Analysis of fault tolerance in Cayley digraphs using forbidden faulty sets

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Abstract

The connectivity and fault tolerance measures of various interconnection networks assume that all the neighboring nodes of any node could be faulty at the same time. The forbidden faulty set analysis of restricted connectivity and fault tolerance assumes that a set of nodes cannot be faulty at the same time. We discuss the difficulties in analyzing the fault-tolerance of directed Cayley graphs using forbidden faulty sets. A new forbidden set is defined with pairs of nodes as elements to study the fault tolerance of the Cayley digraphs. The fault tolerance under this forbidden faulty set is shown to be $(2n-5)$. We also present an algorithm for determining the connectivity of Cayley digraphs under $(2n-4)$ faults and evaluate its time complexity.

1 Introduction

Cayley graphs have been studied by many authors in recent years [4, 3, 12, 15, 21, 28, 10] as efficient interconnection networks for parallel processing. Most of the interconnection networks studied for efficient parallel computation are modeled as undirected graphs. However, in reality a communication link between two processing elements (e.g. optical link) is often realized by two directed links in opposite directions [15, 8, 16, 13]. This has led to the study of symmetric directed interconnection networks as efficient topologies for multiprocessor networks [12, 15]. Machines with directed communications are easy to construct, and they allow faster communication by simplifying the protocols used at the link level [11, 27, 9]. The uni-directional counterparts of the well studied undirected interconnection networks like the star graph [13], the hypercube [9, 16], and mesh [8] have also been proposed and analyzed by many researchers. Recently, a set of directed Cayley graphs called rotator graphs has been introduced in the literature [12]. The cycle prefix digraphs introduced by Faber, Moore, and Chen [15] were similarly defined. These sets of graphs are isomorphic except for reversed directions. The diameter of n-rotator graph and cycle prefix digraphs is lower than that of star and pancake graphs for the same number of nodes. Similar to the star graph these directed Cayley graphs have a degree of $(n-1)$, but the average distance

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between nodes is lower than the star graph. A comparison of the diameter and the average distance of star, pancake, hypercube, and n-rotator has been studied by Corbett [11].

The fault-tolerance of Cayley graphs [2, 1, 5, 6, 12, 1, 11] has been investigated extensively in the literature. Communication structures with high fault tolerance have also been introduced in the literature [19, 26, 20, 23, 7]. It is known that the hierarchical Cayley graphs [2, 12] and hierarchical Cayley digraphs (with some exceptions, see [17, 6]) are optimally fault tolerant. One of the widely used measure is the connectivity of the network. The fault tolerance of the network with connectivity κ is $\kappa - 1$. Apart from the connectivity, many other fault tolerance metrics have been proposed. Pradhan and Meyer [25] discuss the inconsistencies of various fault tolerance measures for interconnection networks. A multiprocessor network is considered to be functional as long as there is a path between any two processors, under the presence of F faults [2]. A directed graph must be strongly connected in order to be F-fault tolerant. A network is said to be maximally fault tolerant when the fault tolerance is one less than the degree of the network. This measure assumes that all the nodes (or links) connected to any node in the interconnection network could fail at the same time. However, in reality the failure of all the nodes connected to any node at the same time is highly unlikely. Therefore, an entirely different fault tolerant analysis is necessary to study the fault tolerant properties of interconnection networks.

Esfahanian [14] introduced the concept of forbidden faulty sets for conditional fault tolerance of interconnection networks, especially for n-cube. Latifi, Hegde, and Naraghi-Pour [22] have recently generalized the forbidden faulty set concept introduced by Esfahanian [14] for n-cubes. The generalized fault tolerant properties of star graphs have also been studied in the literature [29]. In this paper we study the fault tolerance properties of rotator and cycle prefix digraphs under certain forbidden faulty sets. This paper is organized as follows. In section two, the definitions of rotator and cycle prefix digraphs, fault tolerant measures and forbidden faulty sets are given. The maximum fault tolerance of Cayley digraphs using forbidden sets is given in section three. Section four discusses the algorithm for determining conditional connectivity of Cayley digraphs and its time complexity. Section five concludes the paper with the summary of results.

2 Preliminaries

In this section we present definitions of directed Cayley graphs and other fault tolerance parameters which will be used in subsequent sections. The definition of Cayley graphs and other group theoretic terms can be found in Akers and Krishnamurthy [3, 4] and Harary [18]. The rotator graph and the cycle prefix digraphs are denoted by $\mathcal{R}_n$ and $\mathcal{C}_n$, respectively. The notation $\mathcal{D}_n$ is used to denote both $\mathcal{R}_n$ and $\mathcal{C}_n$. The notation G is used to denote a general graph. The set of nodes and edges of G are denoted by V and E respectively. Since $\mathcal{D}_n$ and $\mathcal{R}_n$ are directed graphs the nodes and edges are denoted by V and $\vec{E}$ respectively.

Definition 1 *The generators of the (n, k)-rotator graph $\mathcal{R}_{(n,k)}$ are of the form*

$$x_1x_2\ldots x_ix_{i+1}\ldots x_k \Rightarrow \begin{cases} x_2x_3\ldots x_ix_1x_{i+1}\ldots x_k & \text{if } 2 \le i \le k < n \\ x_2x_3\ldots x_ix_{i+1}\ldots x_kx_j & \text{if } k < j \le n \end{cases}$$

The total number of nodes and degree of $\mathcal{R}_{(n,k)}$ are $n!/(n-k)!$ and $(n-1)$, respectively.

Definition 2 *The generators of the (n, k)-cycle prefix digraph $\mathcal{C}_{(n,k)}$ are of the form*

$$x_1x_2\ldots x_ix_{i+1}\ldots x_k \Rightarrow \begin{cases} x_ix_1\ldots x_{i-1}x_{i+1}\ldots x_k & \text{if } 2 \le i \le k < n \\ x_jx_1x_2\ldots x_ix_{i+1}\ldots x_{k-1} & \text{if } k < j \le n \end{cases}$$

The number of nodes and degree of $\mathcal{C}_{(n,k)}$ are the same as that of $\mathcal{R}_{(n,k)}$.

In this paper we consider special cases of $\mathcal{R}_{(n,k)}$ and $\mathcal{C}_{(n,k)}$, the n-rotator $\mathcal{R}_n$ and the n-cycle prefix digraph $\mathcal{C}_n$. The generators of $\mathcal{R}_n$ and $\mathcal{C}_n$ are of the form $x_1x_2\ldots x_ix_{i+1}\ldots x_n \stackrel{g_i}{\Rightarrow} x_2x_3\ldots x_ix_1x_{i+1}\ldots x_n$ and $x_1x_2\ldots x_ix_{i+1}\ldots x_n \stackrel{g_i}{\Rightarrow} x_ix_1x_2\ldots x_{i-1}x_{i+1}\ldots x_n$, respectively. Both $\mathcal{R}_n$ and $\mathcal{C}_n$ have $n!$ vertices, degree $(n-1)$, connectivity $(n-1)$, and diameter $(n-1)$. The results obtained for $\mathcal{R}_n$ and $\mathcal{C}_n$ can be easily extended to $\mathcal{R}_{(n,k)}$ and $\mathcal{C}_{(n,k)}$. Most of the results presented here for directed Cayley graphs refer to the rotator graph. We use the terms rotation and generator interchangeably, while discussing rotator graphs. It should be noted that the rotator graphs are isomorphic to the cycle prefix digraphs with the direction of the edges reversed [12]. Fig. 1 shows an example of the 4-rotator graph. The direction of the links are indicated by the arrows. Since the generator g_2 is reversible, the resultant directed links between two adjacent vertices are denoted by thick undirected links. The link denoted by an alphabet (o) denotes an outgoing link. This link is connected to the node with a link marked with the same alphabet and (i) (means incoming). We summarize some of the known properties of $\mathcal{R}_n$ [12, 15, 24].

- $\mathcal{R}_n$ is hierarchical and vertex symmetric.
- The diameter of $\mathcal{R}_n$ is $(n-1)$.
- The in-degree and out-degree of $\mathcal{R}_n$ is $(n-1)$.
- The fault diameter of $\mathcal{R}_n$ is less than or equal to $(n+1)$.
- $\mathcal{R}_n$ is one step $(n-1)$-fault diagnosable.

Definition 3 *The vertex connectivity $\kappa(G)$ of G is defined as the minimum number of faulty vertices S required to disconnect G.*

The fault tolerance f is the maximum number of faults that G can tolerate without being disconnected. Therefore, fault tolerance f of G is one less than the connectivity. In the case of Cayley digraphs $\mathcal{D}_n$, if a directed edge $\vec{e} = (u, v) \in \vec{E}$, then the node v is adjacent to node u

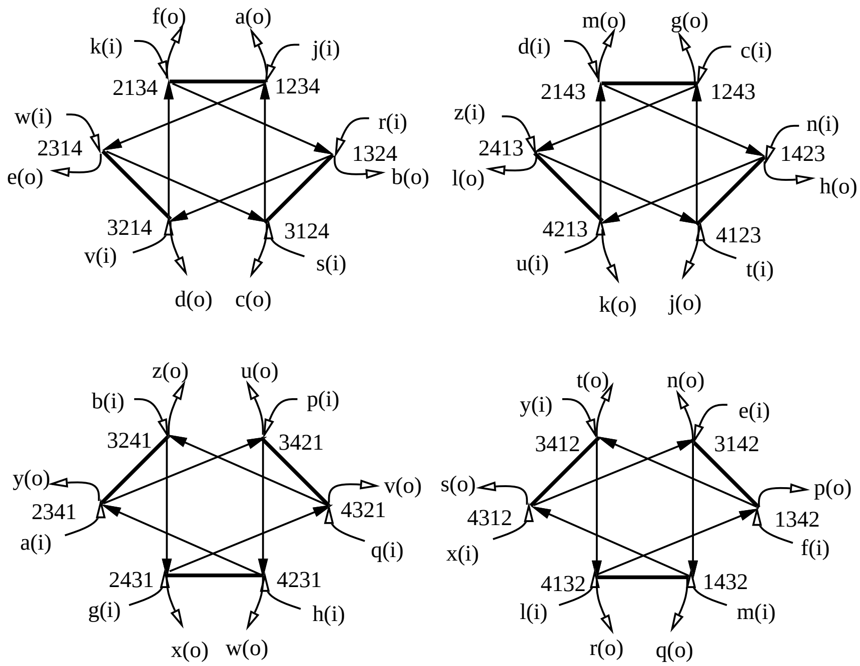

Figure 1: *4-rotator graph ($\mathcal{R}_4$)*

(converse is not necessarily true). Every node has $(n-1)$ incoming links and $(n-1)$ outgoing links. All the nodes connected to the incoming links will be referred to as *incoming nodes* and the nodes connected to the outgoing nodes will be referred to as *outgoing nodes*. The set of all incoming nodes of u are denoted by $In(u)$, and the set of all outgoing nodes of u are denoted by $Out(u)$. There is exactly one node x for every node u in $\mathcal{D}_n$, where $x \in In(u)$ and $x \in Out(u)$, due to the only reversible generator g_2.

Definition 4 *All the incoming and outgoing nodes of any node $u \in V$, (totaling $2n-3$) will be referred to as* neighboring *nodes $N(u)$ of u, where $N(u) = (In(u) \cup Out(u)) - (In(u) \cap Out(u))$.*

A node $v = Out(u)$, can also be represented as $v = g_i \times u$, i.e., v is obtained from u by applying the generator g_i, where $2 \leq i \leq n$. We use the terms vertex, node, and permutation interchangeably. We use the notation ${}^aC_b$ to denote the binomial co-efficient throughout this paper. All $n!$ nodes of $\mathcal{D}_n$ are associated with a unique permutation π of n numbers. The number at the i^{th} position of the permutation π is referred to as $\pi[i]$, where $1 \leq i \leq n$.

3 Fault tolerance of Cayley digraphs

Let $S \in V$, such that $|S| = \kappa(\mathcal{R}_n) = (n-1)$. If the failure (or removal) of the nodes in the set S results in a disconnected graph of size $\mathcal{R}_n - |S|$, the set S is called a minimum cut. There are ${}^{n!}C_\kappa$ distinct subsets of size κ, and only $n!$ of them are minimum cuts. Since the ratio $n!/{}^{n!}C_\kappa$ is very small as n increases, the probability of failure of all elements of $n!$ minimum cut sets is very small.

The fault tolerance and connectivity measures assume that any subset of processors (or links) is equally likely to be faulty. A forbidden set of undirected graphs [14, 29] is defined as the set of all nodes adjacent to any node. The generalized fault tolerance under forbidden sets is usually higher than the fault tolerance of the graphs without forbidden sets. This is due to the fact that any node can communicate in each direction with at least one adjacent node. Therefore, each minimum cut can be a forbidden faulty set, i.e, we can assume that all the neighboring nodes of any node in $\mathcal{R}_n$ cannot fail at the same time. However, the forbidden sets of directed Cayley graphs are different. There are $(2n-3)$ neighbors of a node in $\mathcal{R}_n$. If the forbidden faulty set is the set of all neighboring nodes of any node, then the generalized fault tolerance under this condition will be the same as the fault tolerance of $\mathcal{R}_n$ without forbidden sets. Since all the nodes of a forbidden set cannot fail at the same time, in the worst case, one incoming or outgoing node will be non-faulty. This will not make any difference in the generalized fault tolerance measure of $\mathcal{R}_n$, since the failure of all the $(n-1)$ outgoing or incoming nodes will disconnect the graph. The generalized fault tolerance under this condition is $(n-1)$.

The above selection of forbidden sets is too restrictive and cannot be used for comparison with other undirected graphs. Therefore, the forbidden faulty set $\mathcal{F}_1$ is defined as the the set of $(n-1)$ elements, $\mathcal{F}_1 = \{(x), (x_3, y_3), \ldots, (x_n, y_n)\}$. The element x is the node obtained by the generator g_2 of $\mathcal{R}_n$. There are $(n-2)$ remaining elements in $\mathcal{F}_1$, each consisting of a pair of nodes. An element (x_k, y_k) of $\mathcal{F}_1$, indicates that for any node u, x_k is the incoming node obtained by the generator g_k (i.e., for some node p, $p = g_k \times u$) and y_k is the outgoing node (i.e., for some node q, $u = g_k \times q$), where $3 \le k \le n$. All the elements of $\mathcal{F}_1$ cannot be faulty at the same time, i.e., at most $(n-2)$ elements can be faulty at the same time.

Lemma 1 *There exists cycles of length k in $\mathcal{D}_n$, where $3 \le k \le n$ and $n \ge 3$.*

Proof. The proof of this lemma follows directly from the definition of the directed Cayley graphs [12]. $\square$

Lemma 2 *For any pair of vertices (u, v) in $\mathcal{R}_n$, where $v = g_i \times u$, the number of nodes common to the neighboring nodes of u and v, excluding the nodes u and v is,*

$$|\{N(u) - v\} \cap \{N(v) - u\}| = \begin{cases} 1 & \text{if } i = 3 \\ 0 & \text{otherwise} \end{cases}$$

Proof. The generator g_2 is the only generator of $\mathcal{R}_n$ that is reversible. Consider a pair of vertices $(u, v) \in V$, and $v = g_2 \times u$. Since there are no 3-cycles (cycles of length three) in $\mathcal{R}_n$ with two nodes connected by generator g_2, both nodes u and v have $2(n-2)$ distinct neighbors each. Similar argument holds for nodes connected by generators $g_4, g_5, \ldots, g_n$. However, if $v = g_3 \times u$, then g_3 applied twice to v would result in node u. Therefore one of the outgoing node of v is an incoming node for u. This makes $|\{N(u) - v\} \cap \{N(v) - u\}| = 1$. $\square$

Lemma 3 *For any pair of arbitrary nodes (u, v) in $\mathcal{R}_n$, the total number of distinct neighbors of u and v is $|\{\{N(u) - v\} \cup \{N(v) - u\}\} - \{\{N(u) - v\} \cap \{N(v) - u\}\}| \ge 4n - 10$, for $n \ge 4$.*

Proof. Let us consider five possible pairs of nodes (u, v).

Case I. $v = g_i \times u$*, where* $i = 3$*:* It is known from Lemma 2 that $|\{N(u) - v\} \cap \{N(v) - u\}| = 1$, for $i = 3$. Therefore, $|\{\{N(u) - v\} \cup \{N(v) - u\}\} - \{\{N(u) - v\} \cap \{N(v) - u\}\}| = 4n - 10$ (see Fig. 2).

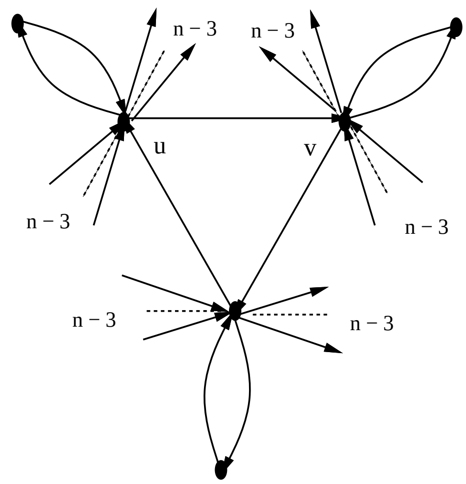

Figure 2: *3-cycle in* $\mathcal{R}_n$

Case II. $v = g_i \times u$*, where* $i = 2, 4, 5, \ldots (n-1)$*:* Since no two neighboring processors in any *3*-cycle of $\mathcal{R}_n$ are connected by any of these generators, $|\{\{N(u) - v\} \cup \{N(v) - u\}\} - \{\{N(u) - v\} \cap \{N(v) - u$ $= 4n - 8$.

Case III. $v \notin In(u)$ *and* $v \notin Out(u)$ *and the pair* (u, v) *is contained in a 4-cycle:* It should be noted that there are only three ways to generate *4*-cycles in $\mathcal{R}_n$ with any node u. The sequence of generators of the three *4*-cycles are $s_1 = (g_4 g_4 g_4 g_4)$, $s_2 = (g_2 g_3 g_2 g_3)$, and $s_3 = (g_3 g_2 g_3 g_2)$. It can be observed from Fig. 3 (for sequence s_1) and Lemma 2 that $|\{\{N(u) - v\} \cup \{N(v) - u\}\} - \{\{N(u) - v\} \cap \{N($ $4n - 10$. Similarly, for the sequences s_2 and s_3, $|\{\{N(u) - v\} \cup \{N(v) - u\}\} - \{\{N(u) - v\} \cap \{N(v) - u\}\}|$ $= 4n - 10$.

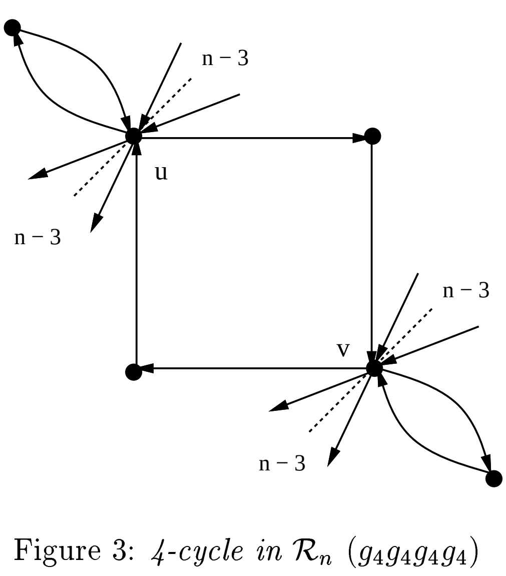

Figure 3: *4-cycle in* $\mathcal{R}_n$ $(g_4 g_4 g_4 g_4)$

Case IV. $v \notin In(u)$, $v \notin Out(u)$, the pair (u, v) is not contained in any 3-cycles or 4-cycles of $\mathcal{R}_n$, and v is two hops away from u: In this case there is only one node which is common to the neighboring sets of u and v. Therefore, the number of distinct neighbors of (u, v) is $4n - 8$.

Case V. $v \notin In(u)$, $v \notin Out(u)$, the pair (u, v) is not contained in any 3-cycles or 4-cycles of $\mathcal{R}_n$, and v is at least three hops away from u: Since there are no common nodes between the neighboring sets of u and v, $|\{\{N(u) - v\} \cup \{N(v) - u\}\} - \{\{N(u) - v\} \cap \{N(v) - u\}\}| = 4n - 6$.

Therefore, any pair of arbitrary vertices (u, v) in $\mathcal{R}_n$ have at least $4n - 10$ distinct neighboring nodes, for $n \geq 4$. □

Theorem 1 *The generalized fault tolerance of $\mathcal{R}_n$ under the forbidden set $\mathcal{F}_1$ is $2n - 5$ for $n \geq 3$.*

Proof: Let us consider a pair of non-faulty nodes $(u, v) \in V$ in $\mathcal{D}_n$. We consider four cases.

Case I: $v = g_3 \times u$ or $u = g_3 \times v$: Nodes u and v have $(n - 3)$ distinct incoming nodes, $(n - 3)$ distinct outgoing nodes, and one node each obtained by generator g_2. One of the incoming nodes of u (or v) is the outgoing node of v (or u). Again, the graph is always connected if the node common to the node sets $\{N(u) - v\}$ and $\{N(v) - u\}$ fails, since there is at least one input and output non-faulty node. However, if all the three nodes in the directed 3-cycle are non-faulty (Fig. 2), then the graph will become disconnected if all the incoming or outgoing nodes of all three nodes in the 3-cycle are faulty. For the graph to be strongly connected, at least one incoming node and one outgoing node should be non-faulty in any of the three non-faulty nodes in the 3-cycle. Therefore, the generalized fault tolerance under this condition is $3(n - 2) - 1 = 3n - 7$.

Case II: $v \notin In(u)$, $v \notin Out(u)$ and (u, v) are elements of a 4-cycle: Similar to Case I, the failure of any of the common neighboring nodes to u and v will not disconnect the graph. When all the nodes in the 4-cycle containing (u, v) are non-faulty, the fault tolerance is $4(n-2) - 1 = 4n - 9$ (Fig. 3).

Case III: $v = g_2 \times u$ and $u = g_2 \times v$: For each node u and v, there are $(2n - 4)$ distinct neighbors ($(n - 2)$ incoming and $(n - 2)$ outgoing nodes), excluding nodes u and v. If all the $(n - 2)$ incoming (or outgoing) nodes of both u and v are faulty, then the graph will not be strongly connected. Therefore, at most $(2n - 5)$ incoming or outgoing nodes can be faulty without disconnecting the network.

Case IV: $v \notin In(u)$, $v \notin Out(u)$, and (u, v) is not contained in any 4-cycle: It can be easily shown that the fault tolerance under this condition is at least $2n - 5$. An example is given in Fig. 4. Consider the case when all the nodes in Fig. 4 are non-faulty. Since all the incoming and outgoing nodes of the nodes between u and v can be faulty, the worst case occurs when all the incoming nodes of u and u' $(= g_2 \times u)$ or all the output nodes of v and v' $(= g_2 \times v)$ are faulty (since the nodes in Fig. 4 form a directed path). Therefore, the fault tolerance under this condition is $2n - 5$. Similar arguments hold when node v is two hops away from node u. □

These Cayley digraphs offer high fault tolerance under the forbidden faulty sets. It can be observed that the fault tolerance is $(2n - 5)$ only under one condition (i.e., when the nodes connected by the generator g_2 are non-faulty), otherwise the fault tolerance is at least $(3n - 7)$. There are $n!/2$ possible pairs of nodes connected by generator g_2 in $\mathcal{D}_n$. Each pair has two possible

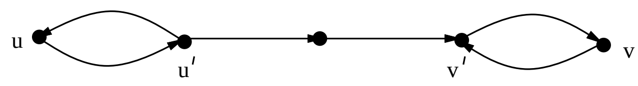

Figure 4: *An example of non-adjacent nodes (u, v) in $\mathcal{R}_n$*

sets of $(2n-4)$ nodes (i.e., either incoming or outgoing nodes). Therefore, there are only $n!$ possible faulty sets with $(2n-4)$ nodes that can disconnect $\mathcal{D}_n$. Since the maximum possible number of $(2n-4)$ processor sets is ${}^{n!}C_{(2n-4)}$, the ratio $n!/{}^{n!}C_{(2n-4)}$ becomes very small as n increases. For example, when $n = 3$, the ratio is $\frac{6}{15}$, and when $n = 4$, the ratio is $\frac{4}{1761}$. Therefore, the probability of failure of such sets is very small in $\mathcal{D}_n$.

4 Algorithm for conditional connectivity

In this section we present algorithms for determining connectivity of the directed Cayley graph in the presence of $(2n-4)$ faults or less. Let f be the set of faulty nodes in $\mathcal{R}_n$, and $|f| \leq 2n-4$.

Algorithm 1:

Step 0: If $|f| < (n-1)$, then $\mathcal{R}_n - |f|$ is connected. Stop.

Step 1: If $(n-1) \leq |f| \leq (2n-4)$, and if there exists a node $u \in V$, such that $u \notin f$ and $In(u) \subseteq f$ or $Out(u) \subseteq f$ then $\mathcal{R}_n - |f|$ is disconnected. Stop.

Step 2: If $|f| < 2n-4$, then $\mathcal{R}_n - |f|$ is connected. Stop.

Step 3: If $|f| = 2n-4$ and ($|f| = \{In(u) \cup In(v)\}$ or $|f| = \{Out(u) \cup Out(v)\}$), for $(u, v) \in V$, and $u = g_2 \times v$, then $\mathcal{R}_n - |f|$ is disconnected; otherwise $\mathcal{R}_n - |f|$ is connected. Stop.

Time complexity analysis: Step 0 and Step 2 require constant time. In Step 1, since a vertex $v \in f$, may be an incoming node or outgoing node for any other node, we need to examine $(2n-3)|f|$ vertices. For each such vertex, at most $(n-1)|f|$ comparisons are necessary to check if $In(v) \subseteq f$ or $Out(v) \subseteq f$. Therefore, the worst case computational requirement is $O((n|f|)^2)$. $\mathcal{R}_n$ with $|f| = 2n-4$ faulty nodes, will be disconnected only when all the faults occur at the input or output nodes of two adjacent nodes (u, v), where $v = g_2 \times u$ (Theorem 1).

Before we discuss the implementation of Step 3 of Algorithm 1, we present some properties of the neighboring sets of u and v. Consider any two nodes of $\mathcal{R}_n$, namely $\pi_u = 1234\ldots(n-1)n$ and $\pi_v = 2134\ldots(n-1)n$, where $\pi_u = g_2 \times \pi_v$. The set of incoming nodes and the set of outgoing nodes of π_u and π_v are of the form,

$$In(\pi_u) = \begin{cases} 31245\ldots(n-1)n, \\ 41235\ldots(n-1)n, \\ 51234\ldots(n-1)n, \\ \vdots \\ n123\ldots(n-2)(n-1) \end{cases}$$

$$Out(\pi_u) = \begin{cases} 23145\ldots(n-1)n, \\ 23415\ldots(n-1)n, \\ 23451\ldots(n-1)n, \\ \vdots \\ 23456\ldots n1 \end{cases}$$

$$In(\pi_v) = \begin{cases} 32145\ldots(n-1)n, \\ 42135\ldots(n-1)n, \\ 52134\ldots(n-1)n, \\ \vdots \\ n2134\ldots(n-2)(n-1) \end{cases}$$

$$Out(\pi_v) = \begin{cases} 13245\ldots(n-1)n, \\ 13425\ldots(n-1)n, \\ 13452\ldots(n-1)n, \\ \vdots \\ 13456\ldots n2 \end{cases}$$

There are few things to note here. All the permutations in the incoming or the outgoing sets of π_u and π_v have the same symbol at the second position. Only one permutation in any set has a last symbol different from other elements of the set, and all other elements have the symbol $\pi_u[n]$ (same as $\pi_v[n]$) at the last position. Once the permutation with a different last symbol is identified, it is easy to identify the permutations (π_u, π_v) and check whether their neighbors are in the faulty set f. Now, Step-3 can be implemented as follows;

1. If $\pi_i[2]$ is not same for all $\pi_i \in f$, then divide the faulty f set into two equal sets f_1 and f_2 with the same $\pi[2]$. If the division is not possible return not disconnected. If $\pi_i[2]$ is the same for all $\pi_i \in f$, divide f into two sets f_1 and f_2 with the same $\pi[1]$, respectively. If not, return not disconnected.

2. If $(n-3)$ permutations of the total $(n-2)$ permutations of f_1 do not have same last symbol and one permutation has a different last symbol, or $(n-3)$ permutations of f_2 do not have same last symbol and one permutation has a different last symbol, then return not disconnected. The permutations with a different n^{th} symbol in f_1 and f_2 are denoted as π_{f_1} and π_{f_2} respectively.

3. Insert $\pi_{f_1}[1]$ and $\pi_{f_2}[1]$ in the n^{th} positions of π_{f_1} and π_{f_2} to generate the permutations π_x and π_y, respectively. If π_x and π_y satisfy the condition, $\pi_x = g_2 \times \pi_y$, and $(\pi_x, \pi_y) \notin f$, and $In(\pi_x) \subset f$ and $In(\pi_y) \subset f$, then return disconnected. If not, insert the symbol $\pi_{f_1}[n]$ and $\pi_{f_2}[n]$ in the first position of the permutations π_{f_1} and π_{f_2} respectively. If the resultant permutations x and y satisfy the condition, $x = g_2 \times y$ and $(x, y) \notin f$, and $Out(\pi_x) \subset f$ and $Out(\pi_y) \subset f$, then return disconnected. If not, return not disconnected.

It requires $O(|f|)$ steps to compare elements $\pi_i[2]$ and $\pi_i[1]$ for all $\pi_i \in f$. The next step to identify π_{f_1} and π_{f_2} also requires $O(|f|)$ steps. The comparisons in part three requires $O(|f|^2)$ steps in the worst case to find out whether the graph is disconnected. So, Step 3 requires $O(|f|^2)$ steps. Therefore, the time complexity of Algorithm 1 is $O((n|f|)^2)$.

5 Conclusion

In this paper we have used the forbidden faulty sets to analyze the fault tolerance of directed Cayley graphs. A new forbidden set is defined with pairs of nodes as elements to study the fault tolerance of the Cayley digraphs. The fault tolerance of Cayley digraphs under this forbidden faulty set is shown to be $(2n-5)$. The fault tolerance of rotator graphs is $(2n-5)$ only when the nodes connected by the generator g_2 are non-faulty. In other cases the fault tolerance is shown to be at least $(3n-7)$. The Cayley digraphs become disconnected only when any two nodes connected by the reversible generator are non-faulty and all their incoming nodes or outgoing nodes are faulty. Therefore, there are only $n!$ sets of size $2n-4$ which can disconnect the directed Cayley graphs. In comparison to the total ${}^{n!}C_{2n-4}$ sets of size $(2n-4)$, $n!$ is very small as n increases. We also presented an algorithm for the determining the connectivity of directed Cayley graphs under $(2n-4)$ faults and evaluated its time complexity.

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