On the Rearrangeability of Shuffle-Exchange Networks

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Abstract

Let \( m(n) \) be the minimum positive integer \( k \) so that the Shuffle-Exchange network with \( k \) stages, \( N = 2^n \) inputs and \( N \) outputs is rearrangeable. Beneš conjectured that \( m(n) = 2n - 1 \). The best bounds known so far are \( 2n - 1 \leq m(n) \leq 3n - 4 \). In this paper, we verify Beneš conjecture for \( n = 4 \), and use this result to show that \( m(n) \leq 3n - 5 \). The \( n = 4 \) case is considerably more complex than the \( n = 3 \) case, which have been done in the literature. We believe that hidden in our proof there is some general technique that would help improve the bound further.

1 Introduction

Shuffle-Exchange networks (SE networks for short) were initially proposed by Stone (1971, [1]) to be an efficient interconnecting architecture for parallel processors. Various applications benefit from this interconnecting pattern such as FFT, matrix transposition, polynomial evaluation, ... A \( k \)-stage SE network with parameter \( n \), denoted by \((SE_n)^k\), is a network with \( N = 2^n \) inputs and \( N \) outputs, consisting of \( k \) SE stages, where each SE stage includes a perfect shuffle pattern (see [1]) followed by an array of \( \frac{N}{2} \times 2 \times 2 \) crossbars. A typical drawing of a 7-stage SE network with \( n = 4 \) (i.e. \((SE_4)^7\)) is shown in Figure 1.

A standard question to be addressed on any multistage interconnection network (MIN) is that if the network is rearrangeable. An \( N \)-input \( N \)-output MIN is rearrangeable if and only if any one to one mapping from the inputs to the outputs is routable by the network. Universality is another term that is often used synonymously with rearrangeability. In the context of SE networks, a long standing question was that how many SE stages are necessary and sufficient for a SE network to be rearrangeable. In fact, it is not entirely clear that increasing the number of stages would increase the rearrangeability of a SE network. There has been a very slow progress toward answering this question. For convenience, let us use \( m(n) \) to denote the minimum positive integer so that \((SE_n)^m(n)\) is rearrangeable. The algorithm discussed by Stone (1971, [1]) showed that \( m(n) \leq n^2 \), thus \( m(n) \) is well defined. Beneš conjectured in 1975 [2] that \( 2n - 1 \) SE stages is necessary and sufficient to route all \( N! \) perfect matchings from the inputs to the outputs, i.e. \( m(n) = 2n - 1 \). Parker (1980,[3]) showed that \( n + 1 \leq m(n) \leq 3n \), where the lower bound was obtained by a counting argument, and the upper bound by group calculations plus the rearrangeability of the Beneš network [4]. Wu and Feng (1981, [5]) gave an explicit algorithm to route all matchings, proving \( m(n) \leq 3n - 1 \). Huang and Tripathi (1986, [6]) improved the bound to \( m(n) \leq 3n - 3 \). Raghavendra and Varma (1987, [7]) verified the conjecture for \( N = 8 \). They used that result to show \( m(n) \leq 3n - 4 \) [8]. They also specified a permutation...
which \((SE_n)^k\) can not route if \(k < 2n - 2\), in effect showing \(2n - 1 \leq m(n)\). With a different formulation, Linial and Tarsi (1989, [9]) also verified the conjecture for \(N = 8\) and showed \(m(n) \leq 3n - 4\). From their formulation it is easy to see that at least \(2n - 1\) stages are needed to route all permutations. Feng and Seo (1994, [10]) gave a proof of the conjecture, which was incomplete as pointed out by Kim, Yoon, and Maeng (1997, [11]).

In this paper, we give a proof that \(m(4) = 7\) using a new method, and then adapting Linial and Tarsi’s results to show that \(m(n) \leq 3n - 5\). As we shall see, the \(n = 4\) case is considerably more difficult than the \(n = 3\) case. We believe that hidden in our proof there is some general technique(s) that would help improve the bound further.

2 Preliminaries

This section presents related concepts and previous results on the problem. Throughout the paper, we shall assume that \(n \in \mathbb{N}\) and \(N = 2^n\). The following definitions and lemmas are from Linial and Tarsi [9].

**Definition 2.1.** For \(k \in \mathbb{N}\), a \(N \times k\) 01-matrix \(A\), denoted by \(A_{N \times k}\) is said to be balanced if

(i) Either \(k \leq n\) and every row vector \(v \in \mathbb{F}_2^k\) occurs exactly \(2^{n-k}\) times as rows of \(A\).

(ii) or \(k > n\) and every \(n\) consecutive column vectors of \(A\) form a balanced matrix.

**Definition 2.2.** Given a balanced matrix \(A_{N \times (n-1)}\), a column vector \(x \in \mathbb{F}_2^N\) is said to agree with \(A\) if appending \(x\) into \(A\) yields an \(N \times n\) balanced matrix (the matrix \([A, x]\)).

**Lemma 2.3.** If \(A\) and \(B\) are two \(N \times (n - 1)\) balanced matrices, then there exists a vector \(x \in \mathbb{F}_2^N\) that agrees with both \(A\) and \(B\).

**Lemma 2.4.** Let \(A_{N \times n}\) be a 01-matrix such that deleting any column of \(A\) yields a balanced \(N \times (n - 1)\) matrix. Then, either (i) \(A\) is balanced, or (ii) each row of \(A\) has an even number of 1’s, or (iii) each row of \(A\) has an odd number of 1’s.

**Lemma 2.5.** Let \(A_{N \times k}\) be a balanced matrix with \(k \leq n\), and let \(T\) be a non-singular \(k \times k\) 01-matrix, then \(AT\) is also balanced, where all the arithmetic is done modulo 2.
Notice that when \( x \) agrees with \( A \), we can insert \( x \) into any position between the columns of \( A \) to get a balanced matrix. It is also easy to see that \((SE_n)^m (m > n)\) is rearrangeable if and only if for every two given balanced matrices \( A_{N \times m} \) and \( B_{N \times l} \) there exists an \( N \times (m - n) \) balanced matrix \( M \) such that the matrix \([A, M, B]\) is balanced. Here the rows of \( A \) are binary representations of the inputs and the corresponding rows of \( B \) are binary representations of the matched outputs.

3 Main Results

To illustrate the idea and introduce notations needed for the main theorem \((m(4) = 7)\), we first reproduce a known result (see [9, 7]) using the new method.

**Lemma 3.1.** \(m(3) = 5\), namely the network \((SE_3)^5\) is rearrangeable.

\[
\begin{array}{c}
000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \\
\end{array}
\]

**Proof.** We use the same approach as that of Raghavendra and Varma [7], namely from first principles. However, the method is different and more intuitive. Figure 2 shows a typical drawing of a 5-stage SE network for \( N = 8 \). For convenience, the network can be redrawn and the switches can be labeled as shown in Figure 3. In the figure, the inputs and outputs have been numbered in decimals for convenience. We write \( x \in I_i \) (\( y \in O_j \)) if input \( x \) (output \( y \)) is connected to input switch \( I_i \) (output switch \( O_j \)). Given a perfect matching \( \pi \) (or permutation) from the inputs \( \{0, 4, 2, 6, 1, 5, 3, 7\} \) to the outputs \( \{0, 1, 2, 3, 4, 5, 6, 7\} \), we first construct a 4x4, 2-regular multi-bipartite graph \( G(\pi) = (I, O, E) \) whose bipartitions are \( I = O = \{0, 1, 2, 3\} \). \( I \) and \( O \) correspond to the set of input and output switches respectively. We shall refer to \( G(\pi) \) as \( G \) if \( \pi \) is clear from the context. \((i, j) \in E \) iff \( \pi(x) = y \) for some \( x \in I_i \) and \( y \in O_j \), introducing multiple edges if necessary. We now need some notations. Suppose we have colored the edges of \( G \) with colors in \( C = \{0, 1, 2, 3\} \). For each \( c \in C \), let \( L(c) \) (\( R(c) \)) be the multi-set of the vertices in \( I \) (\( O \)) which are incident to an edge colored \( c \). For each subset \( S \subseteq C \), let
$L(S) = \bigcup_{e \in S} L(e)$ and $R(S) = \bigcup_{e \in S} R(e)$, where the union is multiset union. For each $e \in E$, let $l(e)$ ($r(e)$) denote the vertex in $f^{-1}(O)$ incident to $e$. Similarly, for any subset $A \subseteq E$, let $L(A) = \{l(e) \mid e \in A\}$ and $R(A) = \{r(e) \mid e \in A\}$.

To this end, we observe from Figure 3 that the realizability of the matching is equivalent to the existence of a coloring of $G$ with colors in $C$ such that

1. For each $c \in C$, $L(c)$ has a representative from each of $\{0,1\}$ and $\{2,3\}$.
2. $L(\{0,1\}) = L(\{2,3\}) = \{0,1,2,3\}$. In other words, $L(\{0,1\})$ and $L(\{2,3\})$ have distinct elements.
3. For each $c \in C$, $R(c)$ has a representative from each of $\{0,1\}$ and $\{2,3\}$.
4. $R(\{0,2\}) = R(\{1,3\}) = \{0,1,2,3\}$. In other words, $R(\{0,2\})$ and $R(\{1,3\})$ have distinct elements.

Note that these conditions imply that each color appears exactly twice. The conditions are chosen so that the two edges colored $c \in \{0,1,2,3\}$ can be routed through middle switch $M_c$. We will not state and prove the correctness of any routing algorithm based on the coloring here, as it is straightforward.

We now describe a procedure to properly color all $4 \times 4$ 2-regular multi-bipartite graphs $G$ as follows. Along the way, we shall also prove that our procedure works.

**Phase 1.** As $G$ is 2 regular and multi-bipartite, it is the union of even cycles. $G$ thus can be decomposed into two $4 \times 4$ perfect matchings by taking alternate edges on each cycle. Let the matchings be $M_1$ and $M_2$ (whose vertex sets are the same as $G$).

**Phase 2.** From each $M_i$ ($i = 1, 2$), construct a $2 \times 2$ 2-regular bipartite graph $G_i$ by combining within each bipartition of $M_i$ the pairs of vertices $\{0,1\}$ and $\{2,3\}$. Figure 4 illustrates the results of our first two phases. Obviously, $L(E(G_i)) = R(E(G_i)) = \{0,1,2,3\}$, for $i = 1, 2$.

Here and henceforth the $L$ and $R$ functions are applied in the context of the original graph $G$.

![Figure 4: An illustration of the first two phases](image)

We call the graphs $G_1$ and $G_2$ the basic components of $G$. Since the basic components are $2 \times 2$ 2-regular bipartite graphs, they can only be either a 4-cycle or a union of two 2-cycles. A basic component is said to be of type 1 if it is a 4-cycle and of type 2 otherwise. In Figure 4, $G_1$ is of type 1 and $G_2$ is of type 2.
**Phase 3.** As each coloring of $G_1$ and $G_2$ induces uniquely a coloring of $G$, we are to color $G_1$ and $G_2$ so that the coloring satisfy conditions $P_i$ and $P'_i$, $1 \leq i \leq 2$. We call an edge whose color is $c \in C$ a $c$-edge. Consider two cases:

**Case 1** Both $G_1$ and $G_2$ are of type 1. In this case we color the graphs as shown in Figure 5a. It is easy to see that the coloring satisfies all prescribed conditions. The basic idea is that as we have used each color exactly twice, to enforce $P_1$ and $P'_1$ we need to make sure that if there is a $c$-edge going from $\{0,1\}$ to $\{0,1\}$, then the other $c$-edge must go from $\{2,3\}$ to $\{2,3\}$ in either basic components, and similarly if a $c$-edge going from $\{0,1\}$ to $\{2,3\}$ then the other $c$-edge must go from $\{2,3\}$ to $\{0,1\}$. To enforce $P_2$ and $P'_2$, on the left side (the $I$ side) we *separate* each color pair $\{0,1\}$ (i.e. $L(0) \cap L(1) = \emptyset$) and $\{2,3\}$ (i.e. $L(2) \cap L(3) = \emptyset$), while on the right (the $O$ side) we separate the pairs $\{0,2\}$ ($R(0) \cap R(2) = \emptyset$) and $\{1,3\}$ ($R(1) \cap R(3) = \emptyset$).

![Figures 5a and 5b](image)

**Figure 5: Illustration of the colorings when $n = 3$**

**Case 2** There is one graph of type 2. Without loss of generality, assume $G_2$ is of type 2 as illustrated in Figures 5b and 5c. In this case we color $G_1$ with $\{0,2\}$ and $G_2$ with $\{1,3\}$. Notice that $P_1$, $P'_1$, and $P_2$ are satisfied even if we switch colors in one (or both) 2-cycles of $G_2$. To ensure $P_2$, we do this switching if necessary at each 2 cycle of $G_2$ to separate each pair $\{0,1\}$ and $\{2,3\}$ on the left.

\[\square\]

Secondly, we use the idea to derive a more elaborate proof for the case where $N = 16$. Firstly, we redraw the network as shown in Figure 6, so that it is easier to derive the conditions similar to the $P_i$ and $P'_i$. Given any perfect matching $\pi$ from the inputs to the outputs, we first construct the $8 \times 8$ 2-regular multi-bipartite graph $G$ in a similar way as the $G$ in Lemma 3.1. The bipartitions of $G$ are $I = O = \{0, \ldots, 7\}$, and $(i,j) \in E(G)$ if for some $x \in \{0, \ldots, 15\}$ we have $x \in I_i$ and $\pi(x) \in O_j$. From Figure 6, the following proposition is easy to see. We reuse all notations introduced in the proof of Lemma 3.1. Again, as a valid coloring induces a routing algorithm in a straightforward way, we shall not describe the algorithm here.

**Proposition 3.2.** The fact that $(SE_4)^T$ is rearrangeable is equivalent to the fact that for any $8 \times 8$ 2-regular multi-bipartite graph $G = (I, O)$ with bipartitions $I = O = \{0, \ldots, 7\}$, there exists an edge coloring of $G$ using colors in $C = \{0, \ldots, 7\}$ satisfying the following conditions:
(\(P_1\)) For each \(c \in C\), \(L(c)\) has a representative from each of \(\{0, 1, 2, 3\}\) and \(\{4, 5, 6, 7\}\).

(\(P_2\)) For each pair \(\{c_1, c_2\} \in \{\{0, 1\}, \{2, 3\}, \{4, 5\}, \{6, 7\}\}\). \(L(\{c_1, c_2\})\) has a representative from each of \(\{0, 1\}, \{2, 3\}, \{4, 5\}\), and \(\{6, 7\}\).

(\(P_3\)) \(L(\{0, 1, 2, 3\}) = L(\{4, 5, 6, 7\}) = \{0, 1, \ldots, 7\}\). In other words, the elements of \(L(\{0, 1, 2, 3\})\) and \(L(\{4, 5, 6, 7\})\) are all distinct.

(\(P'_1\)) For each \(c \in C\), \(R(c)\) has a representative from each of \(\{0, 1, 2, 3\}\) and \(\{4, 5, 6, 7\}\).

(\(P'_2\)) For each pair \(\{c_1, c_2\} \in \{\{0, 4\}, \{2, 6\}, \{1, 5\}, \{3, 7\}\}\). \(R(\{c_1, c_2\})\) has a representative from each of \(\{0, 1\}, \{2, 3\}, \{4, 5\}\), and \(\{6, 7\}\).

(\(P'_3\)) \(R(\{0, 4, 2, 6\}) = R(\{1, 5, 3, 7\}) = \{0, 1, \ldots, 7\}\). In other words, the elements of \(R(\{0, 4, 2, 6\})\) and \(R(\{1, 5, 3, 7\})\) are all distinct.

Note that these conditions imply that each color appears exactly twice. Again, the conditions were specifically chosen so that each pair of edges with the same color \(c \in C\) can be routed through middle switch \(M_c\) without causing any conflict. From now on, we shall refer to a valid coloring of \(G\) as the coloring satisfying the prescribed conditions in Proposition 3.2. The following proposition further explores properties of a graph \(G\) which can be validly colored.

**Proposition 3.3.** \(G\) can be validly colored if and only if the graph \(G'\) obtained from \(G\) by applying one of the following operations can also be validly colored. Let \(X\) be either \(I\) or \(O\), the operations are:

1. Switch labels of the vertices \(\{v_1, v_2\}\) in \(X\), where \(\{v_1, v_2\} \in \{\{0, 1\}, \{2, 3\}, \{4, 5\}, \{6, 7\}\}\).

2. Switch labels of each pair \(\{0, 2\}\) and \(\{1, 3\}\) in \(X\).

3. Switch labels of each pair \(\{4, 6\}\) and \(\{5, 7\}\) in \(X\).

4. Switch labels of each pair \(\{0, 4\}, \{1, 5\}, \{2, 6\}, \text{ and } \{3, 7\}\) in \(X\).
5. Flip $G$ horizontally, i.e. switch labels of each pair $\{0, 7\}$, $\{1, 6\}$, $\{2, 5\}$, and $\{3, 4\}$ in both $I$ and $O$.

6. Flip $G$ vertically, i.e. $G'$ is the mirror image of $G$.

**Proof.** It’s not difficult to see that the valid coloring of $G$ induces a valid coloring of $G'$ under all cases except the operation of flipping $G$ vertically. In this case, from the coloring of $G$ we can obtain a coloring of $G'$ by the following mapping of colors: if in $G$ an edge is colored $c$, whose binary representation is $\overline{c}$, then we use $\bar{c}$ to color the edge in $G'$. The verification that this is indeed a valid coloring of $G'$ is mechanical and we shall not attempt to do so here. \hfill \blacksquare

**Theorem 3.4.** $m(4) = 7$, namely the network $(SE_4)^7$ is rearrangeable.

**Proof.** Let $G$ be an $8 \times 8$ 2-regular multi-bipartite graph. To color $G$ properly, i.e. the coloring satisfies the conditions of Proposition 3.2, we decompose $G$ into 4 basic components. The decomposition is formally described below. Figure 7 illustrates the decomposition procedure.

**Phase 1** Decompose $G$ into two edge disjoint $8 \times 8$ perfect matchings $M_1$ and $M_2$.

**Phase 2** For each $i = 1, 2$, construct the graph $G_i$ by collapsing the pairs of vertices $\{0, 1\}$, $\{2, 3\}$, $\{4, 5\}$, and $\{6, 7\}$ on each bipartition of $M_i$. It is clear that the graphs $G_i$ are $4 \times 4$ 2-regular bipartite graphs.

**Phase 3** For each $i = 1, 2$, decompose $G_i$ into two edge disjoint $4 \times 4$ perfect matchings $M_{i1}$ and $M_{i2}$.

**Phase 4** For each $i = 1, 2$ and $j = 1, 2$, construct the graph $G_{ij}$ by collapsing the pairs of vertices $\{01, 23\}$ and $\{45, 67\}$ on each bipartition of $M_{ij}$. As before, the $G_{ij}$ are called basic components of $G$, and can only be one of two types: (a) type 1 corresponds to a 4-cycle and (b) type 2 corresponds to two 2 cycles. We are now ready to color the basic components so that the (uniquely) induced coloring on $G$ is valid.

As we have seen in the proof of Lemma 3.1, the number of type-2 basic components can roughly be thought of as the degree of flexibility in finding a valid coloring for $G$. When there is no type-2 basic component, we color the edges of $G_{ij}$ as shown in Figure 8. The coloring clearly satisfies conditions $P_i$ and $P'_j$ ($0 \leq i \leq 3, 1 \leq j \leq 3$).

The rest of the cases are considered in Lemmas 3.6, 3.7, and 3.8 with the help of Lemma 3.5 and Proposition 3.3. The basic idea is that as these cases involve at least one type-2 component, Lemma 3.5 allows us to color the other three basic components using certain set of 6 colors, without worrying about coloring the last basic component. Proposition 3.3 helps simplify case analysis. Roughly, the graphs $G$s could be partitioned into “equivalent” classes, where graphs from each of the class can be obtained from one another by applying a sequence of operations described in Proposition 3.3. \hfill \blacksquare

**Lemma 3.5.** Assume $G$ has at least one basic component of type 2, say $G_{22}$. Let

$$C_L = \{\{0, 1\}, \{2, 3\}, \{4, 5\}, \{6, 7\}\},$$

$$C_R = \{\{0, 4\}, \{2, 6\}, \{1, 5\}, \{3, 7\}\}$$

and $\{c_1, c_2\} \in C_L \cup C_R$ be an arbitrary pair of colors in the set. If the other three basic components can be colored using colors in $C - \{c_1, c_2\}$ so that none of the properties $P_i$ ($0 \leq i \leq 3$) and $P'_j$ ($1 \leq j \leq 3$) are violated, then $G_{22}$ can be colored with $c_1$ and $c_2$ to form a completely valid coloring of $G$. 

Figure 7: An illustration of the basic component decomposition

Figure 8: The coloring when there is no type-2 basic component
Proof. We only show the lemma for the case where \( \{c_1, c_2\} = \{0, 1\} \). Other cases are easily seen to be similar. We shall try to color each 2-cycle of \( G_{22} \) with 0 and 1, switching the colors if necessary.

Firstly, we claim that by coloring each 2-cycle of \( G_{22} \) with 0 and 1, properties \( P_i \) (0 ≤ \( i \) ≤ 3), and \( P'_i \) hold, no matter which edge in each 2 cycle gets which color. Indeed, \( P_0 \), and \( P'_0 \) hold trivially. Let \( e_1 \) and \( e_2 \) be two edges in any 2-cycle of \( G_{22} \), then \( P_2 \) holds because \( L(\{e_1, e_2\}) \) has either a representative from each of \( \{0, 1\} \) and \( \{2, 3\} \) or \( \{4, 5\} \) and \( \{6, 7\} \). Since we have assumed that \( P_3 \) was not violated, before the new colors 0 and 1 arrives, \( L(\{4, 5, 6, 7\}) = \{0, 1, \ldots, 7\} \) and \( L(\{2, 3\}) \subset \{0, 1, \ldots, 7\} \). In fact, \( L(\{2, 3\}) \) has 4 distinct members since \( P_2 \) was not violated. Thus, as \( G \) is 2-regular \( L(\{0, 1\}) = \{0, 1, \ldots, 7\} - L(\{2, 3\}) \), no matter how we assign 0 and 1 to the edges of \( G_{22} \). Hence, \( P'_3 \) holds.

Notice that \( P(4) \) and \( P(5) \) each has a representative from each of \( \{0, 1, 2, 3\} \) and \( \{4, 5, 6, 7\} \). We claim that \( R(\{4, 5\}) \) has a representative from each of \( \{0, 1, 2, 3\}, \{4, 5\}, \{6, 7\} \). Assume for contradiction, without loss of generality, that \( R(\{4, 5\}) \cap \{0, 1\} = \emptyset \). Let's look at the 4 edges whose right end points are 2 or 3. Two of them are colored 4 and 5 as \( R(\{4, 5\}) \cap \{0, 1\} = \emptyset \). The third one is one of the 2 edges in a 2-cycle of \( G_{22} \). The fourth edge must have had a color from \( \{2, 6, 3, 7\} \), say 2 or 6. Moreover, of the four edges whose right end points are 0 or 1, one of them is the other edge in the 2-cycle of \( G_{22} \), one of them must have been colored 6 or 2 because \( P'_2 \) was not violated, the last two have to get colors 3 and 7 as \( P'_1 \) holds for 3 and 7. However, this makes \( P'_2 \) invalid for the color pair \( \{3, 7\} \). Contradiction!

Now, we try to switch colors in each 2-cycle of \( G_{22} \) if necessary to achieve \( P'_2 \). Let \( e_1 \), and \( e_2 \) be the two edges at the 2-cycle whose right end points are in \( \{0, 1, 2, 3\} \). By construction, \( R(\{e_1, e_2\}) \) has a representative from each of \( \{0, 1\} \) and \( \{2, 3\} \). Assign colors 0 and 1 to \( e_1 \) and \( e_2 \) so that \( R(\{0, 4\}) \) has a representative from each of \( \{0, 1\} \) and \( \{2, 3\} \). Notice that this implies \( R(\{1, 5\}) \) has a representative from each of \( \{0, 1\} \) and \( \{2, 3\} \), too. The same procedure is done with the other 2-cycle of \( G_{22} \).

Lastly, we show that \( P'_3 \) holds automatically. Notice that \( P'_3 \) is equivalent to the condition that each vertex \( i \) of the bipartition \( O \) of \( G \) is incident to two edges whose colors have different parities. Let \( \{v_0, v_1\} \) be any pair of right side vertices in \( \{\{0, 1\}, \{2, 3\}, \{4, 5\}, \{6, 7\}\} \). It is easy to see that after we have colored the edges of \( G_{22} \) as above, of the four edges whose right end points are in \( \{v_0, v_1\} \), two have even colors and the other two have odd colors. So if after the coloring either \( v_0 \) or \( v_1 \) is incident to colors of the same parity, then so is the other. However, this means that there must have been a violation of \( P'_3 \) even before the coloring of \( G_{22} \), because there is only one edge of \( G_{22} \) whose right end point is in \( \{v_0, v_1\} \).

Lemma 3.6. \( G \) can be validly colored if there is at most 1 basic component of type 1.

Proof. \( G \) could either have one or zero type-1 basic component. Without loss of generality, we assume the three type-2 components are \( G_{12}, G_{21}, \) and \( G_{22} \). The coloring is roughly shown in Figure 9. The idea is to fix the coloring of \( G_{11} \) by two colors \( \{0,4\} \) as shown in the figure, then switch the assigned pair of colors at each 2-cycle if necessary to form a valid coloring of \( G \).

Obviously, properties \( P_1, P'_i \) (1 ≤ \( i \) ≤ 3) hold. We now do the switching on each 2-cycle of \( G_{12} \) so that \( \{\{0, 1\} \cap L(\{0, 2\})\} \) is either 0 or 2, and that \( \{\{4, 5\} \cap L(\{0, 2\})\} \) is either 0 or 2. This is certainly possible. Notice that this implies \( \{\{0, 1\} \cap L(\{4, 6\})\} \) and \( \{\{4, 5\} \cap L(\{4, 6\})\} \) are also either 0 or 2. Secondly, we do the switching on each 2-cycle of \( G_{21} \) and \( G_{22} \) so that \( \{\{0, 1\} \cap L(\{0, 2\})\} + \{\{0, 1\} \cap L(\{1, 3\})\} = 2 \) and \( \{\{4, 5\} \cap L(\{0, 2\})\} + \{\{4, 5\} \cap L(\{1, 3\})\} = 2 \). Intuitively, we want to “separate” the left end points of the edges having colors in \( \{0, 2\} \) from the left end points of edges having colors in \( \{1, 3\} \), in order to maintain properties \( P_2 \). The same assertion holds for the pairs \( \{4, 6\} \) and \( \{3, 5\} \). Lastly, \( P_3 \) is assured by the fact that \( G_{i1} \) and \( G_{i2} \) have edges which form an 8 × 8 matching, for each \( i = 1, 2 \).
The cases when there is at most one type-1 component for each \( \tau \) and \( \omega \), then such that if we let \( \tau \) is an \( \omega \) for a fixed \( \omega \) such that the matrix \( \tau \) of \( \omega \), then for every two \( \omega \) are both balanced. Notice that as \( \omega \). The following steps as follows.

**Figure 9:** The cases when there is at most one type-1 component

The proofs of the following lemmas are put in separate sections, as they are long and thus would distract the reader from the main line of reasoning.

**Lemma 3.7.** \( G \) can be validly colored if there are exactly 2 basic components of type 1.

**Lemma 3.8.** \( G \) can be validly colored if there are exactly 3 basic components of type 1.

Now, we use the formulation of Linial and Tarsi to first show an auxiliary lemma and then combine the lemma with Theorem 3.4 to improve the upper bound of \( m(n) \). The following lemma has been shown by Varma and Raghavendra in [8], however the proof was rather long. We straightforwardly extend Theorem 3.1 in [9] to obtain a much shorter proof.

**Lemma 3.9.** If \( m(k) = 2k - 1 \) for a fixed \( k \in \mathbb{N} \) then \( (SE_n)^{2n-k-1} \) is rearrangeable whenever \( n \geq k \).

**Proof.** The assertion in the lemma is equivalent to the fact that if we know \( m(k) = 2k - 1 \) for some fixed \( k \in \mathbb{N} \), then for every two \( N \times n \) balanced matrices \( A = [a_1, \ldots, a_n] \) and \( B = [b_1, \ldots, b_n] \), there exists an \( N \times (2n - k - 1) \) balanced matrix \( M \) such that the matrix \( [A, M, B] \) is balanced. Here \( a_i \) and \( b_i \) are the \( i^{th} \) columns of \( A \) and \( B \) respectively. We shall construct the \( (2n - k - 1) \) column vectors of \( M \). The construction takes several steps as follows.

**Step 1** Repeatedly apply Lemma 2.3 to construct vectors \( \{u_1, \ldots, u_{n-k}\} \) such that for \( i = 1, \ldots, n-k \), \( u_i \) agrees with \( [a_{i+1}, \ldots, a_n, u_{i+1}, \ldots, u_{n-k} ] \) and \( [u_{i-1}, \ldots, u_1, b_{i}, \ldots, b_{i+1} ] \). Let \( U = [u_1, \ldots, u_{n-k}] \) and \( UR = [u_{n-k}, \ldots, u_1] \), then after this step both \( [A, U] \) and \( [UR, B] \) are balanced.

**Step 2** We want to construct vectors \( x_1, \ldots, x_{k-1} \) such that if we let \( X = [x_1, \ldots, x_{k-1}], \) then \( [A, U, X] \) and \( [X, UR, B] \) are both balanced. Notice that as \( U \) is an \( N \times (n-k) \) balanced matrix, each row of \( U \) occurs exactly \( 2^k \) times, and so do the rows of \( UR \) in the same positions. Hence, the rows of \( U \) and \( UR \) can be partitioned into \( 2^{n-k} \) classes of \( 2^k \) identical row vectors in each partition. For \( v \) be any column of \( U \) or \( UR \), let \( v(i) \) be the subvector of \( v \) with entries in the \( i^{th} \) partition, where \( 0 \leq i \leq 2^{n-k} - 1 \). Notice that \( v(i) \in \mathbb{F}_2 \) for each \( i \). Also, for each \( i = 0, \ldots, 2^{n-k} - 1 \), let

\[
A(i) = [a_{n-k+1}, \ldots, a_n]
\]

and

\[
A(i) = [b_{n-k}, \ldots, b_{n-k+1}]
\]
Then, since Beneš conjecture is true for \( k \) (i.e. \( m(k) = 2k - 1 \)), there exist vectors \( x_1^{(i)}, \ldots, x_{k-1}^{(i)} \) such that \([A^{(i)}, X^{(i)}, B^{(i)}]\) is balanced. The vectors \( x_1, \ldots, x_{k-1} \) are obtained by pasting together the \( x_j^{(i)} \) preserving the positions of the partitions.

After this step, \([A, U, X]\) is balanced because at the positions where the rows of \( U \) are identical we have \([A^{(i)}, X^{(i)}]\) being a \( 2^k \times k \) balanced matrix. The fact that \([X, U^R, B]\) is balanced can be shown similarly.

Step 3 Now we define an \( N \times (n - k) \) matrix \( W \) from \( U \) such that \([A, W, X, U^R, B]\) is balanced. Define \( W \) as follows (all arithmetics are done over \( \mathbb{F}_2 \)).

\[
w_i = \begin{cases} u_i & 1 \leq i \leq \frac{n-k}{2} \\ u_i + u_{n-k-i} & \frac{n-k}{2} + 1 \leq i \leq n - k - 1 \\ u_{n-k} + a_n & i = n - k \end{cases}
\]

We are left to show that \([A, W, X, U^R, B]\) is balanced. The balancedness of \([X, U^R, B]\) has already been established, so we only need to show that \([A, W, X, U^R]\) is balanced. We do this by considering the following types of submatrices:

(a) Submatrices of the form \([a_{i_1}, \ldots, a_{i_1}, w_{1}, \ldots, w_{n-1}]\) where \( 2 \leq i \leq n - k + 1 \). We apply Lemma 2.5 and use the fact that \([a_{i_1}, \ldots, a_{i_1}, u_1, \ldots, u_{k-1}]\) is balanced. \([a_{i_1}, \ldots, a_{i_1}, w_{1}, \ldots, w_{n-1}]\) can be obtained from \([a_{i_1}, \ldots, a_{i_1}, u_1, \ldots, u_{k-1}]\) by an invertible linear transformation with the invert map preserves the \( a_j \) (\( i \leq j \leq n \)) and

\[
u_j = \begin{cases} w_j & 1 \leq j \leq \frac{n-k}{2} \\ w_j + w_{n-i-k} & \frac{n-k}{2} + 1 \leq j \leq n - k - 1 \\ w_{n-k} + a_n & j = n - k \end{cases}
\]

(b) Submatrices of the form \([a_{i_1}, \ldots, a_{i_1}, w_{1}, \ldots, w_{n-k-i}, x_1, \ldots, x_{k+i-n-1}]\) where \( n - k + 2 \leq i \leq n \). Similarly, in this case we use Lemma 2.5 and the balancedness of the matrix \([a_{i_1}, \ldots, a_{i_1}, u_1, \ldots, u_{n-k}, x_1, \ldots, x_{k+i-n-1}]\)

(c) Submatrices of the form \([w_1, \ldots, w_{n-k-i}, x_1, \ldots, x_{k-1}, u_{n-k}, \ldots, u_{n-k-i+1}]\) where \( 1 \leq i \leq n - k \). Here we use the fact that \([a_{n-1}, u_1, \ldots, u_{n-k}, x_1, \ldots, x_{k-1}]\) is balanced.

\[\square\]

**Theorem 3.10.** For \( n \in \mathbb{N} \) and \( n \geq 4 \), a SE network with \( 3n - 5 \) stages is rearrangeable.

**Proof.** This is immediate from Theorem 3.4 and Lemma 3.9. \[\square\]

## 4 Proof of Lemma 3.7

In this section we shall present a proof of Lemma 3.7 and also set up most of the basic techniques and notations needed for the proof of Lemma 3.8, which is more complicated. In the proof of this Lemma and the next, we assume that \( G_{22} \) is of type-2. Without loss of generality, we also assume that the 2-cycles of \( G_{12} \) go horizontally, i.e. one goes from \( 0123 \) to \( 0123 \), and the other from \( 4567 \) to \( 4567 \), applying operation 4 of Proposition 3.3 if necessary.

The basic idea behind the proofs of these Lemmas is to start from a coloring of \( G_{11}, G_{12}, \) and \( G_{21} \) using 6 colors \( \{0, 4, 2, 6, 1, 5\} \) which does not violate \( P_1 \) and \( P'_i \) (\( i = 1, 2, 3 \)), and then
modify this coloring so that none of the properties in Proposition 3.2 is violated. The Lemmas then hold as a consequence of Lemma 3.5.

In the course of modifying the original coloring, we shall need 5 basic color transformations: $UU(c_1, c_2)$, $UL(c_1, c_2)$, $LU(c_1, c_2)$, $LL(c_1, c_2)$, and $A(c_1, c_2)$, where $c_1$ and $c_2$ are two distinct colors in $C$. Transformation $UU(c_1, c_2)$ (for Upper-Upper) switches colors of a $c_1$-edge $e_1$ and a $c_2$-edge $e_2$ where $l(e_i) \in \{0, 1, 2, 3\}$, and $r(e_i) \in \{0, 1, 2, 3\}$, $i = 1, 2$. $UL(c_1, c_2)$ (for Upper-Lower) switches colors of a $c_1$-edge $e_1$ and a $c_2$-edge $e_2$ where $l(e_i) \in \{0, 1, 2, 3\}$, and $r(e_i) \in \{4, 5, 6, 7\}$, $i = 1, 2$. $LU$ and $LL$ are defined similarly in the obvious way. $A(c_1, c_2)$ (for All) changes color of all $c_1$-edges to $c_2$ and vice versa. We will see that the transformations $UU, UL, LU, LL$ are well defined from the associated context where they are applied.

We now need to introduce a concept called the color incidence vectors (or CIV for short) associated with a coloring of $G$. To each subset of vertices $\{0, 1, 2, 3\}$ and $\{4, 5, 6, 7\}$ of either $I$ or $O$, we associate a CIV of 4 components, where the component corresponding to vertex $i$ consists of 2 colors of the edges incident to $i$. For example, Figure 10 shows a coloring of a graph $G$ and the four associated CIVs. Note that the coloring shown is a valid one. It is immaterial if

\[
\begin{align*}
v_1 &= \begin{pmatrix} 5 \\ 2 \\ 3 \\ 0 \\ 1 \\ 4 \\ 6 \\ 7 \\ 2 \end{pmatrix} = u_1 \\
v_2 &= \begin{pmatrix} 5 \\ 2 \\ 3 \\ 0 \\ 1 \\ 4 \\ 6 \\ 7 \\ 2 \end{pmatrix} = u_2
\end{align*}
\]

Figure 10: An example of color incidence vectors

the CIVs are row vectors or column vectors, so we will adopt the convention that in the figures we use column vectors, and in the texts we use row vectors. Moreover, the order of the color pair in each component of a CIV is not important. In the figure, we have specifically chosen the order so that it is easier to see the validity of the coloring.

Corresponding to a coloring of $G$, there are 4 CIVs. We shall use $v_1$ and $v_2$ to denote the CIVs corresponding to the subsets $\{0, 1, 2, 3\}$ and $\{4, 5, 6, 7\}$ of $I$, respectively. Similarly, $u_1$ and $u_2$ are the CIVs corresponding to the subsets $\{0, 1, 2, 3\}$ and $\{4, 5, 6, 7\}$ of $O$, respectively. The following definitions relate the CIVs to the conditions $P_1$ and $P_2$.

**Definition 4.1.** Let $v$ be a CIV corresponding to a coloring of $G$, then

- $v$ is said to respect $P_1$ or $P'_1$ if the components of $v$ contain all 8 colors $\{0, \ldots, 7\}$.
- Let $A$ be the set of 4 colors in the first two components of $v$, and $B$ be the set of 4 colors in the last two components of $v$. Then, $v$ is said to respect $P_2$ if each pair $\{c_1, c_2\} \in \{(0, 1), (2, 3), (4, 5), (6, 7)\}$ has a representative from each of $A$ and $B$. Similarly, $v$ is said to respect $P'_2$ if each pair $\{c_1, c_2\} \in \{(0, 4), (2, 6), (1, 5), (3, 7)\}$ has a representative from each of $A$ and $B$.
- $v$ is said to respect $P_3$ if each component of $v$ has a representative from each of $\{0, 1, 2, 3\}$.
and \( \{4, 5, 6, 7\} \). Similarly, \( v \) respects \( P_3^i \) if each component of \( v \) has a representative from each of \( \{0, 4, 2, 6\} \) and \( \{1, 5, 3, 7\} \).

The following Proposition is fairly straightforward, thus we omit the proof.

**Proposition 4.2.** Let \( v_1, v_2, u_1, u_2 \) be the CIVs corresponding to some coloring of \( G \) as defined above. Then, the coloring is valid if and only if

(i) \( v_1 \) and \( v_2 \) respect \( P_3^i \) (\( i = 1, 2, 3 \)).

(ii) \( u_1 \) and \( u_2 \) respect \( P_3^i \) (\( i = 1, 2, 3 \)).

To show Lemma 3.7, we consider two cases as follows.

**Case 1.** The two type-1 components are \( G_{11} \) and \( G_{21} \) for some \( i \in \{1, 2\} \). Without loss of generality, we assume \( G_{11} \) and \( G_{12} \) are of type-1. We start from the coloring shown in Figure 11, which clearly does not violate \( P_3^1 \) and \( P_3^i \) (\( i = 1, 2, 3 \)). Moreover, applying both \( A(0, 6) \) and \( A(2, 4) \) would yield another coloring where \( P_3^1 \) and \( P_3^3 \) are not violated.

![Figure 11: Case 1 of Lemma 3.7](image)

After the tentative coloring introduced in Figure 11, and after applying a sequence of the first 3 operations in Proposition 3.3, the possible CIVs are the union of two sets:

\[
F_1 = \left\{ \begin{bmatrix} \cdot 0 \\ \diamond 2 \\ \bullet 4 \\ \circ 6 \end{bmatrix}, \begin{bmatrix} \cdot 0 \\ \circ 2 \\ \bullet 4 \\ \diamond 6 \end{bmatrix}, \begin{bmatrix} \cdot 0 \\ \circ 2 \\ \circ 4 \\ \cdot 6 \end{bmatrix}, \begin{bmatrix} \cdot 0 \\ \circ 6 \\ \circ 4 \\ \cdot 6 \end{bmatrix} \right\} \quad F_2 = \left\{ \begin{bmatrix} \circ 0 \\ \cdot 6 \\ \circ 6 \\ \circ 4 \end{bmatrix}, \begin{bmatrix} \circ 0 \\ \circ 6 \\ \circ 2 \\ \circ 4 \end{bmatrix}, \begin{bmatrix} \circ 0 \\ \circ 6 \\ \circ 2 \\ \cdot 4 \end{bmatrix}, \begin{bmatrix} \circ 0 \\ \circ 6 \\ \cdot 2 \\ \circ 4 \end{bmatrix} \right\}
\]

Here, the \( \bullet \)'s stand for the colors of the edges of a 2-cycle of \( G_{21} \), which are to be determined. In each CIV, one \( \bullet \) gets colored 1 and the other gets 5. Obviously, the choice of 5 and 1 on one CIV will affect another. Specifically, the choice of 5 and 1 on one of the CIVs \( v_i \) will affect one of the \( u_j \), but not the other \( v_i \). Similarly, the \( \circ \)'s correspond to colors of \( G_{22} \), which we are not concerned about, as we shall apply Lemma 3.5. Strictly speaking, each \( F_i \) consists of four classes of CIVs. However, we shall not differentiate between individual CIV and its class as this is immaterial.

We shall also extend the notion of a condition \( P_3^i \) or \( P_3^j \) being respected by a CIV to a class of CIV. A class of CIV respects \( P_3^i \) or \( P_3^j \) if the corresponding condition in Definition 4.1 is not violated yet. In this sense, each class of CIVs in \( F_1 \cup F_2 \) respects all \( P_3 \) and \( P_3^j \). Notice that no matter how we assign 1 and 5 in each 2-cycle of \( G_{21} \), the resulting CIVs in \( F_1 \cup F_2 \) (\( u_1 \) and \( u_2 \) in
particular) still respect $P_i'$. Moreover, $u_1$ and $u_2$ keep respecting $P_i'$ even after we apply $A(2,6)$ or $A(0,4)$.  
If both $v_1$ and $v_2$ belong to $F_1$, then in both $v_i$ we choose the $\bullet$ that goes with 0 or 2 to be 5 and the other $\bullet$ to be 1. Clearly, the $P_i$ are respected by $v_1$ and $v_2$.
If both $v_1$ and $v_2$ belong to $F_2$, then we apply $A(2,6)$ which move $v_1$ and $v_2$ to $F_1$ again.

Hence, we could now assume without loss of generality that $v_1 \in F_1$ and $v_2 \in F_2$, flipping $G$ horizontally if necessary (operation 5 of Proposition 3.3). Furthermore, due to operation 6 of flipping $G$ vertically, we can also assume that $u_1$ and $u_2$ don’t belong to the same $F_i$.

As the $\bullet$’s colors in $v_1$ can be chosen easily so that $v_1$ respects $P_i$ ($i = 1, 2, 3$), we try to modify the coloring so that $v_2$ does, too. If $v_2 = [\bullet 0 \bullet 2 \bullet 4]$, we can pick colors so that $v_2 = [\bullet 0 \bullet 6 \bullet 2 \bullet 4]$. When $v_2 = [\bullet 0 \bullet 6 \bullet 2 \bullet 4]$, we apply $A(0,6)$ and $A(2,4)$ to make $v_2 = [\bullet 0 \bullet 6 \bullet 2 \bullet 4]$ and choose $\bullet$’s colors similarly. Note that the availability of the $\bullet$’s colors in $v_1$ are not affected.

If $v_2 = [\bullet 0 \bullet 6 \bullet 2 \bullet 4]$ and $u_1 \in F_1$, we apply $LU(2,4)$, making $v_2 = [\bullet 0 \bullet 6 \bullet 2 \bullet 4]$, but keeping $P_i$ respected by the $v_j$ and $P_i'$ respected by the $u_j$ ($i = 1, 2, 3$, $j = 1, 2$). The $\bullet$’s colors in $v_2$ could now be picked as before. When $v_2 = [\bullet 0 \bullet 6 \bullet 2 \bullet 4]$ and $u_2 \in F_1$, we apply $LU(2,4)$, $UU(0,6)$, $UL(2,4)$, which turns $v_2$ into $[\bullet 0 \bullet 6 \bullet 2 \bullet 4]$ but keeps $v_1$ in $F_1$, still.

Lastly, when $v_2 = [\bullet 0 \bullet 6 \bullet 2 \bullet 4]$, we apply $A(0,6)$ and $A(2,4)$ and return to the previous case.

**Case 2.** It is not the case that the two type-1 components are $G_{11}$ and $G_{22}$ for any $i \in \{1, 2\}$. Without loss of generality, we assume $G_{11}$ and $G_{21}$ are of type-1. We start from the coloring shown in Figure 12, which clearly does not violate $P_1$ and $P_i'$ ($i = 1, 2, 3$).

![Figure 12: Case 2 of Lemma 3.7](image)

Again, there are 8 possible CIVs divided into two classes:

$$F_1 = \left\{ \begin{array}{c}
\begin{bmatrix} 0 & 5 \\ 0 & 1 \\ 4 & 1 \end{bmatrix}, \\
\begin{bmatrix} 0 & 5 \\ 0 & 1 \\ 4 & 1 \end{bmatrix}, \\
\begin{bmatrix} 0 & 5 \\ 0 & 1 \\ 4 & 1 \end{bmatrix}, \\
\begin{bmatrix} 0 & 5 \\ 0 & 1 \\ 4 & 1 \end{bmatrix}\end{array} \right\} = \left\{ \begin{array}{c}
\begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 4 & 1 \end{bmatrix}, \\
\begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 4 & 1 \end{bmatrix}, \\
\begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 4 & 1 \end{bmatrix}, \\
\begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 4 & 1 \end{bmatrix}\end{array} \right\} = F_2$$

**Remark 4.3.** A special property of $F_1$ is that for each vector $v \in F_1$, there is a proper assignment of colors 6 and 2 to the $\bullet$, i.e. one $\bullet$ gets 6, the other gets 2 such that $v$ respect all $P_i$ ($i = 1, 2, 3$). For each $v \in F_2$, a proper assignment exists if 5 and 1 get exchanged in $v$. Moreover, for any vector $v \in F_1 \cup F_2$, if we exchange colors of one of the pairs $\{4, 1\}$ or $\{0, 5\}$, we could properly assign colors 6 and 2 to the $\bullet$’s of $v$ to maintain $P_i$.  

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If both $v_1$ and $v_2$ are in $F_1$, then we properly assign 2 and 6 to the $\bullet$'s. While, if $v_1$ and $v_2$ are in $F_2$ we apply $A(5,1)$ and pick $\bullet$'s colors in the same way.

Now, assume $v_1 \in F_1$ and $v_2 \in F_2$. Also, by flipping $G$ vertically if necessary, we can assume that $u_1$ and $u_2$ are representatives from $F_1$ and $F_2$.

If $v_1 = [0 \bullet 5 4 \bullet 1]$, then apply $A(5,3)$ and $A(1,7)$ so that $v_1 = [0 \bullet 3 4 \bullet 7]$. In this case, $F_2$ becomes

$$F_2 = \begin{cases}
[07] \cdot [07] \cdot [07] \cdot [07] \\
[00] \cdot [00] \cdot [00] \cdot [00]
\end{cases}$$

Next, choose the $\bullet$'s colors in $v_1$ to turn it into $v_1 = [0 \bullet 63 4 \bullet 27]$. While, in $v_2$ the $\bullet$ in the first or second component gets 2 and the other gets 6. We have used up 6 colors $\{0, 4, 2, 6, 3, 7\}$. Only 1 and 5 are left for the type-2 component $G_{22}$, hence Lemma 3.5 applies.

If $v_2 = [0 \bullet 1 4 \bullet 5]$, then apply $A(5,7)$ and $A(1,3)$. The rest is similar.

Consequently, we only need to consider $v_1 \in A_1 := F_1 - \{[0 \bullet 5 4 \bullet 1]\}$, and $v_2 \in A_2 := F_2 - \{[0 \bullet 1 4 \bullet 5]\}$. $u_1$ and $u_2$ can be assumed to be representatives of $A_1$ and $A_2$ for the same reason as before.

If $v_1 = [05 \circ \circ 41 \circ \circ]$, then we flip $G$ vertically so that $u_1 = [05 \circ \circ 41 \circ \circ]$, and that $v_1$ and $v_2$ are now representatives of $A_1$ and $A_2$. Apply $UU(0,5)$ and $LU(1,4)$, then $F_i$ are still respected by $v_1$ and $v_2$. Moreover, 0 and 5 get exchanged in $v_1$, while 1 and 4 get switched in $v_2$. Proper assignments of 2 and 6 now exist for $v_1$ and $v_2$ by Remark 4.3.

Similarly, when $v_2 = [01 \circ \circ 45 \circ \circ]$ we apply $LL(0,5)$ and $UL(1,4)$ and proceed in the same manner.

Hence, we are left to consider the case where

$$v_1 \in B_1 := \begin{cases}
[05] \cdot [05] \\
[00] \cdot [41] \\
[01]
\end{cases}, \quad v_2 \in B_2 := \begin{cases}
[01] \cdot [01] \\
[00] \cdot [45] \\
[05]
\end{cases}$$

and, $u_1$ and $u_2$ are representatives of $B_1$ and $B_2$.

If $v_1 = [05 \circ \circ 4 \circ \bullet 1]$ and $u_1 \in B_2$, $u_2 \in B_1$, then we flip $G$ vertically, apply $UU(0,5)$ and Remark 4.3. While, if $v_1$ is the same but $u_1 = [05 \circ \circ 41 \circ \circ] \in B_1$ and $u_2 \in B_2$, then we apply $LU(1,4)$ and Remark 4.3.

When $v_1 = [0 \circ \circ 5 \circ \circ 41 \circ \circ]$ and $u_1 \in B_1$, $u_2 \in B_2$, then we flip $G$ vertically, apply $LU(1,4)$ and Remark 4.3. While, when $v_1$ is the same but $u_2 = [05 \circ \circ 4 \circ \circ \bullet 1] \in B_1$ and $u_1 \in B_2$, then we apply $LL(0,5)$ and Remark 4.3.

Therefore, there are 8 cases left, 4 of which are when $v_1 = u_1 = [05 \circ \circ 4 \circ \bullet 1]$, and $v_2, u_2 \in B_2$. The other 4 cases are when $v_1 = u_2 = [0 \circ \circ 5 \circ \circ 41 \circ \circ]$, and $v_2, u_1 \in B_2$. We shall consider these 8 cases in turn as follows.

(2a) $v_1 = u_1 = [05 \circ \circ 4 \circ \bullet 1]$, and $v_2 = u_2 = [01 \circ \circ 4 \circ \circ 5]$.

This case introduces a new technique which will be used in later cases and the proof of Lemma 3.8. Let us first take a look at Figure 13. The graph shown represents all graphs considered in this case. The two dashed edges at the upper half is a 2-cycle of $G_{12}$, and the other two dashed edges are from the other 2-cycle. The end points of the dashed edges go to the $\bullet$'s, but we don’t know which edge goes to which $\bullet$.

Figure 14 presents our solution to this case. The figure shows a “proof without words”.
Figure 13: Representative figure for case 2a of Lemma 3.7

Figure 14: Case 2a of Lemma 3.7
Basically, we consider 3 sub-cases represented sequentially by 3 drawings from left to right.

In the first sub-case, we fix the end points (in the square boxes) of a dashed edge in the upper half, letting all other dashed edges go freely. The second dashed edge in the upper half has only one choice to go, of course. The doubly headed arrow specifies that we could exchange colors 0 and 4 of those two dashed edges so that \( v_2 \) looks as shown. It is straightforward to check that in this sub-case \( v_1 \) and \( u_1 \) respect the \( P_1 \) and \( u_2 \) respect the \( P_i^c \).

In the second sub-case, we fix the end points of a dashed edge in the lower half. Assign colors to all edges as shown. The doubly headed arrow on the side means that no matter which dashed edge goes to which \( \bullet \), we still have \( v_1 \) respecting the \( P_i \).

The last drawing considers the only sub-case left. The figure is self-explaining.

\( (2b) \) \[ v_1 = u_1 = [05 \ 01 \ 40 \ 01], \quad v_2 = [01 \ 01 \ 40 \ 05], \quad \text{and} \quad u_2 = [00 \ 01 \ 45 \ 00]. \]

Figure 15 presents our solution to this case.

\( (2c) \) \[ v_1 = u_1 = [05 \ 01 \ 40 \ 01], \quad v_2 = [00 \ 01 \ 40 \ 05], \quad \text{and} \quad u_2 = [01 \ 01 \ 45 \ 00]. \]

Figure 16 presents our solution to this case.

\( (2d) \) \[ v_1 = u_1 = [05 \ 01 \ 40 \ 01], \quad v_2 = u_2 = [00 \ 01 \ 45 \ 00]. \]

Figure 17 presents our solution to this case.
Before proceeding to the proof of case (2e), we need another proposition, which is very useful later on in the proof of Lemma 3.8.

**Proposition 4.4.** Let $G$ be our graph to be colored as usual. For any $j \in \{1, 2\}$, let $\bar{j}$ be the element in $\{1, 2\} - \{j\}$. Assume the following hold:

(i) $G_{22}$ is a type-2 component of $G$ with the two 2-cycles going horizontally, i.e. one 2-cycle of $G_{22}$ goes from $v_1$ to $u_1$ and the other goes from $v_2$ to $u_2$.

(ii) There is a partial coloring of $G$, in which all edges of $G_{11}$, $G_{12}$, and $G_{21}$ are colored using twice each color in $\{0, 4, 2, 6, 1, 5\}$.

(iii) (The partially colored) $u_1$ and $u_2$ respect the $P_i^t$ ($i = 1, 2, 3$.)

(iv) (The partially colored) $v_j$ respects the $P_i$ ($i = 1, 2, 3$)

(v) $v_j$ respects $P_1$ and $P_2$ but does not respect $P_3$ because one component of $v_j$ contains some $c \in \{2, 6\}$ and another color $c'$, but $c$ and $c'$ are not representatives of $\{0, 1, 2, 3\}$ and $\{4, 5, 6, 7\}$.

(vi) The edges get color $c$ go horizontally from $v_j$ to $u_j$ and $v_j$ to $u_7$. Moreover, $c$ goes with $\circ$ in a component of $u_j$, i.e. the edge $e$ that gets colored $c$ has the same right end point as another edge $e'$ in the 2-cycle that goes from $v_j$ to $u_j$.

Then, $G$ can be properly colored.

The last drawing in Figure 18 is an example of such a situation. In this drawing, $j = 2$, $c = 2$ and $c' = 1$.

**Proof.** Firstly, assume $c = 2$. Clearly, just as in the proof of Lemma 3.5, we can assign 3 and 7 to edges of the 2-cycle going from $v_j$ to $u_j$ so that $v_j$ respects the $P_i$ and $u_j$ respects the $P_i^t$. Suppose

$$v_j = [\alpha' \ c_1 \circ \ c_2 c_3 \ c_4 c]$$

where $\{c, c', c_1, c_2, c_3, c_4\} = \{0, 4, 2, 6, 1, 5\} = \{0, 1, 2, 4, 5, 6\}$ because $v_j$ respects $P_1$. As $\alpha'$ is the only component that does not respect $P_3$ (i.e. $c = 2$ and $c'$ are in $\{0, 1, 2, 3\}$), $c_2 c_3$ are representatives of $\{0, 1, 2, 3\}$ and $\{4, 5, 6, 7\}$. Thus, $c_1, c_4 \in \{0, 4, 5\}$. In fact, as $v_j$ respects $P_2$, $c_1 \in \{4, 5\}$, $c' \in \{0, 1\}$ and thus $6 \in \{c_2, c_3, c_4\}$. 

![Figure 17: Case 2d of Lemma 3.7](image-url)
Now, assign 7 to \( e' \) and 3 to the other edge of the 2-cycle that goes from \( v_j \) to \( u_j \). We claim that after exchanging colors (7 and 2) of \( e \) and \( e' \), we have a valid coloring of \( G' \). Indeed, the CIVs \( v_j \) and \( u_j \) are not affected. \( u_j \) respects \( P^t_i \) after the first assignment of 7 and 3, and keeps respecting \( P^t_i \) as we have just exchanged colors of edges having the same right end point. We only need to be concerned about \( v_j \) after this exchange. There are two cases for \( v_j \) before the exchange:

\[
v_j = [2c' \ c_17 \ c_2c_3 \ c_43]
\]

or

\[
v_j = [2c' \ c_13 \ c_2c_3 \ c_47]
\]

The reader can easily check that \( v_j \) does respect \( P^t_i \) after the exchange of 2 and 7.

The case where \( c = 6 \) is done similarly. The only difference is that \( e \) gets 3 this time. \( \square \)

**Remark 4.5.** This proposition will prove to be very useful in the proof of Lemma 3.8. It could be stated in a much more general fashion, however we did not do so because we will need only this instance of the proposition, and because the general statement would be too notationally heavy, thus hard to grasp.

\[(2e) \ v_1 = u_2 = [\bullet \ 5 \ 41 \ \bullet], \text{ and } v_2 = u_1 = [\bullet \ 40 \ 4 \ 5].\]

Figure 18 presents the partial solution to this case. The reason this case was only partially solved is due to the last drawing, i.e. the last sub-case, in which there is a violation of \( P_3 \) in \( v_2 \). To resolve this violation, we reason as follows. If the 2-cycles of \( G_{22} \) go horizontally,

![Figure 18: Case 2e of Lemma 3.7](image)

we apply Proposition 4.4. Otherwise, the 2-cycle going from \( v_1 \) to \( u_2 \) could be colored with 3 and 7 properly, as in Lemma 3.5. The other 2-cycle has edges going from the blacken vertices at \( u_1 \) to the \( \circ \) of \( v_2 \). We assign 7 to the edge whose right end point is the second component of \( u_1 \) and 3 to the other edge.

There are two cases after this assignment, depending on what \( v_2 \) ends up being. If \( v_2 \) ends up to be \( v_2 = [12 \ 74 \ 36 \ 05] \), then we apply \( LU(1, 7) \). Obviously, \( u_1 \) still respects the \( P^t_i \), and \( v_2 \) respects all the \( P_i \) now. If \( v_2 \) becomes \( v_2 = [12 \ 34 \ 76 \ 05] \), then we apply \( LU(1, 7) \) and \( LL(0, 2) \) to get the same result. In these two situations, the relative positions of 4 and 5 are not important.

Lastly, note that we have used 6 colors \{0, 1, 2, 3, 4, 5\} in the first sub-case, leaving \{6, 7\}. Lemma 3.5 still applies.
(2f,2g,2h) When \(v_1 = u_2 = [0\circ \bullet 5 \ 41 \ \circ\circ]\), we have three more cases, which are simple and close enough to be considered at once: (2f) \(v_2 = [0\circ \bullet 4 \bullet 5]\), and \(u_1 = [0\circ \bullet 1 \ 45 \ \circ\circ]\); (2g) \(v_2 = [0\circ \bullet 1 \ 45 \ \circ\circ]\), and \(u_1 = [0\circ \bullet 4 \bullet 5]\); and (2h) \(v_1 = u_2 = [0\circ \bullet 5 \ 41 \ \circ\circ]\), \(v_2 = u_1 = [0\circ \bullet 1 \ 45 \ \circ\circ]\).

Figure 19 presents the solutions all three cases.

Figure 19: Case 2f, 2g and 2h of Lemma 3.7

## 5 Proof of Lemma 3.8

We assume that the 2-cycles of \(G_{22}\) go horizontally, i.e. one goes from \(0123\) to \(0123\), and the other from \(4567\) to \(4567\), applying operation 4 of Proposition 3.3 if necessary. We start from the coloring shown in Figure 20, whose CIV \(v_i\) respect \(P_k\) and \(u_j\) respect \(P^*_2\) \((i = 1, 2, 3, \ j = 1, 2)\). Moreover, applying both \(A(0,2)\) and \(A(4,6)\) would yield another coloring where \(P_k\), and \(P^*_2\) are not violated. So does applying any of \(A(0,4)\), \(A(1,5)\), or \(A(2,6)\).

The possible CIVs are the union of four sets:

\[
F_1 = \left\{ \begin{bmatrix} 50 \\ 14 \\ 06 \end{bmatrix}, \begin{bmatrix} 50 \\ 04 \\ 16 \end{bmatrix}, \begin{bmatrix} 00 \\ 14 \\ 06 \end{bmatrix}, \begin{bmatrix} 00 \\ 04 \\ 16 \end{bmatrix} \right\}
F_2 = \left\{ \begin{bmatrix} 10 \\ 02 \\ 06 \end{bmatrix}, \begin{bmatrix} 10 \\ 02 \\ 06 \end{bmatrix}, \begin{bmatrix} 00 \\ 04 \\ 06 \end{bmatrix}, \begin{bmatrix} 00 \\ 04 \\ 06 \end{bmatrix} \right\}
F_3 = \left\{ \begin{bmatrix} 50 \\ 06 \\ 12 \end{bmatrix}, \begin{bmatrix} 50 \\ 06 \\ 12 \end{bmatrix}, \begin{bmatrix} 00 \\ 04 \\ 12 \end{bmatrix}, \begin{bmatrix} 00 \\ 04 \\ 12 \end{bmatrix} \right\}
F_4 = \left\{ \begin{bmatrix} 10 \\ 02 \\ 06 \end{bmatrix}, \begin{bmatrix} 10 \\ 02 \\ 06 \end{bmatrix}, \begin{bmatrix} 00 \\ 04 \\ 06 \end{bmatrix}, \begin{bmatrix} 00 \\ 04 \\ 06 \end{bmatrix} \right\}
\]
Notice that for all $i = 1, 2, 3$, the vectors in $F_1$ respect $P_i$, $F_2$ respect $P_i$ after applying $A(1,5)$, $F_3$ respect $P_i$ after applying $A(2,6)$, and $F_4$ respect $P_i$ after applying $A(0,4)$. Hence, if $v_1$ and $v_2$ belong to the same $F_i$ then we are done. We could thus also assume $u_1$ and $u_2$ are from different $F_i$'s. Consider 4 cases.

**Case 1.** $v_1 \in F_1$ and $v_2 \in F_3 \cup F_4$. If $v_2 \in F_3$, applying $A(0,2)$ and $A(4,6)$ would keep $v_1$ in $F_1$, while move $v_2$ to $F_3$. Hence, without loss of generality we can assume $v_1 \in F_1$ and $v_2 \in F_3$. Consider 4 sub-cases as follows, which are ordered in increasing in level of complexity.

(1a) $v_2 = [50 \ 06 \ 14 \ 02]$. In this case, $v_2$ and $v_1$ already respect the $P_i$.

(1b) $v_2 = [00 \ 56 \ 14 \ 02]$. In this case, we can not apply Lemma 3.5 directly, but have to go further. There is no problem with picking colors 3 and 7 for the 2 cycle of $G_{22}$ connecting $v_1$ to $u_1$. For the other 2-cycle, we pick colors and do certain transformation as follows. Notice that no matter what $w_2$ is, the 5 in $w_2$ is not in the same component as a 0. There is exactly an edge $e$ of this 2-cycle where $r(e)$ and 5 are both in the first two components or the last two components of $w_2$. Assign 3 to $e$ and 7 to the other edge. There are two cases depending on how $v_2$ ends up to be after this assignment.

If $v_2$ becomes $v_2 = [30 \ 56 \ 14 \ 72]$, then we apply $LL(5,3)$, which keeps $u_2$ respecting the $P_i$ and makes $v_2$ respect the $P_i$.

If $v_2$ becomes $v_2 = [70 \ 56 \ 14 \ 32]$, then we apply $LL(5,3)$, $LU(4,6)$ and $UU(0,2)$. Effectively, this transformation makes $v_2 = [70 \ 34 \ 16 \ 52]$, respecting the $P_i$, exchanges each pair $\{0, 2\}$ and $\{4, 6\}$ in $u_1$, and exchanges $\{0, 2\}$ in $v_1$. Hence, after the transformation $u_1$ still respects the $P_i$, and $v_1$ is still in $F_1$, respecting the $P_i$.

(1c) $v_2 = [50 \ 06 \ 04 \ 12]$. If $w_2$ has a 0 as a component (we will write 02 as $w_2$ for short), then apply Proposition 4.4. If 52 is $w_2$ and 00 is $w_2$, then we apply $LL(5,2)$, then $A(0,2)$ and $A(4,6)$, so that $v_2 = [20 \ 04 \ 06 \ 51]$. Proposition 4.4 could be applied now with $c = 2$.

If 14 is $u_1$, then apply $LU(1,4)$. If 16 is $u_1$, then apply $LU(1,6)$, $LL(0,2)$, and $UL(4,6)$.

Hence, we are left to consider the case where 02 $\notin u_2$, not both 52 and 00 are in $u_2$, and 14 and 16 are not in $u_1$.

As we have seen, the CIVs are useful in classifying how each subset of vertices $\{0, 1, 2, 3\}$ and $\{4, 5, 6, 7\}$ of $I$ and $O$ are connected to the others. There is another way to classify the “shape” of these connecting patterns. Ignoring the edges of $G_{22}$, each subset $\{0, 1, 2, 3\}$ and $\{4, 5, 6, 7\}$ of $I$ and $O$ are connected to three horizontal edges (i.e. edges connecting $v_j$ and $v_j$) and three diagonal edges (i.e. edges connecting $v_j$ and $v_j$). The connecting patterns to each of these vertex subsets can be classified based on the relative end points of the horizontal and diagonal edges, up to the application of the first 3 operations of Proposition 3.3. Figure 21 shows all possible shapes of $u_1$. In the figure, the thicken edges represent edges that go from $u_1$ to $v_1$ (horizontally). The other three edges go from $u_1$ to $v_2$ (diagonally). It is straightforward to check that there are only 5 possible shapes as shown, up to applying the first three operations of Proposition 3.3. For example, if $u_1 = [50 \ 02 \ 04 \ 16]$, then it is of shape 1. While, $[00 \ 12 \ 54 \ 06]$ is of shape 3, and $[00 \ 16 \ 54 \ 02]$ is of shape 2. Similarly, Figure 22 shows all possible shape of $u_2$. These are just mirror images of the shapes of $u_1$. We have tentatively label the edges in the shapes shown. Clearly, some ambiguity exists with the labels of two horizontal or diagonal
edges which share the same end point. However, as we shall see later, these ambiguities are not important when we refer to these labels.

Because 14 and 16 are not in \( u_1 \), \( u_1 \) can only be of shape 3 or 5, because given the way we construct and color \( G_{11} \) and \( G_{12} \), the even-numbered edges can not share an end point. Similarly, as not both \( 52 \) and \( 50 \) are in \( u_2 \) and \( 52 \notin u_2, u_2 \) can only be of shape 3 or 5. For, if \( u_2 \) is of shape 1 or 4, then either \( h_1 \) was colored 2 which makes \( c_2 \in u_2 \), or \( h_1 \) was colored 0 and thus both \( 52 \) and \( 50 \) are in \( u_2 \). If \( u_2 \) is of shape 2, then \( h_2 \) must have been colored 5, which makes \( c_2 \in u_2 \) (2 is the color of \( h_1 \) or \( h_3 \).)

Now, we describe a way to recolor the 6 edges of \( G \) in \( G_{11}, G_{12}, \) and \( G_{21} \). The idea is to use color 1, 2 and 5 to color the horizontal edges, and 0, 4, 6 to color the diagonal edges. The reader might find it useful sketching several figures while we are discussing the coloring.

Let’s start by coloring the edges connected to \( u_1 \). When \( u_1 \) is of shape 3, we assign 2 to \( h_1 \), and \( \{5, 1\} \) to \( \{h_2, h_3\} \). What we mean by this is that one of \( \{h_2, h_3\} \) will get 1 and the other will get 5, but which edge actually gets which color is to be decided later. \( h_1, h_2 \) and \( h_3 \) are connected to the first two components of \( v_1 \) (vertices 0 and 1), because \( v_1 \in F_1 \). If some vertex \( j \in \{0, 1\} \) is incident to two of these \( h_i \) \( (i = 1, 2, 3) \), \( j \) must be incident to some \( h_i \) where \( i \in \{2, 3\} \). Assign 5 to \( h_i \), 1 to the edge in \( \{h_2, h_3\} \) \( \{h_i\} \). Clearly, this way we get a (partially colored) \( v_1 \) respecting the \( P_i \). Now we have to assign 0, 4, and 6 to the diagonal edges connected to \( u_1 \). Assign 0 to the edge \( d_i \) that was originally colored 6. If \( i = 1 \), then assign 4 and 6 to \( d_2 \) and \( d_3 \) arbitrarily. If \( i \neq 1 \), the assign 6 to the edge in \( \{d_2, d_3\} \) \( \{d_i\} \), and 4 to \( d_1 \). Either way, \( u_1 \) respects the \( P_j \) \( (j = 1, 2, 3) \), and \( v_2 \) looks

---

**Figure 21:** All possible shapes of \( u_1 \)

**Figure 22:** All possible shapes of \( u_2 \)
Like

\[
v_2 = \begin{bmatrix}
  c_1 & c_2 \\
  c_0 & c_4 \\
  6c_3 \\
\end{bmatrix}
\] or
\[
v_2 = \begin{bmatrix}
  c_1 & c_2 \\
  c_0 & c_6 \\
  4c_3 \\
\end{bmatrix}
\]

where \(\{c_1, c_2, c_3\} = \{1, 2, 5\}\) are to be precisely determined when we color the horizontal edges connected to \(v_2\). The case where \(u_1\) is of shape 5 is done in exactly the same way.

Secondly, we need to color the edges connected to \(u_2\). When \(u_2\) is of shape 3, we assign 1 to the edge \(h_i\) that was originally colored 2. Clearly, \(i \neq 3\) because \(c_2 \notin u_2\). Assign 2 to \(h_3\) and 5 to the last edge. When \(u_2\) is of shape 5, we also assign 1 to the edge \(h_i\) that was originally colored 2. If \(i \in \{2, 3\}\), then assign 2 to the edge in \(\{h_2, h_3\} = \{h_i\}\) and 5 to the last edge. If \(i = 1\), then assign 2 and 5 to \(h_2\) and \(h_3\) arbitrarily. Either way, \(u_2\) still respects the \(P_j^r\) and \(v_2\) must either be

\[
v_2 = \begin{bmatrix}
  52 \\
  c_0 \\
  c_4 \\
  61 \\
\end{bmatrix}
\] or
\[
v_2 = \begin{bmatrix}
  52 \\
  c_0 \\
  c_6 \\
  41 \\
\end{bmatrix}
\]

which clearly respects the \(P_j\).

The diagonal edges of \(u_2\) can be assigned 0, 4 and 6 in much the same way. \(d_1\) or \(d_2\) will get 6, the other two get 0 or 4 which could be exchanged to ensure \(v_1\) respects the \(P_j\).

(1d) \(v_2 = \begin{bmatrix}
  c_0 & 56 & c_4 & 12 \\
\end{bmatrix}\).

If \(14 \in u_1\), then apply \(LU(1, 4)\) to make \(v_2 = \begin{bmatrix}
  c_0 & 56 & c_1 & 42 \\
\end{bmatrix}\) and then proceed similar to case (1b).

If \(16 \in u_1\), then apply \(LU(1, 6)\), then \(LL(0, 2)\) and \(UL(4, 6)\). Hence, we could now assume that \(u_1\) is of shape 3 or 5.

If \(52 \in u_2\), then applying \(LL(2, 5)\), \(UU(0, 2)\) and \(LU(4, 6)\) would do.

Now, suppose \(c_2 \notin u_2\). As \(52 \notin u_2\), \(u_2\) can only be of shape 3 or 5. Proceeding in the same manner as case (1c) completes the case. Consequently, we could now assume that \(52 \notin u_2\) and \(c_2 \notin u_2\). If \(50 \in u_2\), then apply \(LL(5, 0)\), making \(v_2 = \begin{bmatrix}
  c_5 & 06 & c_4 & 12 \\
\end{bmatrix}\).

Proposition 4.4 proves to be useful here again. When \(52 \notin u_2\) and \(c_2 \notin u_2\) and \(50 \notin u_2\), it is not hard to show that \(u_2\) is of shape 2 or 3.

To summarize, we are left with the cases where \(u_1\) is of shape 3 or 5, and \(u_2\) is of shape 2 or 3. We will consider these 4 possible sub-cases in turn.

Firstly, suppose \(u_1\) is of shape 3 and \(u_2\) is of shape 3. We will color the horizontal edges with 0, 2 and 4, the diagonal edges with 1, 5 and 6. For \(u_1\)'s edges, assign 2 to \(h_1\) and \(\{0, 4\}\) to \(\{h_2, h_3\}\). Which edge gets 0 or 4 can be decided in the same manner as in case (1c) with 1 and 5. Assign 6 to \(d_3\) and \(\{1, 5\}\) to \(\{d_1, d_2\}\). Note that one of \(d_1\) or \(d_2\) must have been colored 1 originally. Assign 1 to that originally-1 edge and 5 to the other. After this assignment, \(v_2\) looks like:

\[
v_2 = \begin{bmatrix}
  c_1 \\
  c_2 & 6 \\
  c_5 \\
  1c_3 \\
\end{bmatrix}
\] or
\[
v_2 = \begin{bmatrix}
  c_1 \\
  c_2 & 5 \\
  c_6 \\
  1c_3 \\
\end{bmatrix}
\]
For $w_2$'s diagonal edges, assign 6 to $d_1$, and $\{1, 5\}$ to $\{d_2, d_3\}$. For $w_2$'s horizontal edges, assign 2 to $h_3$ (which must have been originally colored 2 because $\sigma 2 \in w_2$), and 4 to $h_2$ and 0 to $h_1$. After this assignment, $v_2$ can only be one of four forms:

$$v_2 = \begin{bmatrix} \sigma 4 \\ 06 \\ \sigma 5 \\ 12 \end{bmatrix} \quad \text{or} \quad v_2 = \begin{bmatrix} \sigma 0 \\ 46 \\ \sigma 5 \\ 12 \end{bmatrix} \quad \text{or} \quad v_2 = \begin{bmatrix} \sigma 4 \\ 05 \\ \sigma 6 \\ 12 \end{bmatrix} \quad \text{or} \quad v_2 = \begin{bmatrix} \sigma 0 \\ 45 \\ \sigma 6 \\ 12 \end{bmatrix}$$

The first form is good due to Proposition 4.4. The second form is good after applying $LL(0, 4)$ and Proposition 4.4. The third and fourth form is good after applying $LL(2, 4)$.

Secondly, suppose $u_1$ is of shape 3 and $u_2$ is of shape 2. We can assume that the edge $h_1$ of $u_2$ was originally colored 2, otherwise $h_3$ was and hence the previous trick still applies. Moreover, $h_2$ must have been 5 originally as an originally odd-number edge must have shared an end point with another edge. Let us first try to assign colors to $u_1$’s edges. This time we use 0, 2, and 4 for horizontal and 1, 5, and 6 for diagonal edges. As “usual”, assign 2 to $h_1$ of $u_1$, $\{0, 4\}$ to $\{h_2, h_3\}$. Starting from here, there are two sub-cases: (i) when $d_3$ was originally 4; and (ii) when $d_3$ was originally 6.

If $d_3$ was originally 4, then assign 6 to $d_3$ and $\{1, 5\}$ to $\{d_1, d_2\}$ appropriately so that

$$v_2 = \begin{bmatrix} \sigma c_1 \\ \sigma 2 \\ \sigma 5 \\ \sigma 6 \\ 1c_3 \end{bmatrix}$$

Then, for the diagonal edges of $u_2$, assign 6 to $d_1$ and $\{1, 5\}$ to $\{d_2, d_3\}$ with 1 and 5 exchangeable to make $v_1$ valid. For the horizontal edges of $u_2$, assign 4 to $h_1$ (which was originally 2) and $\{0, 2\}$ to $\{h_2, h_3\}$ arbitrarily. After this assignment, $v_2$ becomes either

$$v_2 = \begin{bmatrix} \sigma 0 \\ 25 \\ \sigma 6 \\ 14 \end{bmatrix} \quad \text{or} \quad v_2 = \begin{bmatrix} \sigma 2 \\ 05 \\ \sigma 6 \\ 14 \end{bmatrix}$$

which obviously respect the $P_k$.

When $d_3$ was originally 6, the situation is more complicated. We will use 0, 6 and 5 for horizontal and 1, 2 and 4 for diagonal edges. For $u_1$’s edges, assign 5 to $h_3$ and $\{0, 6\}$ to $\{h_1, h_2\}$. 0 and 6 could be exchanged to make $v_1$ respect the $P_k$. Next, assign 2 to $d_3$, 1 to $d_1$ and 4 to $d_2$. After this assignment, $v_2$ can be one of two forms:

$$v_2 = \begin{bmatrix} \sigma c_1 \\ \sigma 2 \\ \sigma 1 \\ 4c_3 \end{bmatrix} \quad \text{or} \quad v_2 = \begin{bmatrix} \sigma c_1 \\ \sigma 2 \\ \sigma 4 \\ 1c_3 \end{bmatrix}$$

Now it is $u_2$’s edges’ turn. Assign 2 to $d_3$, $\{1, 4\}$ exchangeable to $\{d_1, d_2\}$. Next, assign 6 to $h_1$, 5 to $h_2$ (which was 5 originally), and 0 to $h_3$. $v_2$ could now only be:

$$v_2 = \begin{bmatrix} \sigma 0 \\ 52 \\ \sigma 1 \\ 46 \end{bmatrix} \quad \text{or} \quad v_2 = \begin{bmatrix} \sigma 0 \\ 52 \\ \sigma 4 \\ 16 \end{bmatrix}$$

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The second form already respects the $P_i$. The second form is good too, due to Proposition 4.4.

Thirdly, suppose $u_1$ is of shape 5 and $u_2$ is of shape 2. We handle this case differently: we will color the edges so that $v_1$ and $v_2$ respect the $P_i$, and $u_1$ and $u_2$ respect the $P_i$. A vertical flip of $G$ would complete the case. Notice that at $u_1$ the original color of $d_2$ must have been 1, and at $u_2$ the original color of $h_2$ must have been 5. We will use 0, 2 and 5 for horizontal edges and 1, 4 and 6 for diagonal edges.

For $u_1$’s diagonal edges, assign 4 to $d_2$, 1 to the edge that was 6 and 6 to the edge that was 4 originally. After this assignment, $u_1$ is of one of two forms:

$$u_1 = \begin{bmatrix} c_1 & c_2 \\ 1 & 6 \\
3 & 4 \end{bmatrix} \quad \text{or} \quad u_1 = \begin{bmatrix} c_1 & c_2 \\ 6 & 0 \\
3 & 4 \end{bmatrix}$$

and $v_2$ must be $[0 \ 2 \ 0 \ 1 \ 06 \ 45]$, which respects the $P_i$. Where $\{c_1, c_2, c_3\} = \{1, 4, 6\}$ whose exact assignment will be decided later. If $u_2$ is of the first form, assign 0 to $h_3$ and $\{2, 5\}$ to $\{h_1, h_2\}$ exchangeable for $v_1$ to respect the $P_i$. If $u_2$ is of the second form, assign 2 to $h_3$ and $\{0, 5\}$ to $\{h_1, h_2\}$ exchangeable for $v_1$ to respect the $P_i$.

For $u_2$’s horizontal edges, assign 0 to $h_2$, 5 to the edge that was 2, and 2 to the edge that was 0 originally. After this assignment, $u_2$ is one of two forms:

$$u_2 = \begin{bmatrix} c_1 & c_2 \\ 2 & 5 \\
3 & 0 \end{bmatrix} \quad \text{or} \quad u_2 = \begin{bmatrix} c_1 & c_2 \\ 5 & 0 \\
3 & 0 \end{bmatrix}$$

Here $\{c_1, c_2, c_3\} = \{0, 2, 5\}$ whose exact assignment will be decided later. If $u_1$ is of the first form, assign 6 to $d_3$ and $\{1, 4\}$ to $\{d_1, d_2\}$ exchangeable for $v_1$ to respect the $P_i$. If $u_1$ is of the second form, assign 4 to $d_3$ and $\{1, 6\}$ to $\{d_1, d_2\}$ exchangeable for $v_1$ to respect the $P_i$. It is also easy to check that $v_2$ respects the $P_i$.

Lastly, suppose $u_1$ is of shape 5 and $u_2$ is of shape 3. In this case, if $d_2$ and $d_3$ of $u_2$ don’t share the left end point, then we can reuse the previous trick. If $d_2$ and $d_3$ of $u_2$ have the same left end point, we start coloring in the usual way, namely for the $v_j$ to respect the $P_i$ and $u_j$ to respect the $P_i$. Note that at $u_1$ the color of $d_2$ originally was 1, and at $u_2$ the original color of $h_3$ was 2 (because $c_2 \in u_2$). We have to consider two cases, depending on the original color of $d_1$ at $u_1$: (i) $d_1$ was 6; and (ii) $d_1$ was 4.

If at $u_1$, $d_1$ was originally 6 then we use 0, 5, and 6 for the horizontal edges and 1, 2, and 4 for the diagonal edges. We start with $u_1$’s horizontal edges, assigning 6 to $h_3$ and $\{5, 0\}$ to $h_1$ and $h_2$. Then, 2 to $d_1$, 1 to $d_2$ and 4 to $d_3$. After this assignment, we must have

$$v_2 = \begin{bmatrix} 0 & c_1 \\ 2 & 0 \\
4 & 1 \end{bmatrix}$$

Now, let’s assign colors to the horizontal edges of $u_2$. We assign 6 to $h_3$, whose original color was 2, and $\{5, 0\}$ to $\{h_1, h_2\}$, whose exact assignment is such that $v_2$ respects $P_i$. 

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This is certainly possible. After the assignment, we either have

\[ u_2 = \begin{bmatrix} \alpha_1 \\ \alpha_0 \\ \alpha_3 \\ \alpha_6 \end{bmatrix} \text{ or } u_2 = \begin{bmatrix} \alpha_1 \\ \alpha_5 \\ \alpha_0 \\ \alpha_6 \end{bmatrix} \]

where, \( \alpha_i \) is the color of \( d_i \) to be chosen from \( \{1, 2, 4\} \). We assign \( 2 \) to \( d_1, 4 \) to \( d_2 \) and \( 1 \) to \( d_3 \) if \( u_2 \) is in the first form. Assign \( 2 \) to \( d_1, 1 \) to \( d_2 \) and \( 4 \) to \( d_3 \) if \( u_2 \) is in the second form. Recall that \( d_2 \) and \( d_3 \) shares the left end point, hence \( v_1 \) keeps respecting the \( P_i \).

If at \( u_1 \), \( d_1 \) was originally \( 4 \) then we use \( 1, 5 \), and \( 6 \) for horizontal and \( 0, 2 \), and \( 4 \) for vertical edges. Starting first with \( u_1 \)'s edges, we assign \( 6 \) to \( h_1 \), \( \{1, 5\} \) to \( \{h_2, h_3\} \), which are exchangeable to make \( v_1 \) respect the \( P_i \). Then, assign \( 2 \) to \( d_3, 4 \) to \( d_1 \) and \( 0 \) to \( d_2 \), making

\[ v_2 = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_4 \\ \alpha_3 \end{bmatrix} \]

With \( u_2 \)'s edges, assign \( 6 \) to \( h_3 \) and \( \{5, 1\} \) to \( \{h_1, h_2\} \) such that

\[ v_2 = \begin{bmatrix} \alpha_1 \\ 52 \\ \alpha_4 \\ 06 \end{bmatrix} \]

which respects the \( P_i \). After this assignment, \( u_2 \) is either

\[ u_2 = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_6 \end{bmatrix} \text{ or } u_2 = \begin{bmatrix} \alpha_1 \\ \alpha_5 \\ \alpha_1 \\ \alpha_6 \end{bmatrix} \]

If \( u_2 \) is in the first form, assign \( 0 \) to \( d_3 \) and \( \{2, 4\} \) to \( \{d_1, d_2\} \), which are exchangeable to make \( v_1 \) valid. If \( u_2 \) is in the second form, assign \( 4 \) to \( d_3 \) and \( \{0, 2\} \) to \( \{d_1, d_2\} \) arbitrarily.

**Case 2.** \( v_1 \in F_1 \) and \( v_2 \in F_2 \). In this case, we flip \( G \) vertically so that \( u_1 \in F_1 \) and \( u_2 \in F_2 \). Clearly \( u_1 \) and \( u_2 \) respect \( P_i \). We need to modify the coloring so that \( v_1 \) and \( v_2 \) respect \( P_i \). (\( P_i \) is already respected.) Since \( u_1 \in F_1 \) and \( u_2 \in F_2 \), we can apply any sequence of \( UU(0, 2) \), \( LU(4, 6) \), \( UL(4, 6) \) and \( LL(0, 2) \) while keeping \( P_i \) respected by the \( u_j \). As case 1 has been done, we can assume that \( v_1 \) and \( v_2 \) are not representatives of \( F_1 \) and \( F_3 \cup F_4 \). Consider 3 sub-cases left:

(2a) \( v_1 \in F_1 \) and \( v_2 \in F_2 \). In this case, apply \( UU(0, 2) \) and/or \( UL(4, 6) \) if necessary to make \( v_1 = [50 \ 02 \ 14 \ 06] \). Apply \( LU(4, 6) \) as needed so that \( 14 \) is one component of \( u_1 \). Apply \( LL(0, 2) \) if necessary so that \( v_2 \) is either \( [10 \ 02 \ 54 \ 06] \) or \( [10 \ 02 \ 4 \ 56] \). Next, as \( 14 \) is a component of \( u_1 \), we can apply \( LU(1, 4) \), so that \( v_2 \) is either \( [40 \ 02 \ 51 \ 06] \) or \( [40 \ 02 \ 1 \ 56] \). The first vector already respects \( P_i \). The second vector respects \( P_i \) after we switch \( 2 \) and \( 6 \), so does \( v_1 \). Hence, we apply \( A(2, 6) \) if necessary to make both \( v_1 \) and \( v_2 \) respect \( P_i \).

(2b) \( v_1 \in F_2 \) and \( v_2 \in F_1 \). This is quite similar to the previous case. Apply \( LL(0, 2) \) and/or \( LU(4, 6) \) if necessary to make \( v_2 = [50 \ 02 \ 14 \ 06] \). Apply \( UU(0, 2) \) as needed so that \( 50 \) is one component of \( u_1 \). Apply \( UL(4, 6) \) if necessary so that \( v_1 \) is either \( [10 \ 02 \ 54 \ 06] \) or \( [00 \ 12 \ 54 \ 06] \). Next, as \( 50 \) is a component of \( u_1 \), we can apply \( UU(0, 5) \), so that \( v_1 \) is either \( [15 \ 02 \ 04 \ 06] \) or \( [05 \ 12 \ 04 \ 06] \). The first vector already respects \( P_i \). The second vector respects \( P_i \) after we switch \( 2 \) and \( 6 \), so does \( v_2 \). Hence, we apply \( A(2, 6) \) if necessary to make both \( v_1 \) and \( v_2 \) respect \( P_i \).
and
and
and
and
and
and
and
and
and
and
and
and
and
and
and
and
and
to,
we
do
is
but
,
This
would
,mak
, and
,as
needed
, Finally,
we
apply
A(0,4),
making
v1
and
v2
both
respect
P3.
The
dition
when
v1
F2
and
v2
F3
F4
is
d.

Case 3. v1 F2 and v2 F3 F4. As A(0,2) and A(4,6) transform a vector from F4 to F3,
while
keeping
a
vector
from
F2
in
F2,
we
only
need
to
consider
v1
F2
and
v2
F3.
If
v1
[ 0 12 4 56],
then
we
first
apply
A(0,2)
and
A(4,6).
This
would
make
v1
[ 10 0 2 54 0 6],
and
move
v2
into
F4.
Secondly,
A(0,4)
makes
both
v1
and
v2
respect
the
P2.
If
v1
[ 10 0 2 54 0 6]
and
u1
F2,
then
apply
these
in
order:
LU(4,6),
LL(0,2),
UL(4,6),
and
A(0,4).
While
if
v1
is
the
same
but
u2
F2,
we
do
UL(4,6),
and
A(0,4).

If
v1
[ 00 12 54 0 6]
and
u1
F2,
we
apply
a
sequence
of
transformations:
UU(0,2),
then
A(0,2)
and
A(4,6),
then
A(0,4).
While
if
v1
is
the
same
but
u2
F2,
we
do
UU(0,2),
LU(4,6),
LL(0,2),
and
lastly
A(0,4).

Consequently,
the
cases
left
to
be
considered
are:

• v1
[ 10 0 2 54 0 6],
v2
F3,
u1
and
v2
are
representatives
of
some
Fi
and
Fj
with
i
j;

• v1
[ 10 0 2 54 0 6],
u1
and
v2
are
representatives
of
F3
and
F4.

We
consider
all
these
sub-cases
in
turn,
using
the
technique
“proof
without
words”
introduced
in
the
previous
section.

(3a) v1
[ 10 0 2 54 0 6],
v2
F3,
u1
F2,
and
v2
F3.
Figure
23
shows
our
solution
to
this
case.
The
dashed
edges
are
from
G21,
the
thicken
edges
are
from
G11,
and
the
rest
of
the
edges
are
from
G12.
As
we
are
considering
the
case
where
v2
F3,
u1
F2,
and
v2
F3,
there
are
6
degrees
of
flexibility
for
the
1
and
5
to
move
within
either
the
first
two
components
or
the
last
two
components
of
the
associated
CIVs.
This
corresponds
precisely
to
moving
the
end
points
of
the
dashed
edges,
within
the
first
two
components
or
the
last
two
components
of
their
CIVs.

In
reality,
there
are
totally
26
= 64
cases.
In
Figure
23,
we
consider
only
three
cases
where
the
end
points
of
the
dashed
edges
in
the
boxes
are
fixed.
The
doubly
headed
arrows indicate that no matter if this end point is at one head or the other, this coloring is still good. Occasionally, we have to explain why when we move the end point of the dashed edge to the other head of the arrow, the coloring is still OK. The explanations are given right on the figure itself, and they are self-explaining.

The CIVs \( v_1 \) and \( v_2 \) have been put next to the colorings for the ease of referencing which edge gets which color, and checking if the coloring is a valid one. Note that we only color the edges not in \( G_{22} \) as usual.

(3b) \( v_1 = [10 \ 02 \ 54 \ 06], v_2 \in F_3 u_1 \in F_3, \) and \( u_2 \in F_2. \) Figure 24 shows our solution to this case.

![Figure 24: Case 3b of Lemma 3.8](image)

(3c) \( v_1 = [10 \ 02 \ 54 \ 06], v_2 \in F_3 u_1 \in F_2, \) and \( u_2 \in F_4. \) Figure 25 shows our solution to this case.

![Figure 25: Case 3c of Lemma 3.8](image)

(3d) \( v_1 = [10 \ 02 \ 54 \ 06], v_2 \in F_3 u_1 \in F_4, \) and \( u_2 \in F_2. \) This sub-case is a little special, as we have to consider 4 cases shown in Figure 26.

(3e) \( v_1 = [10 \ 02 \ 54 \ 06], v_2 \in F_3 u_1 \in F_3, \) and \( u_2 \in F_4. \) This case is even more special. There are 4 sub-cases to be considered as shown in Figure 27. In the last sub-case (see the last drawing of Figure 27), we have provided the coloring so that the \( v_j \) respect the \( P_i \) and the \( u_j \) respect the \( P_i. \) Flipping \( G \) vertically would complete the proof. The vectors \( u_1 \)
and \( u_2 \) have been put on the right of the drawing, indicating that we are trying to flip \( G \) vertically.

(3f) \( v_1 = [10, 0, 2, 54, 0, 6] \), \( u_2 \in F_3 \), \( u_1 \in F_4 \), and \( u_2 \in F_3 \). Figure 28 shows our solution to this case. In the first and second drawings, we use the vertical flip of \( G \).

(3g) \( v_1 = [10, 0, 2, 0, 4, 56] \), \( u_2 \in F_3 \), \( u_1 \in F_3 \), and \( u_2 \in F_4 \). Figure 29 shows our solution to this case.

(3h) \( v_1 = [10, 0, 2, 0, 4, 56] \), \( u_2 \in F_3 \), \( u_1 \in F_4 \), and \( u_2 \in F_3 \). The coloring shown in case (3f) is also valid here.

(3i) \( v_1 = [0, 0, 12, 54, 0, 6] \), \( u_2 \in F_3 \), \( u_1 \in F_4 \), and \( u_2 \in F_3 \). Figure 30 shows our solution to this case. We use the vertical flip of \( G \) in the second, third and fifth drawings.

(3j) \( v_1 = [0, 0, 12, 54, 0, 6] \), \( u_2 \in F_3 \), \( u_1 \in F_3 \), and \( u_2 \in F_4 \). The coloring in case (3e) is also valid here.

**Case 4.** \( v_1 \in F_3 \) and \( u_2 \in F_4 \). Now that all cases 1, 2, and 3 have been proven, we can assume that \( u_1 \) and \( u_2 \) are also representatives of \( F_3 \) and \( F_4 \). Otherwise, we flip \( G \) vertically and apply one of the previous cases. This case is surprisingly simple. There are 2 sub-cases depending on which of \( u_1 \) and \( u_2 \) comes from which of \( F_3 \) and \( F_4 \). Figure 31 shows the solution to both the sub-cases.

### 6 Discussions

In this paper, we have verified that the 7-stage SE network for \( n = 4 \) is rearrangeable. This result and an extension of another formulation were used to show that \( 3n - 5 \) SE stages are sufficient for the rearrangeability of the SE network with \( 2^n \) inputs and \( 2^n \) outputs.
Figure 27: Case 3e of Lemma 3.8

Figure 28: Case 3f of Lemma 3.8

Figure 29: Case 3g of Lemma 3.8
Figure 30: Case 3i of Lemma 3.8

Figure 31: Case 4 of Lemma 3.8
It was conjectured that $2k - 1$ SE stages are necessary and sufficient for the SE network to be rearrangeable. However, there has been very slow progress on proving the conjecture. Although the proof of the main theorem is tedious and fairly tricky, there are several tricks that are used very often. Some of the them were general enough to be put as lemmas and propositions. We were not able to generalize the others, which are interesting in their own right. The proof, in some sense, also “shows” why this conjecture is so difficult. There is no particular technique that could be used throughout, we need different tricks to solve different subcases. That is not to say there is no nice proof of the main theorem or the conjecture, we just were not able to see the “right” formulation, yet. In particular, any connection of our proof to the formulation of Linial and Tarsi should be helpful, beside the fact that the binary representation of our colors are precisely the row vectors of the matrix $M$ to be constructed in Linial and Tarsi’s formulation.

We hope that our work, besides improving the bound, contributes to the effort of attacking this difficult problem. We believe that an algebraic formulation of the proof would yield better upper bound for $\eta(n)$.

References


