B$ to $A$, we
have $d^2 \leq |C| \leq d - 1$, a contradiction. Therefore, $X$ and $Y$ form a partition of the vertex set of $G$. Note that every link from $X$ to $Y$ belongs to $U$. Moreover, $X \cap Y$ must be empty, as the non-emptiness of $X \cap Y$ implies the existence of a link from $A$ to $B$ in $L(G)$ which is not in $C$. Therefore, $U$ is a line-cut of $G$. Since $|U| = 1$, we have $d - 1 \leq 1$ and hence $d = 2$. This falls into the case that we wanted to avoid.

If $(v, w)$ is a loop, i.e., $(v, w) = (v, v)$, then $(v, v)$ must belong to $U$. Otherwise, from $(v, v)$ to other out-links at $v$ would induce more links of $L(G)$ from $A$ to $B$, but not in $C$, a contradiction. Summarizing the above arguments, we conclude that $U$ contains only one element $(v, v)$ which is a loop in $G$ and all out-links at $v$ except the loop belong to $W$. Therefore, $C$ contains $d - 1$ links from $U$ to $W$ and at $(v, v)$ there is a loop $((v, v), (v, v))$. Hence, $C$ is natural.

The case when $|W| = 1$ can be done similarly.

$$|C| = \begin{cases} 
  d & \text{if } G \text{ has no loop} \\
  d - 1 & \text{otherwise.}
\end{cases}$$

**Proof.** If $G$ has no loop, then $L(G)$ has no loop. Thus, every natural line-cut of $L(G)$ has cardinality $d$. By Claim 4, $|C| = d$. If $G$ has a loop, then this loop will induce a loop for $L(G)$. Therefore, the line-connectivity of $L(G)$ is at most $d - 1$. However, since each vertex of $G$ has at most one loop, so does each vertex of $L(G)$. Thus, every natural line-cut of $L(G)$ has cardinality at least $d - 1$. By Claim 4, $|C| = d - 1$.

By Claims 4 and 5, if $G$ has no loop, then $L(G)$ has super line-connectivity $d$; if $G$ has a loop, then $L(G)$ has super line-connectivity $d - 1$, i.e., $L(G)$ satisfies condition (a). Thus, it remains to show that if $G$ has a loop and $d > 2$, $L(G)$ satisfies condition (b). To do so, consider a line-cut $C^*$ of size $d$ in $L(G)$. Suppose $C^*$ is not natural and $C^*$ breaks the vertex set of $L(G)$ into two parts $A^*$ and $B^*$ such that no link other than those in $C^*$ is from $A^*$ to $B^*$. Let

$$
U^* = \{ (u, v) \mid ((u, v), (v, w)) \in C^*, \text{ for some vertex } w \text{ of } G \} \\
W^* = \{ (v, w) \mid ((u, v), (v, w)) \in C^*, \text{ for some vertex } u \text{ of } G \} \\
V^* = \{ v \mid ((u, v), (v, w)) \in C^*, \text{ for some vertices } u, w \text{ of } G \}
$$

We show the following claims.
Claim 6. $A^* - U^* \neq \emptyset$ and $B^* - W^* \neq \emptyset$.

Proof. If $|V^*| \geq 2$, then by an argument similar to the proof of Claim 1, we can show that $A^* - U^* \neq \emptyset$ and $B^* - W^* \neq \emptyset$. If $|V^*| = 1$, then as $C^*$ is not natural, it must be the case that $|U^*| \geq 2$ and $|W^*| \geq 2$. Since $|C^*| = |U^*| \cdot |W^*|$, we have $|C^*| - |U^*| \geq |U^*|$ and $|C^*| - |W^*| \geq |W^*|$. Hence, $(d-1) - |U^*| \geq |U^*| - 1 \geq 1$ and $(d-1) - |W^*| \geq |W^*| - 1 \geq 1$. Assume $V^* = \{v\}$. Then at $v$, there exist at least $(d-1) - |U^*|$ in-links not in $U^*$, which must belong to $B^* - W^*$, and there exist at least $(d-1) - |W^*|$ out-links not in $W^*$, which must belong to $A^* - U^*$. Therefore, $A^* - U^* \neq \emptyset$ and $B^* - W^* \neq \emptyset$. \qed

Claim 7. If $A^* - U^* \neq \emptyset$ and $B^* - W^* \neq \emptyset$, then both $A^*$ and $B^*$ contain at least one loop-vertex.

Proof. Define

$$X = \{x \mid (u,x) \in A^* - U^*, \text{ for some vertex } u \text{ of } G\}$$

$$Y = \{y \mid (y,w) \in B^*, \text{ for some vertex } w \text{ of } G\}.$$

Similar to the proof of Claim 4, we can show that every link from $X$ to $Y$ belongs to $U^*$, that $X$ and $Y$ form a partition for the vertex set of $G$, so that $U^*$ is a line-cut of $G$. Note that $|U^*| \leq |C^*| = d$. Since $G$ has properties (a) and (b), we have $|U^*| \geq d-1$ and either

(u1) $U^*$ is a natural line-cut in $G$, or

(u2) $|U^*| = d$ and both $X$ and $Y$ contain at least one loop-vertex. (When either $X$ or $Y$ is a singleton, case (u1) applies.)

If (u2) holds, then both $A^*$ and $B^*$ contain a loop-vertex. Thus, we may assume that (u1) holds.

Similarly, we may assume that $W^*$ is a natural line-cut of $G$. Note that $d - 1 \leq |U^*| \leq d$ and $d - 1 \leq |W^*| \leq d$. Therefore, we have four cases as Figure 4 illustrated.

Case 1. $|U^*| = |W^*| = d - 1$. Since $|C^*| = d$, in $L(G)$ there is exactly one element of $U^*$ incident to two links of $C^*$. Meanwhile, there is exactly one element of $W^*$ incident to two links of $C^*$. This happens only if $d \geq 4$. Note that at each $v \in V^*$, all in-links not in $U^*$ must belong to $B^*$ and all out-links not in $W^*$ must belong to $A^*$. Therefore, there are totally

$$(d-4)(d-1)^2 + 2(d-2)(d-1) = (d^2 - 3d)(d-1)$$
Figure 4: The proof of Claim 7.

links in $L(G)$ from $B^*$ to $A^*$ located at vertices in $V^*$. Since the number of links from $A^*$ to $B^*$ equals the number of links from $B^*$ to $A^*$, we have

$$(d^2 - 3d)(d - 1) \leq |C^*| = d.$$  

This inequality cannot hold for $d \geq 4$.

Case 2. $|U^*| = d - 1$ and $|W^*| = d$. In this case there is exactly one element in $U^*$ incident to two links in $C^*$ and none of the elements of $W^*$ is incident to more than one link in $C^*$. This occurs only if $d \geq 3$ (recall that $d > 2$). In this case, there are totally $(d - 2)(d - 1)^2 + (d - 2)(d - 1) (= d(d - 2)(d - 1))$ links in $L(G)$ from $B^*$ to $A^*$ located at vertices in $V^*$. Since the number of links from $A^*$ to $B^*$ equals the number of links from $B^*$ to $A^*$, we have

$$d(d - 2)(d - 1) \leq |C^*| = d.$$  

This inequality cannot hold for $d \geq 3$.

Case 3. $|U^*| = d$ and $|W^*| = d - 1$. A contradiction can be found by an argument similar to that in Case 2.

Case 4. $|U^*| = |W^*| = d$. By an argument similar to the proof of Claim 2, we can find a contradiction.

By Claims 6 and 7, $L(G)$ satisfies condition (b), completing the proof of Theorem 2.2.

The following is a special case of Theorem 2.2, since every cyclic modification of a $d$-regular digraph $G$ with super line-connectivity $d$ is the same as $G$.

**Corollary 2.3.** If a $d$-regular digraph $G$ has super line-connectivity $d$, then $L^k(G)$ has super line-connectivity $d$ for every $k \geq 1$.

What would happen to Theorem 2.2 if $d = 2$ and $G$ contains a loop? In this exceptional case, Theorem 2.2 does not hold. A counterexample is shown in Fig. 5. However, with certain additional condition, we can still establish the same result.
Corollary 2.4. Consider a 2-regular digraph $G$ with some loops. Suppose $G$ has super line-connectivity one and no path of length two is between two loop-vertices. If every cyclic modification of $G$ has super line-connectivity two, then every cyclic modification of $L^k(G)$ $(k \geq 1)$ has super line-connectivity two.

Proof. Going over the proof of Theorem 2.2, we may find that only in the proof of Claim 4 we need to avoid the exceptional case that $d = 2$ and $G$ contains a loop. The proof cannot proceed because in this case $|U| = |W| = 1$. Assume $U = \{(u, v)\}$ and $W = \{(v, w)\}$. Then, both $U$ and $W$ can be natural line-cuts of $G$ while $u$ and $w$ are loop-vertices, but $v \in V$ is not. This produces a path of length two between two loop-vertices. \hfill \Box

3 Applications

We look at several examples in this section.

Example 3.1. The Kautz digraph $K(d, 1)$ is the complete digraph on $d + 1$ vertices without loop and in general $K(d, D) = L^{D-1}(K(d, 1))$ [12]. We claim that $K(d, 1)$ has super line-connectivity $d$. Consider a line-cut $C$ of size $d$ in $K(d, 1)$, which breaks the vertex set of $K(d, 1)$ into two parts $A$ and $B$ such that every link from $A$ to $B$ belongs to $C$. Note that there are $|A|(|A| - 1)$ links from $A$ to $A$ and each vertex has $d$ out-links. Therefore, $|A|d - |A|(|A| - 1) = d$. That is, $(|A| - 1)(d - |A|) = 0$. Thus, $|A| = 1$ or $|A| = d$. Since $|A| = d$ implies $|B| = 1$, $C$ is a natural line-cut.

Corollary 3.2. The Kautz digraph $K(d, D)$ has super line-connectivity $d$.

Example 3.3. The de Bruijn digraph $B(d, 1)$ is the complete digraph on $d$ vertices with all loops and in general $B(d, D) = L^{D-1}(B(d, 1))$. We claim that every cyclic modification of $B(d, 1)$
has super line-connectivity \( d \). In fact, every vertex of \( B(d, 1) \) has a loop and hence it has property (b). Moreover, removal all loops of \( B(d, 1) \) results in \( K(d - 1, 1) \) and hence \( B(d, 1) \) has super line-connectivity \( d - 1 \). By Lemma 2.1, every cyclic modification of \( B(d, 1) \) has super line-connectivity \( d \) for \( d \geq 2 \). By Theorem 2.2, every cyclic modification of \( B(d, D) \) has super line-connectivity \( d \) for \( d \geq 3 \). For \( d = 2 \), we may directly verify that every cyclic modification of \( B(2, 2) \) and \( B(2, 3) \) (Fig. 6) have super line-connectivity 2. Note that the distance between two loop-vertices in \( B(2, 2) \) is at least \( D - 1 \). By Corollary 2.4, every cyclic modification of \( B(2, D) \) for \( D \geq 4 \) also has super line-connectivity 2. Therefore, we have

**Corollary 3.4.** Every cyclic modification of the de Bruijn digraph \( B(d, D) \) has super line-connectivity \( d \).

**Example 3.5.** Fiol and Yebra [8] defined a family of bipartite digraphs \( BD(d, n) \) as follows: The vertex set is \( \mathbb{Z}_2 \times \mathbb{Z}_n = \{ (\alpha, i) \mid \alpha \in \mathbb{Z}_2, i \in \mathbb{Z}_n \} \). There is a link from \( (\alpha, i) \) to \( (1 - \alpha, (-1)^\alpha (i + \alpha) + t) \) for every \( t = 0, 1, \ldots, d - 1 \). This family of digraphs has a property that \( BD(d, d^D) = L(BD(d, n)) \). We will show the following.

**Corollary 3.6.** For \( d \geq 3 \) and \( D \geq 1 \), the bipartite digraph \( BD(d, d^D) \) has super line-connectivity \( d \).

**Proof.** It is easy to see that \( BD(d, d) \) is the complete bipartite digraph. For \( d \geq 3 \), \( BD(d, d) \) has super line-connectivity \( d \). Note that a simple digraph without loop has super line-connectivity \( d \) if and only if every line-cut of size at most \( d \) is natural. Thus, it suffices to show that every line-cut of size at most \( d \) in \( BD(d, d) \) is natural. To do it, suppose \( BD(d, d) \) has a line-cut \( C \) with cardinality at most \( d \). We will prove that \( C \) is a natural line-cut and hence \( C \) must have cardinality \( d \). Suppose \( C \) breaks the vertex set of \( BD(d, d) \) into two parts \( A \) and \( B \) such that every link from \( A \) to \( B \) belongs to \( C \). Let \( (P_1, P_2) \) be the partition of the vertex set such that

![Figure 6: B(2, 2) and B(2, 3).](image)
every link is between $P_1$ and $P_2$. Denote $x = |A \cap P_1|$ and $y = |A \cap P_2|$. Then $d - x = |B \cap P_1|$ and $d - y = |B \cap P_2|$. Note that there are $x(d - y)$ links from $A \cap P_1$ to $B \cap P_2$ and there are $y(d - x)$ links from $A \cap P_2$ to $B \cap P_1$. Therefore

$$x(d - y) + y(d - x) \leq d.$$ 

We claim that one of $x, y, d - x,$ and $d - y$ must be 0. For contradiction, suppose $x > 0$, $y > 0$, $d - x > 0$, and $d - y > 0$. Note that $y \geq 2$ or $d - y \geq 2$. Thus,

$$x(d - y) + y(d - x) > x + (d - x) = d,$$

a contradiction. Now, without loss of generality, assume $y = 0$. Then $d - y = d$ and $x = |A| > 0$. Note that $x(d - y) \leq d$. This implies $x = 1$. Hence, $C$ is a natural line-cut.

It is worth mentioning that $BD(2, 2)$ does not have super line-connectivity 2.

The bipartite digraph $BD(d, d^2 + 1) = (P_1, P_2, E)$ has the property that for each vertex $v \in P_1$ (or $v \in P_2$),

$$\{w \mid \exists u \text{ such that } (v, u), (u, w) \in E\} = P_1 - \{v\} \text{ (or } P_2 - \{v\} \}.$$

Now, we show that for $d \geq 3$, $BD(d, d^2 + 1)$ has super line-connectivity $d$. To do so, let $C$ be a line-cut of size at most $d$, which breaks the vertex set into two nonempty parts $A$ and $B$ such that every link from $A$ to $B$ belongs to $C$. Denote $x = |A \cap P_1|$ and $y = |A \cap P_2|$. Then $(d^2 + 1) - x = |B \cap P_1|$ and $(d^2 + 1) - y = |B \cap P_2|$. Note that $x + y = |A| \geq 1$. Without loss of generality, we may assume $x \geq 1$. For each $v \in A \cap P_1$, let $t_v$ be the number of links from $A \cap P_1$ to $B \cap P_2$. Denote $t = \min_{v \in A \cap P_1} t_v$. Suppose $v^* \in A \cap P_1$ achieves $t_{v^*} = t$. Then there are $(d - t)$ vertices in $A \cap P_2$ adjacent to $v^*$. From those $(d - t)$ vertices, there are at least $d(d - t) - (x - 1)$ links to $B \cap P_1$. Therefore, there are at least $xt + d(d - t) - (x - 1)$ links from $A$ to $B$. This means that

$$xt + d(d - t) - (x - 1) \leq d.$$ 

If $t \geq 1$, then $(x - 1)(t - 1) \geq 0$, i.e., $xt \geq x + t - 1$. It follows that

$$x + t - 1 + d(d - t) - (x - 1) \leq d.$$ 

Therefore, $(d - 1)(d - t) \leq 0$. Hence, $t = d$. So, $xd - (x - 1) \leq d$, that is, $(x - 1)(d - 1) \leq 0$. This implies $x = 1$. 

13
If \( t = 0 \), then \( d^2 + 1 - d \leq x \). Similarly, if \( (d^2 + 1) - x > 0 \), then either \( (d^2 + 1) - x = 1 \) or \( (d^2 + 1) - x \geq (d^2 + 1) - d \). This implies that \( x \) has only three possible values \( 1, d^2 \) and \( d^2 + 1 \). Similarly, each of \( y, (d^2 + 1) - x, \) and \( (d^2 + 1) - y \) has four possible values \( 0, 1, d^2, \) and \( (d^2 + 1) \).

Suppose \( x = 1 \). Then \( t = d \) and hence \( (d^2 + 1) - y = d^2 \) or \( (d^2 + 1) - y = d^2 + 1 \). If \( (d^2 + 1) - y = d^2 \), then \( y = 1 \). This would imply the existence of \( d \) links from \( A \cap P_2 \) to \( B \cap P_1 \). Therefore, there are totally \( 2d \) links from \( A \) to \( B \), a contradiction. This means that \( (d^2 + 1) - y = d^2 + 1 \). Hence, \( y = 0 \). Thus, \( C \) is natural.

Note that by the same argument, we can show that \( y = 1 \) or \( (d^2 + 1) - x = 1 \) or \( (d^2 + 1) - y = 1 \) implies that \( C \) is natural. Thus, it remains to prove that we must have \( x = 1 \) or \( y = 1 \) or \( (d^2 + 1) - x = 1 \) or \( (d^2 + 1) - y = 1 \). For contradiction, suppose none of them equals 1. Then they must equal 0 or \( d^2 + 1 \). That is, \( x = d^2 + 1 \), \( (d^2 + 1) - y = 0 \) (since \( t = 0 \)), and \( (d^2 + 1) - x = 0 \). Hence, \( |B| = (d^2 + 1) - x + (d^2 + 1) - y = 0 \), a contradiction.

**Corollary 3.7.** For \( d \geq 3 \) and \( D \geq 2 \), the bipartite digraph \( BD(d, d^D + d^{D-2}) \) has supper line-connectivity \( d \).

**Example 3.8.** Ferrero and Padró [7] studied two families of digraphs \( BGC(p, d, n) = C_p \otimes B(d, n) \) and \( KGC(p, d, n) = C_p \otimes K(d, n) \) where \( C_p \) is a directed cycle of length \( p \) and operation \( \otimes \) is defined as follows. Let \( G = (V, E) \) and \( G' = (V', E') \). Then \( G \otimes G' \) has vertex set \( V \times V' \) and link set \( \{((u, u'), (v, v')) \mid (u, v) \in E, (u', v') \in E'\} \).

With arguments similar to those in Example 3.5, we can show the following:

**Corollary 3.9.** For \( d \geq 3 \), \( BGC(p, d, d^k) \) has supper line-connectivity \( d \).

**Corollary 3.10.** For \( d \geq 3 \), \( KGC(p, d, d^{p+k} + d^k) \) (\( = L^k(KGC(p, d, d^p + 1)) \)) has supper line-connectivity \( d \).

### 4 Discussion

The line digraph iteration preserves the degree, that is, the line digraph of a \( d \)-regular digraph is still \( d \)-regular. This is a very important property different from line graph iteration. This property enable the line digraph iteration to become a very useful tool to study interconnection networks. Many important properties can be preserved through line digraph iterations [8, 6, 4, 14] under certain conditions. Those conditions should be carefully established.
References


