Knowledge Representation and Reasoning
Logics for Artificial Intelligence

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2.3 Clause Form Propositional Logic

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2.3.1 Clause Form Syntax

part 1

Atomic Propositions:
- Any letter of the alphabet
- Any letter with a numeric subscript
- Any alphanumeric string.

Literals:
If $P$ is an atomic proposition, $P$ and $\neg P$ are literals.
$P$ is called a positive literal
$\neg P$ is called a negative literal.
2.3.1 Clause Form Syntax

part 2

Clauses: If $L_1, \ldots, L_n$ are literals
then the set $\{L_1, \ldots, L_n\}$ is a clause.

Sets of Clauses: If $C_1, \ldots, C_n$ are clauses
then the set $\{C_1, \ldots, C_n\}$ is a set of clauses.
2.3.2 Clause Form Semantics

Atomic Propositions

**Intensional:** $[P]$ is some proposition in the domain.

**Extensional:** $\llbracket P \rrbracket$ is either True or False.
2.3.2 Clause Form Semantics

Literal

Positive Literals: The meaning of $P$ as a literal is the same as it is as an atomic proposition.

Negative Literals:

Intensional:

$[\neg P]$ means that it is not the case that $[P]$.

Extensional: $[\neg P]$ is True if $[P]$ is False; Otherwise, it is False.
2.3.2 Clause Form Semantics

Clauses

Intensional:
\[[\{L_1, \ldots, L_n\}] = [L_1] \text{ and/or } \ldots \text{ and/or } [L_n].\]

Extensional:
\[[\{L_1, \ldots, L_n\}] \text{ is True if at least one of } [L_1], \ldots, [L_n] \text{ is True; Otherwise, it is False.}\]

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2.3.2 Clause Form Semantics

Sets of Clauses

Intensional:

\[ \{C_1, \ldots, C_n\} = [C_1] \text{ and } \ldots \text{ and } [C_n]. \]

Extensional:

\[ \{C_1, \ldots, C_n\} \text{ is True if } [C_1] \text{ and } \ldots \text{ and } [C_n] \text{ are all True;} \]

Otherwise, it is False.
Clause Form Proof Theory: Resolution

Notion of Proof: None!

Notion of Derivation: A set of clauses constitutes a derivation.

Assumptions: The derivation is initialized with a set of assumption clauses $AC_1, \ldots, AC_n$.

Rule of Inference: A clause may be added to a set of clauses if justified by resolution.

Derived Clause: If clause $CQ$ has been added to a set of clauses initialized with the set of assumption clauses $AC_1, \ldots, AC_n$ by one or more applications of resolution, then $AC_1, \ldots, AC_n \vdash CQ$. 

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Resolution

\[
\begin{array}{c}
\{P, L_1, \ldots, L_n\}, \{-P, L_{n+1,} \ldots, L_m\} \\
\hline
\{L_1, \ldots, L_n, L_{n+1,} \ldots, L_m\}
\end{array}
\]

Resolution is sound, but not complete!
Example Derivation

1. \{\neg TomIsTheDriver, \neg TomIsThePassenger\} Assumption
2. \{TomIsThePassenger, BettyIsThePassenger\} Assumption
3. \{TomIsTheDriver\} Assumption

4. \{\neg TomIsThePassenger\} R,1,3
5. \{BettyIsThePassenger\} R,2,4
Example of Incompleteness

\[ \{P\} \models \{P, Q\} \]

but

Resolution does not apply to \(\{\{P\}\}\).
Resolution Refutation

- Notice that \{\{P\}, \{\lnot P\}\} is contradictory.

- Notice that resolution applies to \{P\} and \{\lnot P\} producing \{\}\, the empty clause.

- If a set of clauses is contradictory, repeated application of resolution is guaranteed to produce \{\}.
Implications

- Set of clauses \( \{ P_1, \ldots, P_n, Q_1, \ldots, Q_m \} \) is contradictory.
  - means \( (P_1 \land \ldots \land P_n \land Q_1 \land \ldots \land Q_m) \) is False in all models.
  - means whenever \( (P_1 \land \ldots \land P_n) \) is True, \( (Q_1 \land \ldots \land Q_m) \) is False.
  - means whenever \( (P_1 \land \ldots \land P_n) \) is True \( \neg(Q_1 \land \ldots \land Q_m) \) is True.
  - means \( P_1, \ldots, P_n \models \neg(Q_1 \land \ldots \land Q_m) \).
Negation and Clauses

\( \neg \{L_1, \ldots, L_n\} = \{\neg L_1, \ldots, \neg L_n\}. \)

\( \neg L = \begin{cases} 
\neg A & \text{if } L = A \\
A & \text{if } L = \neg A 
\end{cases} \)
Resolution Refutation

To decide if $C_1, \ldots, C_n \models CQ$:

1. Let $S = \{C_1, \ldots, C_n\} \cup \neg CQ$

2. Repeatedly apply resolution to clauses in $S$.
   
   (Determine if $\{C_1, \ldots, C_n\} \cup \neg CQ \vdash \{\}$)

3. If generate $\{\}$, $C_1, \ldots, C_n \models CQ$.
   
   (If $\{C_1, \ldots, C_n\} \cup \neg CQ \vdash \{\}$ then $C_1, \ldots, C_n \models CQ$)

4. If reach point where no new clause can be generated, but $\{\}$ has not appeared, $C_1, \ldots, C_n \not\models CQ$.
   
   (If $\{C_1, \ldots, C_n\} \cup \neg CQ \not\vdash \{\}$ then $C_1, \ldots, C_n \not\models CQ$)

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Example 1

To decide if \( \{P\} \models \{P, Q\} \)

\[ S = \{\{P\}, \{\neg P\}, \{\neg Q\}\} \]

1. \( \{P\} \quad \text{Assumption} \)
2. \( \{\neg P\} \quad \text{From query clause} \)
3. \( \{\} \quad R, 1, 2 \)
Example 2

To decide if

\{\neg TomIsTheDriver, \neg TomIsThePassenger\},
\{TomIsThePassenger, BettyIsThePassenger\},
\{TomIsTheDriver\} \models \{BettyIsThePassenger\}

1. \{\neg TomIsTheDriver, \neg TomIsThePassenger\} Assumption
2. \{TomIsThePassenger, BettyIsThePassenger\} Assumption
3. \{TomIsTheDriver\} Assumption
4. \{\neg BettyIsThePassenger\} From query clause
5. \{TomIsThePassenger\} \quad R, 2, 4
6. \{\neg TomIsTheDriver\} \quad R, 1, 5
7. \{\} \quad R, 3, 6
Resolution Efficiency Rules

Tautology Elimination: If clause $C$ contains literals $L$ and $\neg L$, delete $C$ from the set of clauses.

Pure-Literal Elimination: If clause $C$ contains a literal $A$ ($\neg A$) and no clause contains a literal $\neg A$ ($A$), delete $C$ from the set of clauses.

Subsumption Elimination: If the set of clauses contains clauses $C_1$ and $C_2$ such that $C_1 \subseteq C_2$, delete $C_2$ from the set of clauses.

These rules delete unhelpful clauses.
Resolution Strategies

Unit Preference: Resolve shorter clauses before longer clauses.

Set of Support: One clause in each pair being resolved must descend from the query.

Many others

These are heuristics for finding {} faster.
Example 1 Using prover

prover(6): (prove '(P) '(P or Q))

1 (P) Assumption
2 (~ P) From Query
3 (~ Q) From Query
4 nil R,2,1,{}

QED
Example 2 Using prover

prover(5): (prove '(((~ TomIsTheDriver) or (~ TomIsThePassenger))
(TomIsThePassenger or BettyIsThePassenger)
TomIsTheDriver)
'BettyIsThePassenger)

1 (TomIsTheDriver) Assumption
2 ((~ TomIsTheDriver) (~ TomIsThePassenger)) Assumption
3 (TomIsThePassenger BettyIsThePassenger) Assumption
4 ((~ BettyIsThePassenger)) From Query
5 (TomIsThePassenger) R,4,3,{}
Deleting 3 (TomIsThePassenger BettyIsThePassenger) because it’s subsumed by 5 (TomIsThePassenger)

6 ((~ TomIsTheDriver)) R,5,2,{}
Deleting 2 ((~ TomIsTheDriver) (~ TomIsThePassenger)) because it’s subsumed by 6 ((~ TomIsTheDriver))

7 nil R,6,1,{}
QED
Example 1 Using SNARK

snark-user(29): (assert 'P)
nil
snark-user(30): (prove '(or P Q))
(Refutation
(Row 1
  P
  assertion)
(Row 2
  false
  (rewrite ~conclusion 1))
)
:proof-found

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Properties of Resolution Refutation

Resolution Refutation is sound, complete, and a decision procedure for Clause Form Propositional Logic.

It remains so when Tautology Elimination, Pure-Literal Elimination, Subsumption and the Unit-Preference Strategy are included.

It remains so when Set of Support is used as long as the assumptions are not contradictory.
Translating Standard Wfps into Clause Form

Every set of clauses,

\[ \{\{L_{1,1}, \ldots, L_{1,n_1}\}, \ldots, \{L_{m,1}, \ldots, L_{m,n_m}\}\} \]

has the same semantics as the standard wfp

\[ ((L_{1,1} \lor \cdots \lor L_{1,n_1}) \land \cdots \land (L_{m,1} \lor \cdots \lor L_{m,n_m})) \]

That is, there is a translation from any set of clauses into a well-formed proposition of standard propositional logic.

Question: Is there a translation from any well-formed proposition of standard propositional logic into a set of clauses?

Answer: Yes!
Translating Standard Wfps into Clause Form

Conjunctive Normal Form (CNF)

A standard wfp is in CNF if it is a conjunction of disjunctions of literals.

\[((L_{1,1} \lor \cdots \lor L_{1,n_1}) \land \cdots \land (L_{m,1} \lor \cdots \lor L_{m,n_m}))\]

Translation technique:

1. Turn any arbitrary wfp into CNF.

2. Translate the CNF wfp into a set of clauses.
Translating Standard Wfps into Clause Form
Useful Meta-Theorem:
The Subformula Property

If $A$ is (an occurrence of) a subformula of $B$,
and $\models A \iff C$,
then $\models B \iff B\{C/A\}$
Translating Standard Wfps into Clause Form

Step 1

Eliminate occurrences of $\Leftrightarrow$ using

$$\models (A \Leftrightarrow B) \Leftrightarrow ((A \Rightarrow B) \land (B \Rightarrow A))$$

From: $(LivingThing \Leftrightarrow (Animal \lor Vegetable))$

To:

$((LivingThing \Rightarrow (Animal \lor Vegetable))$

$\land((Animal \lor Vegetable) \Rightarrow LivingThing))$
Translation Step 2

Eliminate occurrences of $\Rightarrow$ using

$$\models (A \Rightarrow B) \iff (\neg A \lor B)$$

From:

$$(\langle \text{LivingThing} \Rightarrow (\text{Animal} \lor \text{Vegetable})\rangle \land (\langle \text{Animal} \lor \text{Vegetable} \rangle \Rightarrow \text{LivingThing}))$$

To:

$$((\neg \text{LivingThing} \lor (\text{Animal} \lor \text{Vegetable})) \land (\neg (\text{Animal} \lor \text{Vegetable}) \lor \text{LivingThing}))$$
Translation Step 3

Translate to miniscope form using

\[ \models \neg(A \land B) \iff (\neg A \lor \neg B) \]
\[ \models \neg(A \lor B) \iff (\neg A \land \neg B) \]
\[ \models \neg(\neg A) \iff A \]

From:

\[ ((\neg\text{LivingThing} \lor (\text{Animal} \lor \text{Vegetable})) \]
\[ \land (\neg(\text{Animal} \lor \text{Vegetable}) \lor \text{LivingThing})) \]

To:

\[ ((\neg\text{LivingThing} \lor (\text{Animal} \lor \text{Vegetable})) \]
\[ \land ((\neg\text{Animal} \land \neg\text{Vegetable}) \lor \text{LivingThing})) \]
Translation Step 4

CNF: Translate into Conjunctive Normal Form, using

\[ \models (A \lor (B \land C)) \iff ((A \lor B) \land (A \lor C)) \]

From:

\[ (\neg LivingThing \lor (Animal \lor Vegetable)) \]
\[ \land ((\neg Animal \land \neg Vegetable) \lor LivingThing) \]

To:

\[ (\neg LivingThing \lor (Animal \lor Vegetable)) \]
\[ \land ((\neg Animal \lor LivingThing) \land (\neg Vegetable \lor LivingThing)) \]
Translation Step 5

Discard extra parentheses using the associativity of ∧ and ∨.

From:

\[(\neg\text{LivingThing} \lor (\text{Animal} \lor \text{Vegetable})) \land ((\neg\text{Animal} \lor \text{LivingThing}) \land (\neg\text{Vegetable} \lor \text{LivingThing})))\]

To:

\[(\neg\text{LivingThing} \lor \text{Animal} \lor \text{Vegetable}) \land (\neg\text{Animal} \lor \text{LivingThing}) \land (\neg\text{Vegetable} \lor \text{LivingThing}))\]
Translation Step 6

Turn each disjunction into a clause, and the conjunction into a set of clauses.

From:
\[ ((\neg \text{LivingThing} \lor \text{Animal} \lor \text{Vegetable}) \]
\[ \land (\neg \text{Animal} \lor \text{LivingThing}) \]
\[ \land (\neg \text{Vegetable} \lor \text{LivingThing}) \]

To:
\[ ((\neg \text{LivingThing} \land \text{Animal} \land \text{Vegetable}) \]
\[ (\neg \text{Animal} \land \text{LivingThing}) \]
\[ (\neg \text{Vegetable} \land \text{LivingThing}) \]
Use of Translation

\[ A_1, \ldots, A_n \models_{Standard} B \]

iff

The translation of \( A_1 \land \cdots \land A_n \land \neg B \) into a set of clauses is contradictory.
## Connections

**Modus Ponens**

\[
\begin{align*}
A, A & \Rightarrow B \\
B
\end{align*}
\]

**Modus Tollens**

\[
\begin{align*}
A & \Rightarrow B, \neg B \\
\neg A
\end{align*}
\]

**Disjunctive Syllogism**

\[
\begin{align*}
A \lor B, \neg A \\
B
\end{align*}
\]

**Chaining**

\[
\begin{align*}
A & \Rightarrow B, B \Rightarrow C \\
A \Rightarrow C
\end{align*}
\]

**Resolution**

\[
\begin{align*}
\{A\}, \{\neg A, B\} \\
\{B\}
\end{align*}
\]

\[
\begin{align*}
\{\neg A, B\}, \{\neg B\} \\
\{\neg A\}
\end{align*}
\]

\[
\begin{align*}
\{A, B\}, \{\neg A\} \\
\{B\}
\end{align*}
\]

\[
\begin{align*}
\{\neg A, B\}, \{\neg B, C\} \\
\{\neg A, C\}
\end{align*}
\]
# More Connections

<table>
<thead>
<tr>
<th>Clause</th>
<th>Rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>( { \neg A_1, \ldots, \neg A_n, C_1, \ldots, C_m } )</td>
<td>((A_1 \land \cdots \land A_n) \Rightarrow (C_1 \lor \cdots \lor C_m))</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Horn Clause</th>
<th>Rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>( { \neg A_1, \ldots, \neg A_n, C } )</td>
<td>((A_1 \land \cdots \land A_n) \Rightarrow C)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Prolog Clause</th>
<th>Back-chaining</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C : \neg A_1, \ldots, A_n )</td>
<td></td>
</tr>
</tbody>
</table>
prover Example

prover(57): (prove '((LivingThing <=> (Animal or Vegetable))
    (LivingThing & (~ Animal)))
    'Vegetable)

1 (LivingThing) Assumption
2 ((~ Animal)) Assumption
3 ((~ Animal) LivingThing) Assumption
4 ((~ Vegetable) LivingThing) Assumption
5 ((~ LivingThing) Animal Vegetable) Assumption
6 ((~ Vegetable)) From Query

Deleting 3 ((~ Animal) LivingThing) because it’s subsumed by 1 (LivingThing)
Deleting 4 ((~ Vegetable) LivingThing) because it’s subsumed by 1 (LivingThing)
prover Example, continued

1 (LivingThing) Assumption
2 (¬ Animal) Assumption
5 (¬ LivingThing) Animal Vegetable) Assumption
6 (¬ Vegetable) From Query

7 (¬ LivingThing) Animal) R,6,5,{}
Deleting 5 (¬ LivingThing) Animal Vegetable)
because it’s subsumed by 7 (¬ LivingThing) Animal)
8 (Animal) R,7,1,{}
9 (¬ LivingThing)) R,7,2,{}
10 nil R,9,1,{}
QED